

AN IMPROVED INITIAL VALUE METHOD FOR SINGULARLY PERTURBED CONVECTION DIFFUSION DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, convection diffusion type singularly perturbed delay differential equations are considered. An asymptotic expansion approximation of the solution is constructed. Further the asymptotic expansion approximation is numerically approximated using the Runge Kutta methods and hybrid finite difference methods. The error estimate is obtained and it is of almost second order. Numerical examples are given to illustrate the present method.

1. INTRODUCTION

In many branches of applied mathematics and engineering, Singularly Perturbed Delay Differential Equations (SPDDEs) are commonly used. In the mathematical modeling of various practical phenomena, certain forms of equations often occur, such as in the modeling of the human pupil-light reflex [1], The mathematical model for calculating the expected time by random synaptic inputs in the dendrites [2] to generate action potential in nerve cells and variational problems in control theory [3]. It is well known that the classical uniform

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mesh techniques do not yield satisfactory results for these equations. Therefore, the direction needs to be shifted towards non-classical methods. In the last three decades, an excellent number of papers have emerged for non-classical approaches covering second order equations and higher order equations. Several numerical methods for solving the various types of Delay Differential Equations (DDEs) are focused in [4]. The author in [5], proposed an exponentially fitted operator method for first order SPDDEs, hybrid finite difference method in [6, 7], etc. Quite a lot of article for solving singularly perturbed small DDEs are available in the literature, to cite some, [8, 9]. For non vanishing DDEs there are number of articles available in the literature, to cite a few [10–13].

An improved Asymptotic Expansion Approximation (AEA) is constructed in this article. In addition, using the Runge-Kutta (R-K) method of fourth order and hybrid finite difference method, asymptotic expansion is approximated numerically. The proposed method is shown to be almost second-order convergence.

2. STATEMENT OF THE PROBLEM

We assume throughout the article that C and C_1 represent arbitrary constants irrespective of parameters ε and N . The index set $I_{2N} = \{1, 2, \dots, 2N\}$. To study the convergence of the numerical solution to the exact solution of a singular perturbation problem, the supremum norm is used: $\|w\|_{\Omega} = \sup_{x \in \Omega} |w(x)|$.

Consider the following Boundary Value Problem (BVP) [12, 14]: Find $y \in Y = C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$(2.1) \quad \begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), & x \in \Omega, \\ y(x) = \phi(x), & x \in [-1, 0], \quad y(2) = l, \end{cases}$$

where $a(x) \geq \alpha_1 > \alpha > 0$, $b(x) \geq \beta_0 \geq 0$, $\gamma_0 \leq c(x) \leq \gamma < 0$, $2\alpha_1 + 5\beta_0 + 5\gamma_0 \geq \eta > 0$, a , b , c , f , and ϕ are sufficiently differentiable functions on $\overline{\Omega}$, $\Omega = (0, 2)$, $\overline{\Omega} = [0, 2]$, $\Omega^- = (0, 1)$, $\Omega^+ = (1, 2)$, $\Omega^* = \Omega^- \cup \Omega^+$.

The above problem can be written as

$$(2.2) \quad \begin{aligned} Py(x) &:= -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) \\ &= \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega^-, \\ f(x) - c(x)y(x-1), & x \in \Omega^+, \end{cases} \\ y(0) &= \phi(0), \quad y(1-) = y(1+), \quad y'(1-) = y'(1+), \quad y(2) = l, \end{aligned}$$

where $y(1-)$ and $y(1+)$ denote the left and right limits of y at $x = 1$, respectively. The problem (2.1) exhibits a strong boundary layer at $x = 2$, [12, 14].

3. ANALYTICAL RESULTS

Let $y_0 \in C^0(\overline{\Omega}) \cap C^1(\Omega \cup \{2\})$ be the reduced problem solution of (2.1) given by

$$(3.1) \quad \begin{cases} a(x)y_0'(x) + b(x)y_0(x) + c(x)y_0(x-1) = f(x), & x \in \Omega \cup \{2\}, \\ y_0(x) = \phi(x), & x \in [-1, 0]. \end{cases}$$

Further, we assume that, $\|y_0''\|_{\Omega^*} \leq C$. Let $y_1 \in C^0(\overline{\Omega}) \cap C^1(\Omega^* \cup \{2\})$ be the solution of

$$(3.2) \quad \begin{cases} a(x)y_1'(x) + b(x)y_1(x) + c(x)y_1(x-1) = y_0''(x), & x \in \Omega^* \cup \{2\}, \\ y_1(x) = 0, & x \in [-1, 0], \end{cases}$$

the second derivative of y_0 can be written as

$$(3.3) \quad y_0''(x) = \frac{1}{a(x)} \left[f'(x) - \left(a'(x) + b'(x) + \frac{b^2(x)}{a(x)} \right) y_0(x) - \frac{b(x)}{a(x)} f(x) \right. \\ \left. + \left(\frac{c(x)b(x)}{a(x)} - c'(x) \right) \times y_0(x-1) - \frac{c(x)}{a(x-1)} [f(x-1) \right. \\ \left. - b(x-1)y_0(x-1) - c(x-1)y_0(x-2)] \right], \quad x \in \Omega^*,$$

and assume that $\|y_1''\|_{\Omega^*} \leq C$. Let v_1 and v_2 be two functions satisfy the following Terminal Value Problems (TVPs)

$$(3.4) \quad \varepsilon v_1'(x) - a(x)v_1(x) = 0, \quad x \in [0, 1), \quad v_1(1) = 1$$

and

$$(3.5) \quad \varepsilon v_2'(x) - a(x)v_2(x) = 0, \quad x \in [0, 2), \quad v_2(2) = 1.$$

Define an AEA of $y(x)$

$$(3.6) \quad y_{as}(x) = \begin{cases} y_0(x) + \varepsilon y_1(x) + k_1(v_1(x) - v_1(0)), & x \in [0, 1], \\ y_0(x) + \varepsilon y_1(x) + k_2(v_2(x) - 1) - y_0(2) + l - \varepsilon y_1(2), & x \in [1, 2], \end{cases}$$

where

$$k_1 = \frac{1}{(1 - v_1(0))(1 - v_1(0)v_2(1))} \cdot \left[v_2(1)(1 - v_1(0))(l - y_0(2) - \varepsilon y_1(2)) \right. \\ \left. + \frac{\varepsilon^2}{a(1)} \{ (y_1'(1+) - y_1'(1-))(1 - v_1(0))(1 - v_2(1)) \} \right] \\ k_2 = \frac{l - y_0(2) - \varepsilon y_1(2) - \frac{\varepsilon^2}{a(1)} (y_1'(1+) - y_1'(1-))(1 - v_1(0))}{1 - v_1(0)v_2(1)}.$$

It is easy to see that $k_1 = O(\varepsilon^2)$, $k_2 = O(1)$.

Theorem 3.1. *If the solutions y and its AEA y_{as} of (2.1) and (3.6), respectively. Then, $|y(x) - y_{as}(x)| \leq C\varepsilon^2$, $x \in \bar{\Omega}$.*

Proof. Using the following barrier function and adapting the procedure given in [10, Theorem 3.1], one can prove the desired result: $\varphi^\pm(x) = C_1\varepsilon^2\psi(x) \pm (y(x) - y_{as}(x))$, $x \in \Omega$, where

$$\psi(x) = \begin{cases} \frac{1}{8} + \frac{x}{2} + e^{-\frac{\alpha}{\varepsilon}(1-x)} + \frac{(2-x)^2}{\varepsilon} e^{-\frac{\alpha}{\varepsilon}(2-x)}, & x \in [0, 1] \\ \frac{3}{8} + \frac{x}{4} + 1 + \frac{(2-x)^2}{\varepsilon} e^{-\frac{\alpha}{\varepsilon}(2-x)}, & x \in [1, 2]. \end{cases}$$

□

4. DISCRETE PROBLEM

The mesh $\bar{\Omega}^{2N}$ defined in [10, Section 5.1] is used to compute the numerical solution.

4.1. Numerical Methods for Initial Value Problems. R-K method of fourth order with piecewise cubic Hermite interpolation is applied on the mesh $\bar{\Omega}^{2N}$ [4, 10], then we have

$$(4.1) \quad \begin{aligned} Y_0(x_0) &= \phi(x_0), \\ Y_0(x_{i+1}) &= Y_0(x_i) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \quad i = 0, 1, \dots, 2N - 1, \end{aligned}$$

where

$$\begin{cases} K_1 = h_i \left[f(x_i) - b(x_i)Y_0(x_i) - c(x_i)Y_0^I(x_i) \right] \frac{1}{a(x_i)}, \\ K_2 = h_i \left[f(x_i + \frac{h_i}{2}) - b(x_i + \frac{h_i}{2})(Y_0(x_i) + \frac{K_1}{2}) - c(x_i + \frac{h_i}{2})Y_0^I(x_i + \frac{h_i}{2}) \right] \frac{1}{a(x_i + \frac{h_i}{2})}, \\ K_3 = h_i \left[f(x_i + \frac{h_i}{2}) - b(x_i + \frac{h_i}{2})(Y_0(x_i) + \frac{K_2}{2}) - c(x_i + \frac{h_i}{2})Y_0^I(x_i + \frac{h_i}{2}) \right] \frac{1}{a(x_i + \frac{h_i}{2})}, \\ K_4 = h_i \left[f(x_i + h_i) - b(x_i + h_i)(Y_0(x_i) + K_3) - c(x_i + h_i)Y_0^I(x_i + h_i) \right] \frac{1}{a(x_i + h_i)}, \end{cases}$$

$$h_i = x_i - x_{i-1},$$

$$Y_0^I(x) = \begin{cases} \phi(x-1), & x \in [x_i, x_{i+1}], \quad i = 0, 1, 2, \dots, N-1, \\ Y_0(x_{i-N})A_{i-N}(x-1) + Y_0(x_{i-N+1})A_{i+1-N}(x-1) + B_{i-N}(x-1)\tilde{f}(x_{i-N}) \\ \quad + B_{i+1-N}(x-1)\tilde{f}(x_{i-N+1}), \\ x \in [x_i, x_{i+1}], \quad i = N, N+1, \dots, 2N-1, \end{cases},$$

$A_i(x)$ and $B_i(x)$ are called Hermite polynomials, they are defined in [10, Section 5.2].

Theorem 4.1. [4] *The solution $y_0(x)$ of (3.1) and its discrete problem solution $Y_0(x_i)$ of (4.1) satisfies $\|y_0 - Y_0\|_{\bar{\Omega}^{2N}} \leq CN^{-4}$.*

Lemma 4.1. *If $y_0(x)$ is the solution of (3.1) and its numerical solution is given by (4.1), further its interpolant is $\tilde{Y}_0(x) = \sum_{i=0}^{2N} \phi_i(x)Y_0(x_i)$, then $\|y_0 - \tilde{Y}_0\|_{\bar{\Omega}^{2N}} \leq CN^{-2}$.*

Proof. By the triangle inequality $|y_0(x) - \tilde{Y}_0(x)| \leq |y_0(x) - \tilde{y}_0(x)| + |\tilde{y}_0(x) - \tilde{Y}_0(x)|$, Theorem 4.1 and by [10, 15] we have the desired result. Here $\tilde{y}_0(x) = \sum_{i=0}^{2N} \phi_i(x)y_0(x_i)$ and $\phi_i(x)$ is usual hat function,

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1}-x}{h_{i+1}}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

□

Using (3.3) the $y_0''(x)$ can be approximated as $p(x) = \frac{1}{a(x)}[f'(x) - (a'(x) + b'(x) + \frac{b^2(x)}{a(x)})\tilde{Y}_0(x) - \frac{b(x)}{a(x)}f(x) + (\frac{c(x)b(x)}{a(x)} - c'(x))\tilde{Y}_0(x-1) - c(x)\tilde{Y}_0'(x-1)]$, $x \in \Omega^*$. Then, the

problem (3.2) can be written as,

$$\begin{aligned}
 & a(x)y_1^{*'}(x) + b(x)y_1^*(x) + c(x)y_1^*(x-1) \\
 &= \frac{1}{a(x)} \left[- \left(a'(x) + b'(x) + \frac{b^2(x)}{a(x)} \right) \tilde{Y}_0(x) + f'(x) - \frac{b(x)}{a(x)} f(x) \right. \\
 & \quad + \left(\frac{c(x)b(x)}{a(x)} - c'(x) \right) \tilde{Y}_0(x-1) - \frac{c(x)}{a(x-1)} [f(x-1) \\
 (4.2) \quad & \left. - b(x-1)\tilde{Y}_0(x-1) - c(x-1)\tilde{Y}_0(x-2)] \right],
 \end{aligned}$$

$$x \in \Omega^*, y_1^*(x) = 0, x \in [-1, 0].$$

Lemma 4.2. Let y_1 and y_1^* be the solutions of (3.2) and (4.2), respectively, then $|y_1(x_i) - y_1^*(x_i)| \leq CN^{-2}$, $\forall x_i$.

Proof. From the equations (3.2) and (4.2), Lemma 4.1 and by [6] we have, the desired result. \square

The R-K method of fourth order with piecewise cubic Hermite interpolation is applied on Shishkin mesh $\bar{\Omega}^{2N}$.

$$\begin{aligned}
 & Y_1^*(x_0) = 0, \\
 (4.3) \quad & Y_1^*(x_{i+1}) = Y_1^*(x_i) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \quad i = 0, \dots, 2N-1,
 \end{aligned}$$

where

$$\begin{cases} K_1 = h_i \left[p(x_i) - b(x_i)Y_1^*(x_i) - c(x_i)Y_1^I(x_i) \right] / a(x_i), \\ K_2 = h_i \left[p(x_i + \frac{h_i}{2}) - b(x_i + \frac{h_i}{2})(Y_1^*(x_i) + \frac{K_1}{2}) - c(x_i + \frac{h_i}{2})Y_1^I(x_i + \frac{h_i}{2}) \right] / a(x_i + \frac{h_i}{2}), \\ K_3 = h_i \left[p(x_i + \frac{h_i}{2}) - b(x_i + \frac{h_i}{2})(Y_1^*(x_i) + \frac{K_2}{2}) - c(x_i + \frac{h_i}{2})Y_1^I(x_i + \frac{h_i}{2}) \right] / a(x_i + \frac{h_i}{2}), \\ K_4 = h_i \left[p(x_i + h_i) - b(x_i + h_i)(Y_1^*(x_i) + K_3) - c(x_i + h_i)Y_1^I(x_i + h_i) \right] / a(x_i + h_i), \end{cases}$$

$$Y_1^I(x) = \begin{cases} 0, & x \in [x_i, x_{i+1}], \quad i = 0, 1, \dots, N-1, \\ Y_1(x_{i-N})A_{i-N}(x-1) + Y_1(x_{i-N+1})A_{i+1-N}(x-1) + B_{i-N}(x-1)\tilde{p}(x_{i-N}) \\ \quad + B_{i+1-N}(x-1)\tilde{p}(x_{i-N+1}), & x \in [x_i, x_{i+1}], \quad i = N, N+1, \dots, 2N-1, \end{cases}$$

$\tilde{p}(x_{i-N})$ and $\tilde{p}(x_{i-N+1})$ are defined like $\tilde{f}(x_{i-N})$ and $\tilde{f}(x_{i-N})$.

Lemma 4.3. [4] Let $y_1^*(x)$ and $Y_1^*(x_i)$ be the solutions of the problem (4.2) and (4.3). Then, $\|y_1^* - Y_1^*\|_{\bar{\Omega}^{2N}} \leq CN^{-4}$.

Theorem 4.2. If the solution y_1 of (3.2) and its discrete problem solution $Y_1^*(x_i)$ of (4.3), then $|y_1(x_i) - Y_1^*(x_i)| \leq CN^{-2}$, $i \in I_{2N}$.

Proof. By the triangle inequality, $|y_1(x_i) - Y_1^*(x_i)| \leq |y_1(x_i) - y_1^*(x_i)| + |y_1^*(x_i) - Y_1^*(x_i)| \leq CN^{-2} + CN^{-4} \leq CN^{-2}$. Hence the proof. \square

4.2. Numerical Methods for Terminal Value Problems (TVPs). The numerical solutions of v_1 and v_2 are defined in the following equations:

$$(4.4) \quad \begin{cases} \frac{V_1(x_i) - V_1(x_{i-1})}{h_i} - a(x_i)V_1(x_i) = 0, \\ i = 1, \dots, \frac{N}{2}, \quad V_1(x_N) = 1, \\ \frac{V_1(x_i) - V_1(x_{i-1})}{h_i} - a\left(\frac{x_i + x_{i-1}}{2}\right) \frac{V_1(x_i) + V_1(x_{i-1})}{2} = 0, \\ i = \frac{N}{2} + 1, \dots, N - 1 \end{cases}$$

and

$$(4.5) \quad \begin{cases} \frac{V_2(x_i) - V_2(x_{i-1})}{h_i} - a(x_i)V_2(x_i) = 0, \quad i = 1, \dots, \frac{3N}{2}, \\ V_2(x_{2N}) = 1, \\ \frac{V_2(x_i) - V_2(x_{i-1})}{h_i} - a\left(\frac{x_i + x_{i-1}}{2}\right) \frac{V_2(x_i) + V_2(x_{i-1})}{2} = 0, \\ i = \frac{3N}{2} + 1, \dots, 2N - 1 \end{cases}$$

Theorem 4.3. [6] Let v_1 and v_2 be the solutions of (3.4) and (3.5) and its discrete problem solutions defined by (4.4) and (4.5) respectively, then $|v_k(x_i) - V_k(x_i)| \leq CN^{-2} \ln^2 N$, $i \in I_{2N}$, $k = 1, 2$.

5. NUMERICAL SOLUTION OF (2.2) AND ERROR ANALYSIS

A discrete problem solution of (2.2) is defined as follows

$$(5.1) \quad Y_{as}(x_i) = \begin{cases} Y_0(x_i) + \varepsilon Y_1^*(x_i) + k_1[V_1(x_i) - v_1(x_0)], & i \leq N \\ Y_0(x_i) + \varepsilon Y_1^*(x_i) + k_2[V_2(x_i) - 1] + l - Y_0(x_{2N}) - \varepsilon Y_1^*(x_{2N}), \\ i \geq N + 1. \end{cases}$$

Theorem 5.1. *If the solution y of (2.2) and discrete problem solution Y_{as} defined by (5.1) and if $\varepsilon \leq CN^{-1}$, then $|y(x_i) - Y_{as}(x_i)| \leq CN^{-2} \ln^2 N$.*

Proof. $|y(x_i) - Y_{as}(x_i)| \leq |y(x_i) - y_{as}(x_i)| + |y_{as}(x_i) - Y_{as}(x_i)| \leq C\varepsilon^2 + CN^{-2} \ln^2 N \leq CN^{-2} \ln^2 N$. Hence the proof. \square

6. NUMERICAL ILLUSTRATION

In this section, using the two mesh principle given in [10], the maximum error $D^M = \max_{\varepsilon} D_{\varepsilon}^M$, $D_{\varepsilon}^M = \max_{0 \leq i \leq M} |Y_i^M - Y_{2i}^{2M}|$ and $p^M = \log_2 \left(\frac{D^M}{D^{2M}} \right)$ are calculated.

Example 1. *Consider the BVP (2.2) with $a(x) = 5 + x$; $b(x) = 2$; $c(x) = -\frac{x}{2}$; $f(x) = e^x$; $\phi(x) = 1 + x$; $l = 2$. Table 1 presents the values of D^M and p^M . Figure 1 represents the numerical solution, Figure 2 presents the LogLog plot for this example.*

TABLE 1. Numerical results of y for the Example 1

N (Number of grid points)							
$\varepsilon \downarrow$	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
2^{-4}	1.9829e-02	8.0161e-03	2.7182e-03	9.8252e-04	3.1607e-04	9.4653e-05	2.8835e-05
2^{-5}	1.9942e-02	8.0574e-03	2.7302e-03	9.8462e-04	3.1648e-04	9.4696e-05	2.8752e-05
2^{-6}	1.9998e-02	8.0778e-03	2.7360e-03	9.8540e-04	3.1654e-04	9.4661e-05	2.8678e-05
2^{-7}	2.0027e-02	8.0879e-03	2.7388e-03	9.8573e-04	3.1655e-04	9.4630e-05	2.8633e-05
2^{-8}	2.0041e-02	8.0929e-03	2.7402e-03	9.8587e-04	3.1654e-04	9.4611e-05	2.8609e-05
2^{-9}	2.0048e-02	8.0954e-03	2.7409e-03	9.8594e-04	3.1653e-04	9.4601e-05	2.8596e-05
2^{-10}	2.0055e-02	8.0979e-03	2.7416e-03	9.8601e-04	3.1653e-04	9.4590e-05	2.8583e-05
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
2^{-23}	2.0055e-02	8.0979e-03	2.7416e-03	9.8601e-04	3.1653e-04	9.4590e-05	2.8583e-05
D^M	2.0055e-2	8.0979e-3	2.7416e-3	9.8601e-4	3.1655e-4	9.4696e-05	2.8835e-5
p^M	1.3083	1.5626	1.4753	1.6392	1.7410	1.7155	-

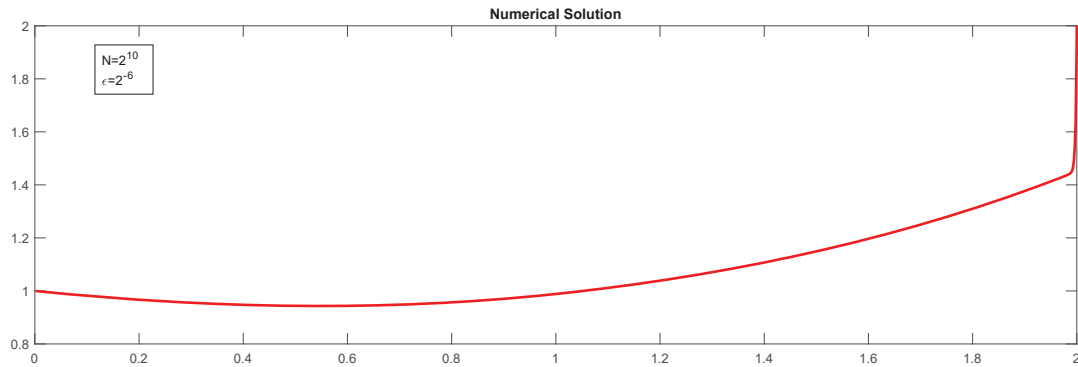


FIGURE 1. Numerical solution of Example 1.

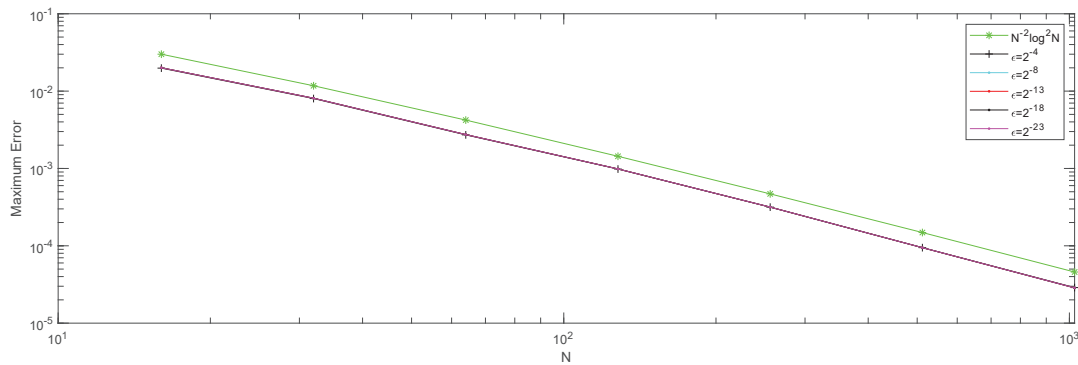


FIGURE 2. Loglog plot for problem stated in Example 1

7. CONCLUDING REMARKS

In this article we considered a class of singularly perturbed convection diffusion type second order delay differential equations. An asymptotic expansion approximation of the solution is constructed by the perturbation technique using the zeroth order and the first order asymptotics. Further the zeroth order and the first order asymptotics are approximated numerically by the Runge-Kutta methods with Hermite interpolation technique. The TVPs are approximated by the hybrid finite difference methods. The present initial value method is almost second order convergent provided $\varepsilon \leq CN^{-1}$. In [10] the authors presented zeroth order asymptotic approximation and applied initial value method. They obtained $(\varepsilon + N^{-2} \ln^2 N)$ order of convergence. Here, it has been improved to the first order asymptotic expansion and the order of convergence is $(\varepsilon^2 + N^{-2} \ln^2 N)$.

Table 1 presents the numerical error for the Example 1. Further, it shows that the maximum error and order of convergence is two. The Figure 1 represents the numerical solution of the Example 1 and loglog plot is drawn in Figure 2.

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