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ANALYSIS OF SERIES QUEUES IN FUZZY ENVIRONMENTS USING QUADRATIC FUZZY NUMBERS AND THE DSW ALGORITHM

H. Merlyn Margaret[1] and P. Thirunavukarasu

ABSTRACT. In this paper, we analyze systems consisting of a series of k service facilities in fuzzy environments. These are a special class of queueing systems called series queues. We use quadratic fuzzy numbers to model fuzziness in the parameters of such systems, namely the arrival rate and the service rates of the k servers. We propose a solution procedure that uses the DSW algorithm to arrive at the fuzzy system characteristics of the queueing system. Finally, we present a numerical example to illustrate the solution procedure.

1. INTRODUCTION

Zadeh in 1965 introduced fuzzy set theory to deal with imprecision and uncertainty in data. Significant research in this field has been carried out by researchers like Dubois and Prade [8], Moore [11], Dong et al. [10], Mizumoto and Tanaka [13], etc. The theory finds wide applications in optimization, queueing theory, communication, neural networks, soft computing, decision making and so on.

Queueing theory is essentially a probabilistic approach to analyzing queueing systems. System parameters, viz. the nature of arrivals, service completion

[1]*corresponding author*

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times etc. are assumed to follow certain probability distributions based on application. Fuzzy set theory resolves the intrinsic uncertainty associated with these parameters.

Extensive research has been carried out in the area of fuzzy queues. Buckley [2] used possibility theory to analyze queueing models with intrinsic fuzziness. Negi and Lee [17] and Li and Lee [6] analyzed several fuzzy queueing models. Kao et al. [16] used nonlinear programming to study fuzzy queues. Prade [5] considered possibilistic queueing models. Pardo and David [9] used fuzzy techniques to analyze queueing models on priority discipline.

Recent studies on fuzzy queueing networks like queues connected in series (tandem queues) find many applications in manufacturing systems, supermarkets, and in assembling parts. Zhang and Phillis [14] studied parallel queueing systems in a fuzzy environment with two heterogeneous servers. Madan and Abu-Dayyel [3] studied two server queues under vacation policy. Chen [12] analyzed fuzzy tandem queues using nonlinear programming. This paper studies the system characteristics of a fuzzy queueing model in which several service facilities are connected in series by making use of quadratic fuzzy numbers and the Dong-Shah-Wong (DSW) algorithm.

The summary of this article is as follows: Sec. 2 of the paper discusses basic fuzzy set theory. Sec. 3 introduces the reader to quadratic fuzzy numbers. Sec. 4 discusses series queues in brief. Sec. 5 describes the DSW algorithm. Sec. 6 describes the proposed solution procedure, and Sec. 7 discusses a numerical example. Sec. 8 concludes the study.

2. Preliminaries

2.1. **Fuzzy set theoretic definitions.** A *fuzzy set* $\tilde{A}$ in the (crisp) universe U, written (U, A), is a function $A : U \to [0, 1]$ that maps each element of the universe to a number in [0,1], interpreted as its degree of membership in U. We say that $\tilde{A}$ is a *fuzzy subset* of U, and that A is its *membership function*.

There are certain useful crisp sets associated with a fuzzy set, namely the *weak α-cut*, the *strong α-cut*, the *support* and the *core*.

The *weak α-cut* of $\tilde{A}$, written $\tilde{A}_\alpha$, is defined by

$$\tilde{A}_\alpha := \{u \in U : A(u) \geq \alpha\}.$$

The *strong* α*-cut* of $\tilde{A}$, written $\tilde{A}_{\alpha+}$, is defined similarly as

$$\tilde{A}_{\alpha+} := \{u \in U : A(u) > \alpha\},$$

for each $\alpha \in [0,1]$. The term α*-cut* will be used to refer to the weak α-cut. Observe that the α-cuts form a nested collection of crisp sets. For $0 \leq \alpha \leq \beta \leq 1$, the inclusion $\tilde{A}_\beta \subseteq \tilde{A}_\alpha$ holds.

The *support*, denoted $\operatorname{supp}(\tilde{A})$ and the *core*, denoted $\operatorname{core}(\tilde{A})$ are merely special α-cuts, given by

$$\operatorname{supp}(\tilde{A}) := \tilde{A}_{0+} \text{ and } \operatorname{core}(\tilde{A}) := \tilde{A}_1.$$

An important measure associated with fuzzy sets $\tilde{A}$ is the *height*, denoted $h(\tilde{A})$.

$$h(\tilde{A}) := \sup_U A.$$

The fuzzy set $\tilde{A}$ is called *normal* if its height is equal to 1, and is called *convex* if all of its α-cuts are convex. A useful characterization of convexity is the following: $\tilde{A}$ is convex iff

$$A(\lambda u_1 + (1-\lambda)u_2) \geq \min\{A(u_1), A(u_2)\}\ \forall\, u_1, u_2 \in U \text{ and } \ \forall\, \lambda \in [0,1].$$

2.2. **Fuzzy numbers.** Special fuzzy subsets of $\mathbb{R}$ qualify as *fuzzy numbers* if the following hold:

(i) $\tilde{A}$ has bounded support and is normal;
(ii) α-cuts, for $\alpha \in (0,1]$ are closed intervals.

In most applications, core$(\tilde{A})$ is a singleton. This element is often referred to as the *mean* or the *modal value* of $\tilde{A}$.

Observe that (ii) implies fuzzy numbers are convex, since intervals are convex. A *positive* fuzzy number is one whose support is a subset of the positive reals. The collection of all positive fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R}^+)$.

2.3. **Binary interval analysis.** Let $I_1, I_2 \subset \mathbb{R}$ be closed intervals in $\mathbb{R}$ of finite length. Therefore, we can write

$$I_1 = [p_1, q_1] \text{ and } I_2 = [p_2, q_2] \text{ for some reals } p_1, p_2, q_1, q_2 \in \mathbb{R}.$$

For $\alpha \in \mathbb{R}$, we define $\alpha I_1 = \begin{cases} [\alpha p_1, \alpha q_1] & \alpha \geq 0 \\ [\alpha q_1, \alpha p_1] & \alpha < 0 \end{cases}.$

Also, $I_1 * I_2$ for $* \in \{+, -, \cdot, \div\}$ is an interval in $\mathbb{R}$, given by the following equations:

$$\begin{aligned} I_1 + I_2 &= [p_1 + p_2, q_1 + q_2] \\ I_1 - I_2 &= I_1 + (-1) \cdot I_2 \\ I_1 \cdot I_2 &= \left[\min\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\}, \max\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\}\right] \\ I_1 \div I_2 &= I_1 \cdot [1/q_2, 1/p_2]\,, \text{ provided } 0 \notin [p_2, q_2] \end{aligned}$$

2.4. **Operations on fuzzy numbers.** Let $\tilde{M}, \tilde{N} \in \mathcal{F}(\mathbb{R}^+)$, and let $*$ denote any one of the operations $+, -, \cdot, \div$. Then, the fuzzy number $\tilde{M} * \tilde{N}$ is well defined, and is given by

$$(\tilde{M} * \tilde{N})_\alpha = \tilde{M}_\alpha * \tilde{N}_\alpha \text{ for each } \alpha \in [0, 1].$$

These α-cuts are clearly intervals of $\mathbb{R}^+$, and standard binary interval analysis techniques (as described in the previous section) apply.

3. Quadratic fuzzy numbers (QFNs)

A fuzzy number $\tilde{A}$ is said to be a *quadratic fuzzy number* [4] if

$$A(x) = \begin{cases} 1 - \left(\dfrac{\mu - x}{\beta_l}\right)^2, & \mu - \beta_l \le x \le \mu \\ 1 - \left(\dfrac{x - \mu}{\beta_r}\right)^2, & \mu \le x \le \mu + \beta_r \\ 0, & \text{otherwise}, \end{cases},$$

where $\mu \in \mathbb{R}$ is the mean of $\tilde{A}$ and $\beta_l, \beta_r \in \mathbb{R}^+$ are called the left-hand and right-hand spreads respectively. We will write $\tilde{A} = (\mu, \beta_l, \beta_r)_q$ for notational brevity.

For illustration, a plot of the membership function of the quadratic fuzzy number $(3, 2, 1)_q$ is as shown in Fig. 1.

It is easy to see that $\mathrm{supp}(\tilde{A}) = (\mu - \beta_l, \mu + \beta_r)$, $\mathrm{core}(\tilde{A}) = \{\mu\}$. Also, the α-cut of $\tilde{A}$ is given by

$$\tilde{A}_\alpha = \left[\mu - \beta_l\sqrt{1 - \alpha}, \mu + \beta_r\sqrt{1 - \alpha}\right]$$

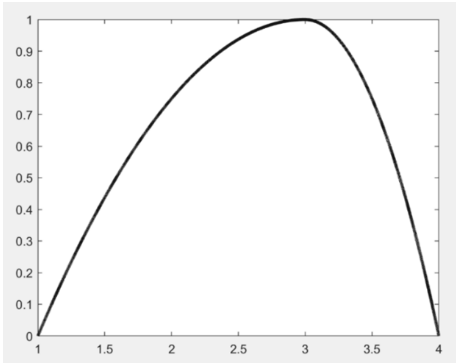

FIGURE 1. The quadratic fuzzy number $(3, 2, 1)_q$

4. SERIES QUEUES

4.1. **Basic description.** The system in consideration is a special type of a series queue. Series queues [1] are queueing models consisting of a series of k nodes, where each node consists of a certain number of parallel service facilities. The arrivals are neither allowed to visit previously visited nodes, nor leave the system before advancing through each of the nodes in sequence.

Suppose that node i consists of s_i service facilities in parallel. In our analysis, we assume that $s_i = 1$ for all $1 \leq i \leq k$. Thus, the system in consideration is essentially a series of k service facilities, where facility i functions with a prescribed service rate. Furthermore, arrivals to the system are assumed to be Poisson, with mean λ, and the service times are assumed to be exponentially distributed. We also assume that facility i functions with service rate μ_i. Equivalently, the time taken by facility i to process a unit is exponentially distributed with mean $1/\mu_i$. The service stations are also assumed to have infinite capacity, meaning that there is no restriction on the size of the queue formed between two consecutive service facilities.

The analysis of such systems is greatly simplified due to a result attributed to Burke [15]. Informally, it states that the inter-departure times from an $M/M/c/\infty$ service facility are distributed in a fashion identical to that of the inter-arrival times. We state the result here without proof.

Theorem 4.1. *(Burke, 1956) If the arrivals to an $M/M/c/\infty$ service facility are Poisson with mean λ and the service times are exponentially distributed with mean*

$1/\mu$ *with* $(\lambda/c\mu) < 1$*, then the departures out of the queue are Poisson with mean* λ.

Put differently, the output distribution is identical to the input distribution, and is unaffected by the exponential service mechanism. In effect, this allows us to treat the k queues in series as k completely independent $M/M/1/\infty$ queues.

In our analysis, we also take fuzziness in the parameters of the system into account, viz. the arrival and service rates. We shall assume that the rates are fuzzy numbers $\tilde{\lambda}$, $\tilde{\mu}_1, \dots, \tilde{\mu}_k$. Our objective is to arrive at the fuzzy system characteristics of the system in discussion.

4.2. **Relevant results.** Burke's theorem allows one to use the results that are known for the $FM/FM/1/\infty$ model and extend them. Immediately, it follows that in steady state, we must have

(1) The average number in the system, denoted $\tilde{N}$, is given by

$$\tilde{N} = \frac{\tilde{\lambda}}{\tilde{\mu}_1 - \tilde{\lambda}} + \frac{\tilde{\lambda}}{\tilde{\mu}_2 - \tilde{\lambda}} + \cdots + \frac{\tilde{\lambda}}{\tilde{\mu}_k - \tilde{\lambda}} = \sum_{i=1}^{k} \frac{\tilde{\lambda}}{\tilde{\mu}_i - \tilde{\lambda}}.$$

(2) The average time spent by the customer in the system, denoted $\tilde{W}$, is given by

$$\tilde{W} = \frac{1}{\tilde{\mu}_1 - \tilde{\lambda}} + \frac{1}{\tilde{\mu}_2 - \tilde{\lambda}} + \cdots + \frac{1}{\tilde{\mu}_k - \tilde{\lambda}} = \sum_{i=1}^{k} \frac{1}{\tilde{\mu}_i - \tilde{\lambda}}.$$

These are the system characteristics that we shall be interested in. Also, one must keep in mind that a steady state solution exists only if $\tilde{\lambda} < \min\{\tilde{\mu}_1, \dots, \tilde{\mu}_k\}$.

5. The Dong-Shah-Wong Algorithm

In fuzzy analysis of systems, in principle, one can use Zadeh's *extension principle* to arrive at the fuzzy solution to the problem. In practice, it is not uncommon to discretize the continuous support of the fuzzy input parameters (usually a continuum of reals) into a finite set of points to arrive at an approximate fuzzy solution. The reason for doing so is that, oftentimes, an analytic solution cannot be determined, and thus one solves the problem on a computer, numerically – computers handle discrete data very well.

Using the extension principle (following discretization of the input parameters) to propagate fuzziness in the input parameters to the output gives erroneous results – these errors do not arise due to inherent problems in the extension principle itself; they are due to the discretization that was done for numerical convenience.

To get around this, one employs the *Dong-Shah-Wong* (often abbreviated to *DSW*) algorithm, which uses α-cuts to arrive at reasonable estimates. The DSW algorithm [7] is as described below:

(1) Fix a value of α in $[0, 1]$.
(2) Evaluate the α-cuts of the fuzzy input parameters for this value of α – these are intervals in $\mathbb{R}$.
(3) Use binary interval analysis as described in Sec. 2.3 to arrive at an interval for the output – this is the α-cut of the output.
(4) Repeat the first three steps for different values of α (equispaced values in $[0, 1]$ are usually chosen for convenience) to arrive at an approximate plot of the membership function of the output.

6. Solution procedure

We first model the fuzziness in the parameters of the system in discussion (here, these are the rate parameters) using quadratic membership functions. Put differently, we assume that the rate parameters are quadratic fuzzy numbers.

Then, we employ the DSW algorithm. For each $\alpha \in \{0.0, 0.1, 0.2, \ldots, 1.0\}$, we compute the α-cuts of the fuzzy quadratic rate parameters (see Sec. 3) and use binary interval analysis to determine the α-cuts of the system characteristics of the queueing system, using the formulae in Sec. 4.2. We end up with intervals for the system characteristics at each confidence level. This then allows us to obtain an approximate plot of the membership functions associated with the fuzzy system characteristics by means of interpolation.

We remark that this procedure works for any kind of membership, since no properties specific to quadratic fuzzy numbers have been exploited. We now present a numerical example to illustrate the solution procedure.

7. Numerical example

We consider a series queueing model with $k = 3$, $\tilde{\lambda} = (20, 2, 2)_q$, $\tilde{\mu}_1 = (25, 2, 2)_q$, $\tilde{\mu}_2 = (40, 5, 5)_q$ and $\tilde{\mu}_3 = (30, 3, 3)_q$, where all rates are in units per hour. We have modelled the fuzziness in the rate parameters of the system using quadratic memberships as stated before. With the above assumptions, our objective is to arrive at the fuzzy performance measures of the system.

We find the intervals for $\tilde{\lambda}$ and the $\tilde{\mu}_i$ corresponding to each α in $\{0, 0.1, \ldots, 1\}$; that is, we find $\tilde{\lambda}_\alpha$ and $(\tilde{\mu}_i)_\alpha$ for each α. We use these intervals to compute intervals for the system characteristics. We demonstrate the procedure for $\alpha = 0.1$ and compute $\tilde{N}_{0.1}$.

The 0.1-cuts of the rates are $\tilde{\lambda}_{0.1} = [18.10, 21.90]$, $(\tilde{\mu}_1)_{0.1} = [23.10, 26.90]$, $(\tilde{\mu}_2)_{0.1} = [35.26, 44.74]$, $(\tilde{\mu}_3)_{0.1} = [27.15, 32.85]$. Now, we use the formula for $\tilde{N}$ and arrive at

$$\tilde{N}_{0.1} = \frac{[18.10, 21.90]}{[23.10, 26.90] - [18.10, 21.90]} + \frac{[18.10, 21.90]}{[35.26, 44.74] - [18.10, 21.90]} + \frac{[18.10, 21.90]}{[27.15, 32.85] - [18.10, 21.90]}.$$

This gives $\tilde{N}_{0.1} = [3.96, 23.97]$. Other computations are carried out similarly. The results are tabulated below in Table 1.

TABLE 1. Intervals associated with the system characteristics of the system for $\alpha \in \{0, 0.1, \ldots, 1\}$

	$\tilde{N}_\alpha$	$\tilde{W}_\alpha$
$\alpha = 0.0$	$[3.86, 28.09]$	$[0.21, 1.27]$
$\alpha = 0.1$	$[3.96, 23.97]$	$[0.21, 1.09]$
$\alpha = 0.2$	$[4.07, 20.84]$	$[0.22, 0.95]$
$\alpha = 0.3$	$[4.19, 18.36]$	$[0.22, 0.84]$
$\alpha = 0.4$	$[4.33, 16.32]$	$[0.23, 0.75]$
$\alpha = 0.5$	$[4.49, 14.59]$	$[0.24, 0.68]$
$\alpha = 0.6$	$[4.67, 13.08]$	$[0.24, 0.61]$
$\alpha = 0.7$	$[4.90, 11.71]$	$[0.25, 0.55]$
$\alpha = 0.8$	$[5.20, 10.43]$	$[0.27, 0.49]$
$\alpha = 0.9$	$[5.63, 9.13]$	$[0.29, 0.44]$
$\alpha = 1.0$	$[7.00, 7.00]$	$[0.35, 0.35]$

Now, using a graphing utility like MATLAB, one can arrive at approximate plots of the membership functions associated with the system characteristics. The plots of the membership functions are as shown in Figures 2 and 3.

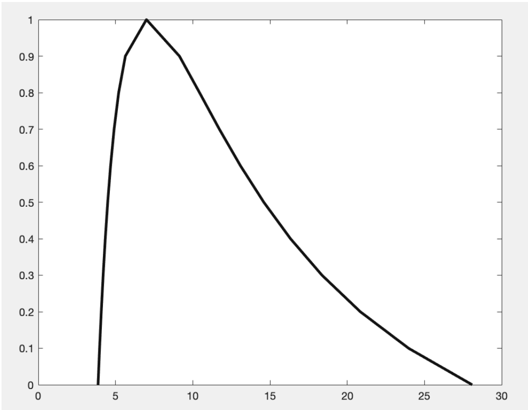

FIGURE 2. Membership function of $\tilde{N}$

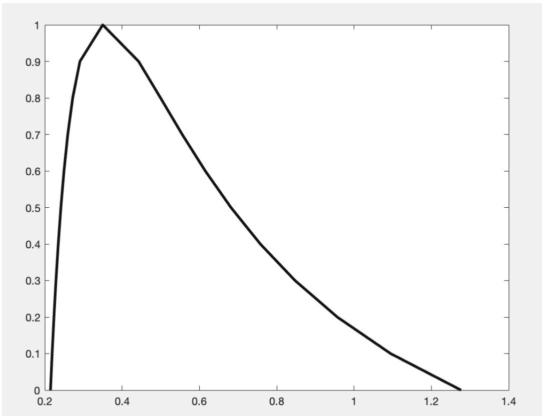

FIGURE 3. Membership function of $\tilde{W}$

From the above results, one can draw the following meaningful conclusions in steady state:

(1) The average number in the system lies between 4 and 28, with mean 7;
(2) The average time spent by a customer in the system lies between 12.6 min to 76.2 min, with mean 21 min.

8. Conclusion

By incorporating fuzziness in queueing models, all inherent uncertainty and imprecision that occur in real-life situations are taken into account, and thereby accuracy is improved. Various measures that help one to analyze fuzzy queueing systems, viz. (fuzzy) mean time spent for getting service, (fuzzy) mean number of customers in the queue, etc. can be calculated using the procedure outlined in this paper. Also, this study applies to a plethora of queueing models that incorporate various other schemes – for instance, optimizing the fuzzy cost associated with an additional service facility and balancing the cost associated with waiting. These data prove useful in the design of efficient queueing systems.

References

[1] D. Gross, C. M. Harris: *Fundamentals of Queueing Theory*, 2nd ed., John Wiley and Sons, New York, NY, USA, 2000.

[2] J. J. Buckley: *Elementary Queueing Theory Based on Probability Theory*, Fuzzy Sets and Systems, **37** (1990), 43–52.

[3] K. C. Madan, W. Abu-Dayyel: *A two server queue with Bernoulli schedules and a single vacation policy*, Applied Mathematics and Computation, **145** (2003), 5971.

[4] M. Hanss: *Applied Fuzzy Arithmetic, An Introduction with Engineering Applications*, Springer-Verlag, Berlin Heidelberg, 2005.

[5] H. M. Prade: *An Outline of Fuzzy or Possibilistic Models for Queuing Systems*, In: Wang P.P., Chang S.K. (eds) Fuzzy Sets. Springer, Boston, MA, 1980.

[6] R. J. Li, E. S. Lee: *Analysis of Fuzzy Queues*, Computers and Mathematics with Applications, **17** (1989), 1143–1147.

[7] W. M. Dong, H. C Shah, F. S. Wongt: *Fuzzy computations in risk and decision analysis*, Civil Engineering Systems, **2**(4) (1987), 201–208.

[8] D. Dubois, H. Prade: *Fuzzy Sets and Systems*, Academic Press, Inc, 1980.

[9] M. J. Pardo, D. de la Fuente; *Optimizing a priority–discipline queueing model using fuzzy set theory*, Computers and Mathematics with Applications, **54** (2007), 267–281.

[10] W. M. Dong, H. C. Shah: *Vertex Method for computing functions on fuzzy variables*, Fuzzy Sets and Systems, **24** (1987), 65–78.

[11] R. E. Moore: *Methods and applications of Interval Analysis*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia PA, 1979.

[12] S.-P. Chen: *A Mathematical Programming Approach to Fuzzy Tandem Queues*, International Journal of Uncertainty, Fuzziness and Knowledge–Based Systems, **13**(4) (2005), 425–436.

[13] M. MIZUMOTO, K. TANAKA: *Some properties of fuzzy sets of type 2*, Information Control, **34**(4) (1976), 312–340.

[14] R. ZHANG, Y. A. PHILLIS: *Fuzzy assignment of customers for a parallel queueing system with two heterogeneous servers*, Journal of Intelligent and Fuzzy Systems, **11** (2001), 163–169.

[15] P.J. BURKE: *The Output of a Queuing System*, Operations Research, **4**(6) (1956), 699–704.

[16] C. KAO, C. C. LI, S. P. CHEN: *Parametric programming to the analysis of fuzzy queues*, Fuzzy sets and Systems, **107** (1999), 93–100.

[17] D. S. NEGI, E. S. LEE: *Analysis and simulation of Fuzzy queues*, Fuzzy Sets ans Systems, **46**(3) (1992), 321–330.

DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING AND TECHNOLOGY
SRM INSTITUTE OF SCIENCE AND TECHNOLOGY, SRM NAGAR
KATTANKULATHUR - 603203
KANCHIPURAM, CHENNAI, TN, INDIA
Email address: merlynh@srmist.edu.in

P.G. AND RESEARCH DEPARTMENT OF MATHEMATICS
PERIYAR E.V.R. COLLEGE (AUTONOMOUS)
TIRUCHIRAPPALLI - 620023, TN, INDIA
Email address: ptavinash1967@gmail.com