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QUASI-CARTESIAN COMPOSITIONS IN AFFINE SPACE CONNECTION, FOUR DIMENSIONAL WITHOUT TENSOR

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ABSTRACT. Let A_N be an affine space connected without tensor in A_4 ([1–4]), where are presented reciprocal compositions as $X_2 \times \overline{X}_2, Y_2 \times \overline{Y}_2$ and $Z_2 \times \overline{Z}_2$. Each of these compositions are of (qC, qC) (Quasi-Cartesian) kind, each of these has in its own the manifold base. Spaces which accept these kinds of compositions are significant. New types of compositions $X_n \times \overline{X}_n$ are created by using parallel and Quasi parallel oriented conditions with $P(X_n)$ and $P(\overline{X}_n)$ positions, as well as the transformations of connections. These kinds of tenses are gained and proved. In this work symmetric affinor connections are counted correctly, which will signify Riemannian's connections and Equi-affine as well as the transformations of connections.

1. INTRODUCTION

Let be A_4 a space with symmetric connected affinor without tensor defined with $\Gamma^{\nu}_{\alpha\beta}(\alpha\beta\gamma = 1, 2, 3, 4)(\alpha\beta\gamma$ - are coefficients of connections). Let us consider the composition $X_n \times X_m$, two different manifolds X_n and X_m $(n+m = 4 = N)$ in space A_N. For any point of space composition $A_N(X_n \times X_m)$, there are two positions of manifolds which we can notice with $P(X_n)$ and $P(X_m)$, ([1,2,5,7,8]).

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We know that the composition is thoroughly defined by the space affinor a_α^β which completes the conditions ([8–11]). We have studied the transformation of symmetric connections, composition of affinor which completes the condition from complete components of the curve by adopting the coordinates of compositions which we notice the special Riemannian and Equi affine.

2. Preliminaries

Let us take $A_N(N=4)$ a space with symmetric affine connection presented with $\Gamma_{\alpha\beta}^{\nu}(\alpha\beta\gamma=1,2,3,4)$. With A_4, we consider the topological compositions $X_n \times X_m$ of two basic manifolds X_n and X_m. We have put down above with $P(X_n)$ and $P(X_m)$ are parallel or Quasi-parallel turned to any line of manifolds A_N and we can say $X_n \times \overline{X}_n$ is of the type (qK, qK) ([1, 2, 11–13]).

The definition of composition is called a special composition in A_4 of an affinor field a_α^β which completes ([5–9, 11])

$$a_\alpha^\nu \cdot a_\nu^\beta = \delta_\alpha^\beta,$$

and contract condition

$$a_\alpha^\nu \nabla_\delta a_\nu^\gamma - a_\rho^\delta \nabla_\delta a_\delta^\nu = 0,$$

where ∇ is a covariant of derivation by respecting the connection $\Gamma_{\alpha\beta}^{\gamma}(\alpha\beta\gamma=1,2,3,4)$. Affinor a_α^β is called affinor of composition ([1–4, 7, 11, 13, 15, 16]). Projected affinor are given:

$$\overset{n\beta}{a}_\alpha = \frac{1}{2}(\delta_\alpha^\beta + \alpha_\alpha^\beta), \overset{n\beta}{a}_\alpha = \frac{1}{2}(\delta_\alpha^\beta + \alpha_\alpha^\beta),$$

which complete the equations

$$\begin{aligned} &\overset{n\beta}{a}_\alpha + \overset{n\beta}{a}_\alpha = \delta_\alpha^\beta, \quad \overset{n\beta}{a}_\alpha - a_\alpha^\beta = a_\alpha^\beta, \\ &\overset{n\beta}{a}_\alpha \cdot \overset{n\nu}{a}_\beta = \overset{n\nu}{a}_\alpha, \quad \overset{m\beta}{a}_\alpha \cdot \overset{m\delta}{a}_\beta = \overset{m\delta}{a}_\alpha, \quad \overset{n\beta}{a}_\alpha \cdot \overset{m\nu}{a}_\beta = 0. \end{aligned} \tag{2.1}$$

With adopted coordinates, affinor's matrix a_α^β, $\overset{n\beta}{a}_\alpha$ and $\overset{m\beta}{a}_\alpha$ have this form ([1–4]):

$$(a_\alpha^\beta) = \begin{pmatrix} \delta_j^i & 0 \\ 0 & -\delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \quad (\overset{n\beta}{a}_\alpha) = \begin{pmatrix} \delta_j^i & 0 \\ 0 & 0 \end{pmatrix}, \quad (\overset{m\beta}{a}_\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \tag{2.2}$$

where δ_α^β is a unit affinor.

For one arbitrary vector v^α we get decomposition $v_\alpha = \overset{n\beta}{a}_\alpha v^\beta + \overset{m\beta}{a}_\alpha v^\beta$ then $\overset{n\beta}{a}_\alpha v^\beta \in P(X_n)$ and $\overset{n\beta}{a}_\alpha v^\beta \in P(X_m)$.

Let us take the composition $X_n \times X_m$ of the type (qK, qK) where positions $P(X_n)$ and $P(X_m)$ are turned Quasi-parallel along every line of A_4 respectively ([1,4,6–9,13]). According [6] the compositions $X_n \times X_m$ is of the type (qK, qK), then and only then if it is worth

$$\nabla_\sigma a^\beta_\alpha - 2\tau_\nu(\overset{n\nu}{a}_\alpha \overset{m\beta}{a}_\sigma - \overset{m\nu}{a}_\sigma \overset{n\beta}{a}_\alpha) = 0. \tag{2.3}$$

According the equation (2.3) we have τ_ν which is equally with

$$\tau_\nu = \frac{-1}{m}\overset{n\beta}{a}_\alpha \Psi_\sigma - \frac{1}{n}\overset{m\beta}{a}_\alpha \Psi_\sigma, \tag{2.4}$$

then

$$\Psi_\sigma = \frac{1}{2} a^\beta_\alpha \nabla_\sigma a^\nu_\sigma, \tag{2.5}$$

because we notice Ψ_i and $\Psi_{\bar{i}}$.

Vector τ_ν is called the vector turned Quasi–parallel where as Ψ_σ - is called Cartesian vector. With adopted coordinates of the equation (2.3) is in [1,2,9,11] and we have:

$$\Gamma^{\bar{k}}_{\alpha i} = \delta^{\bar{k}}_\alpha \Psi_i, \quad \Gamma^k_{\alpha\bar{i}} = \delta^k_\alpha \Psi_{\bar{i}}. \tag{2.6}$$

Ψ_i and $\Psi_{\bar{i}}$ in equation (2.6) are given in the equation (2.5) and the expression $\nabla_\sigma a^\beta_\alpha$ in equation (2.3) is

$$\nabla_\sigma a^\beta_\alpha = \nabla_\sigma \left(\underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha - \underset{3}{v}^\beta \overset{3}{v}_\alpha - \underset{4}{v}^\beta \overset{4}{v}_\alpha \right),$$

whereas the expression in brackets of the equation (2.3) is:

$$\overset{n\nu}{a}_\alpha \overset{m\beta}{a}_\sigma - \overset{n\nu}{a}_\sigma a^\beta_\alpha = \left(\underset{1}{v}^\nu \overset{1}{v}_\alpha + \underset{2}{v}^\nu \overset{2}{v}_\alpha \right) \cdot \left(\underset{3}{v}^\beta \overset{3}{v}_\beta + \underset{4}{v}^\beta \overset{4}{v}_\beta \right) - \left(\underset{3}{v}^\nu \overset{3}{v}_\sigma + \underset{4}{v}^\nu \overset{4}{v}_\sigma \right) \cdot \left(\underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha \right)$$

or

$$\overset{n\nu}{a}_\alpha \overset{n\beta}{a}_\sigma - \overset{\nu}{a}_\sigma a^\beta_\alpha = \underset{1}{v}^\nu \left(\underset{3}{v}^\beta \overset{3}{v}_\sigma + \underset{4}{v}^\beta \overset{4}{v}_\sigma \right) - \underset{1}{v}^\beta \left(\underset{3}{v}^\nu \overset{3}{v}_\sigma + \underset{4}{v}^\nu \overset{4}{v}_\sigma \right).$$

Then the equation (2.3) has the form:

$$\begin{aligned} &\underset{1}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^\beta - \underset{1}{\overset{1}{T}}_\sigma \underset{1}{v}^\beta - \underset{1}{\overset{2}{T}}_\sigma \underset{2}{v}^\beta + \underset{1}{\overset{3}{T}}_\sigma \underset{3}{v}^\beta + \underset{1}{\overset{4}{T}}_\sigma \underset{4}{v}^\beta - \\ &- 2\tau_\nu \left[\underset{1}{v}^\nu \left(\underset{3}{v}^\beta \overset{3}{v}_\sigma + \underset{4}{v}^\beta \overset{4}{v}_\alpha \right) - \underset{1}{v}^\beta \left(\underset{3}{v}^\nu \overset{3}{v}_\sigma + \underset{4}{v}^\nu \overset{4}{v}_\sigma \right) \right] = 0. \end{aligned} \tag{2.7}$$

According the equation (2.7) curvature has the form $R^{\sigma}_{\alpha\beta\gamma}=0$, in space A_4 as usually of Riemannian:

$$R^{\nu}_{\alpha\beta\gamma}=\delta_{\alpha}\Gamma^{\nu}_{\beta\sigma}-\delta_{\beta}\Gamma^{\nu}_{\alpha\sigma}+\Gamma^{\nu}_{\alpha\sigma}\Gamma^{\tau}_{\beta\sigma}-\Gamma^{\nu}_{\beta\tau}\Gamma^{\tau}_{\alpha\sigma}, \tag{2.8}$$

and A_4 is equi-affine with the square base $N=4$. $\varrho_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ according [8,9,12–15]. So, space A_4 is equi-affine then and only then where:

$$R^{\sigma}_{\alpha\beta\gamma}=0.$$

It stands the condition, because the space A_4 is from the composition $X_n\times X_m$ of the type (qK,qK) with suited compositions will be:

$${}^1R^{\sigma}_{ijk},\quad {}^1R^{\sigma}_{\overline{i}\overline{j}k}.$$

3. Special Products

If compositions X_n and $\overline{X}_n$ are of the type (qC,qC), then positions $P(X_n)$ and $P(\overline{X}_n)$ are parallel turned along every line of base manifold A_4. The lack of parallel curve or Quasi parallel along one of the compositions $X_n\times\overline{X}_m$ is signed with (-,-).

Example 1. *Composition $X_n\times\overline{X}_n$ is of the type $(-,qC)$ if positions $(\overline{X}_n)$ are parallel turned along to every line of base manifold A_4. We get types off compositions $X_n\times\overline{X}_n$ which has the form (-,qC) dhe (qC,-) which are given: $(qY_n,-)\Rightarrow$ characteristic of e invariants will be:*

$$\underset{j}{\overset{\bar{i}}{T}}{}_{\sigma}\underset{s}{v}{}^{\sigma}+\underset{j}{\overset{\bar{i}}{T}}{}_{\sigma}\underset{\bar{s}-n}{v}{}^{\sigma}=\delta^{\bar{i}}_{\bar{s}}\underset{j}{v}{}^{\sigma}\underset{i}{\varphi}{}_{\alpha},$$

whereas the characteristic of me parameters of net's vector $(\underset{1}{v},\underset{2}{v},\underset{3}{v},\underset{4}{v})$ is:

$$\underset{j}{\overset{\bar{i}}{T}}{}_{\bar{s}}-\underset{j}{\overset{i}{T}}{}_{\bar{s}-n}=\delta^{i}_{\bar{s}}\underset{1}{\varphi}{}^{j}\quad and\quad \Gamma^{\bar{i}}_{\bar{s}j}+\Gamma^{\bar{i}}_{\bar{s}-nj}=\delta^{i}_{\bar{s}}\underset{1}{\varphi}{}_{j},$$

but $(-,qY_n)\Rightarrow$ characteristic of invariants will be:

$$\underset{\bar{s}}{\overset{i}{T}}{}_{\sigma}\underset{s}{v}{}^{\sigma}+\underset{\bar{j}}{\overset{i}{T}}{}_{\sigma}\underset{n+s}{v}{}^{\sigma}=\underset{s}{\delta^{i}}\underset{\bar{j}}{v}{}^{\sigma}\underset{2}{\varphi}{}_{\sigma},$$

whereas characteristics with vector's net $(\underset{1}{v},\underset{2}{v},\underset{3}{v},\underset{4}{v})$ *is:*

$$\overset{i}{\underset{\bar{j}}{T}}{}_{s} + \overset{i}{\underset{\bar{j}}{T}}{}_{s+n} = \underset{s}{\delta^{i}}\ \underset{2}{\varphi_{\bar{j}}}, \tag{3.1}$$

and coefficients of connections have the form:

$$\Gamma^{i}_{s\bar{j}} + \Gamma^{i}_{n+s\bar{j}} = \underset{s}{\delta^{i}}\ \underset{2}{\varphi_{\bar{j}}}.$$

4. Quasi-Cartesian compositions in A_4

According [1, 5, 7–9] we have the different equations which follow

$$\nabla_{\sigma}\underset{\alpha}{v^{\beta}} = \overset{\nu}{\underset{\alpha}{T}}{}_{\sigma}\underset{\nu}{v^{\beta}}, \quad \nabla_{\sigma}\overset{\alpha}{v}_{\beta} = -\overset{\alpha}{\underset{\nu}{T}}{}_{\sigma}\overset{\nu}{v}_{\beta}.$$

Let us take the composition $X_2 \times \overline{X}_2$ with signs of connected coefficients α, β, γ, σ, $\nu \in \{1,2,3,4\}$, $i,j,s,k,l \in \{1,2\}$, $\bar{i},\bar{j},\bar{k},\bar{l} \in \{3,4\}$.

Theorem 4.1. *Composition* $X_2 \times \overline{X}_2$ *is of the type* (qC, qC) *if coefficients of equation complete the condition*

$$\Gamma^{k}_{i\bar{j}} = \delta^{k}_{i}\Psi_{j}, \quad \Gamma^{k}_{\bar{i}j} = \delta^{\bar{k}}_{\bar{i}}\Psi_{j}.$$

Proof. According the equation

$$\overset{n\beta}{a}_{\alpha}\nabla_{\alpha}\overset{n\nu}{a}_{\sigma} - \overset{n\sigma}{a}_{\alpha}\nabla_{\sigma}\overset{n\nu}{a}_{\beta} - \Psi_{\sigma}\left(\overset{n\sigma}{a}_{\beta}\overset{m\nu}{a}_{\alpha} + \overset{n\sigma}{a}_{\beta}\overset{m\nu}{a}_{\alpha}\right) = 0, \tag{4.1}$$

according $\overset{n\beta}{a}_{\alpha}$ and [5, 7, 9, 11, 13, 14] it is worth

$$\begin{aligned}\nabla_{\sigma}a^{\beta}_{\alpha} = {} & \overset{\nu}{\underset{1}{T}}{}_{\sigma}\underset{1}{v^{\beta}}\overset{1}{v}_{\alpha} - \overset{1}{\underset{\nu}{T}}{}_{\sigma}\underset{1}{v^{\beta}}\overset{\nu}{v}_{\alpha} + \overset{\nu}{\underset{2}{T}}{}_{\sigma}\underset{\nu}{v^{\beta}}\overset{2}{v}_{\alpha} - \overset{2}{\underset{\nu}{T}}{}_{\sigma}\underset{2}{v^{\beta}}\overset{\nu}{v}_{\alpha} \\ & - \overset{3}{\underset{\nu}{T}}{}_{\sigma}\underset{\nu}{v^{\beta}}\overset{3}{v}_{\alpha} + \overset{3}{\underset{\nu}{T}}{}_{\sigma}\underset{3}{v^{\beta}}\overset{\nu}{v}_{\alpha} - \overset{\nu}{\underset{4}{T}}{}_{\sigma}\underset{4}{v^{\beta}}\overset{\nu}{v}_{\alpha} + \overset{2}{\underset{\nu}{T}}{}_{\sigma}\underset{4}{v^{\beta}}\overset{\nu}{v}_{\alpha}.\end{aligned}$$

Then equation (4.1) we contract with independent net vectors $(\underset{1}{v},\underset{2}{v},\underset{3}{v},\underset{4}{v})$ then we get the equation:

$$a)\ \begin{array}{l}\overset{3}{\underset{1}{T}}{}_{\alpha} - \underset{\alpha}{\overset{n\sigma}{a}}\overset{3}{\underset{1}{T}}{}_{\sigma} - \Psi_{\sigma}\underset{1}{v^{\sigma}}\overset{3}{v}_{\alpha} = 0 \\ \overset{3}{\underset{1}{T}}{}_{\alpha}\underset{n}{v^{\alpha}} = \Psi_{\sigma}\underset{1}{v^{\alpha}}, \ \ \overset{3}{\underset{1}{T}}{}_{\sigma}\underset{4}{v^{\alpha}} = 0\end{array} \iff \begin{array}{l}\overset{4}{\underset{1}{T}}{}_{\alpha} - \underset{\alpha}{\overset{3\sigma}{a}}\overset{4}{\underset{1}{T}}{}_{\sigma} - \Psi_{\sigma}\underset{1}{v^{\sigma}}\overset{4}{v}_{\alpha} = 0 \\ \overset{4}{\underset{1}{T}}{}_{\alpha}\underset{3}{v^{\alpha}} = 0, \overset{4}{\underset{1}{T}}{}_{\alpha}\underset{4}{v^{\alpha}} = \Psi_{\sigma}\underset{1}{v^{\sigma}}\end{array} \tag{4.2}$$

$$b)\quad \begin{array}{l} \overset{3}{\underset{2}{T}}_\alpha - \overset{4}{\underset{\alpha}{a}}{}^{\sigma} \overset{3}{\underset{2}{T}}_\sigma - \Psi_\sigma \underset{2}{v}^\sigma \overset{3}{v}_\alpha = 0 \\ \overset{4}{\underset{2}{T}}_\alpha - \overset{n}{\underset{\alpha}{a}}{}^{\sigma} \overset{4}{\underset{2}{T}}_\sigma - \Psi_\sigma \underset{2}{v}^\sigma \overset{4}{v}_\alpha = 0 \end{array} \iff \begin{array}{l} \overset{3}{\underset{2}{T}}_\alpha \underset{3}{v}^\alpha = \Psi_\sigma \underset{2}{v}^\sigma, \;\; \overset{3}{\underset{2}{T}}_\alpha \underset{2}{v}^\alpha = 0 \\ \overset{4}{\underset{2}{T}}_\alpha \underset{3}{v}^\alpha = 0, \;\; \overset{4}{\underset{2}{T}}_\alpha \underset{4}{v}^\alpha = \Psi_\sigma \underset{2}{v}^\sigma \end{array}$$

$$c)\quad \begin{array}{l} \overset{n}{a}{}^{\sigma}_{\alpha} \overset{1}{\underset{3}{T}}_\sigma - \Psi_\sigma \underset{3}{v}^\sigma \overset{1}{v}_\alpha = 0 \\ \overset{n}{a}{}^{\sigma}_{\alpha} \overset{2}{\underset{3}{T}}_\sigma - \Psi_\sigma \underset{3}{v}^\sigma \overset{2}{v}_\alpha = 0 \end{array} \iff \begin{array}{l} \overset{1}{\underset{3}{T}}_\sigma \underset{1}{v}^\sigma = \Psi_\sigma \underset{3}{v}^\sigma, \;\; \overset{1}{\underset{3}{T}}_\sigma \underset{3}{v}^\sigma = 0 \\ \overset{2}{\underset{3}{T}}_\sigma \underset{1}{v}^\sigma = 0; \;\; \overset{2}{\underset{3}{T}}_\sigma \underset{2}{v}^\sigma = \Psi_\sigma \underset{3}{v}^\sigma \end{array}$$

$$d)\quad \begin{array}{l} \overset{n}{a}{}^{\sigma}_{\alpha} \overset{1}{\underset{4}{T}}_\alpha - \Psi_\sigma \underset{4}{v}^\sigma \overset{1}{v}_\alpha = 0 \\ \overset{n}{a}{}^{\sigma}_{\alpha} \overset{2}{\underset{4}{T}}_\alpha - \Psi_\sigma \underset{4}{v}^\sigma \overset{2}{v}_\alpha = 0 \end{array} \iff \begin{array}{l} \overset{1}{\underset{4}{T}}_\sigma \underset{1}{v}^\sigma = \Psi_\sigma \underset{4}{v}^\sigma, \;\; \overset{1}{\underset{4}{T}}_\alpha \underset{2}{v}^\sigma = 0 \\ \overset{2}{\underset{4}{T}}_\alpha \underset{1}{v}^\sigma = 0; \;\; \overset{2}{\underset{4}{T}}_\sigma \underset{2}{v}^\sigma = \Psi_\sigma \underset{4}{v}^\sigma \end{array}$$

From equation (4.2) cases (a-d) it is worth

$$\text{(4.3)}\qquad \overset{\bar{i}}{\underset{s}{T}}_\alpha \underset{\bar{k}}{v}^\alpha = \delta^{\bar{i}}_{\bar{k}} \Psi_\alpha \underset{s}{v}^\alpha; \quad \overset{i}{\underset{s}{T}}_\alpha \underset{k}{v}^\alpha = \delta^{i}_{k} \widetilde{\Psi}_\alpha \underset{\bar{s}}{v}^\alpha; \quad \overset{\bar{i}}{\underset{s}{T}}_{\bar{k}} = \delta^{\bar{i}}_{\bar{k}} \Psi_s; \quad \overset{i}{\underset{\bar{s}}{T}}_{k} = \delta^{i}_{k} \Psi_{\bar{s}}.$$

Equation (4.3) can be written according (3.1) in this form

$$\Gamma^i_{k\bar{s}} = \delta^i_k \Psi_{\bar{s}}, \quad \Gamma^{\bar{i}}_{\bar{k}s} = \delta^{\bar{i}}_{\bar{k}} \Psi_s,$$

which even proves the theorem. □

Theorem 4.2. *Composition $Y_2 \times \overline{Y}_2$ is of the type (qC, qC). If $\nabla_\sigma W^\beta_\alpha = \overset{\nu}{\underset{\alpha}{P}}_\sigma \underset{\nu}{W}^\beta$, then $\overset{i}{\underset{\alpha}{P}}_s \underset{\bar{k}}{W}^\alpha = \delta^i_{\bar{k}} \Psi_\alpha \underset{1}{W}^\alpha$, $\overset{i}{\underset{\bar{s}}{P}}_\alpha \underset{k}{W}^\alpha = \delta^i_k \Psi_\alpha \underset{\bar{s}}{W}^\alpha$.*

Proof. We act identically as with the Theorem 4.1 with the contraction independent net vectors $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v})$ solved and we have:

$$\text{(4.4)}\qquad \begin{array}{l} \overset{3}{\underset{1}{P}}_\alpha \underset{3}{W}^\alpha = \Psi_\alpha \underset{1}{W}^\alpha; \quad \overset{3}{\underset{1}{P}}_\alpha \underset{4}{W}^\alpha = 0 \\ \overset{3}{\underset{2}{P}}_\alpha \underset{3}{W}^\alpha = \Psi_\alpha \underset{2}{W}^\alpha; \quad \overset{3}{\underset{2}{P}}_\alpha \underset{4}{W}^\alpha = 0 \\ \overset{4}{\underset{1}{P}}_\alpha \underset{3}{W}^\alpha = 0; \quad \overset{4}{\underset{1}{P}}_\alpha \underset{4}{W}^\alpha = \Psi_\alpha \underset{1}{W}^\alpha \\ \overset{4}{\underset{2}{P}}_\alpha \underset{3}{W}^\alpha = 0; \quad \overset{4}{\underset{2}{P}}_\alpha \underset{4}{W}^\alpha = \Psi_\alpha \underset{2}{W}^\alpha \\ \overset{1}{\underset{3}{P}}_\alpha \underset{1}{W}^\alpha = \Psi_\alpha \underset{3}{W}^\alpha; \quad \overset{1}{\underset{4}{P}}_\alpha \underset{1}{W}^\alpha = 0 \\ \overset{1}{\underset{3}{P}}_\alpha \underset{2}{W}^\alpha = 0; \quad \overset{1}{\underset{4}{P}}_\alpha \underset{2}{W}^\alpha = 0 \end{array}$$

$$\underset{3}{\overset{2}{P}}_\alpha \underset{1}{W}^\alpha = 0; \quad \underset{3}{\overset{2}{P}}_\alpha \underset{2}{W}^\alpha = \Psi_\alpha \underset{3}{W}^\alpha$$

$$\underset{4}{\overset{2}{P}}_\alpha \underset{1}{W}^\alpha = 0; \quad \underset{4}{\overset{2}{P}}_\alpha \underset{2}{W}^\alpha = \Psi_\alpha \underset{4}{W}^\alpha$$

In equation (4.4) expressions $\underset{\alpha}{W}^\beta$ will be replaced with

$$\underset{1}{W}^\alpha = \underset{1}{v}^\alpha + \underset{3}{v}^\alpha, \ \ \underset{2}{W}^\alpha = \underset{2}{v}^\alpha + \underset{4}{v}^\alpha, \ \ \underset{3}{W}^\alpha = \underset{1}{v}^\alpha - \underset{3}{v}^\alpha, \ \ \underset{4}{W}^\alpha = \underset{2}{v}^\alpha - \underset{4}{v}^\alpha,$$

then equation (4.4) has the form:

$$\begin{aligned}
&\underset{1}{\overset{3}{P}}_\alpha \left(\underset{1}{v}^\alpha - \underset{3}{v}^\alpha\right) = \Psi_\alpha \left(\underset{1}{v}^\alpha + \underset{3}{v}^\alpha\right); \quad \underset{1}{\overset{3}{P}}_\alpha \left(\underset{2}{v}^\alpha - \underset{4}{v}^\alpha\right) = 0\\
&\underset{2}{\overset{3}{P}}_\alpha \left(\underset{1}{v}^\alpha - \underset{3}{v}^\alpha\right) = \Psi_\alpha \left(\underset{2}{v}^\alpha + \underset{4}{v}^\alpha\right); \quad \underset{2}{\overset{3}{P}}_\alpha \left(\underset{2}{v}^\alpha - \underset{4}{v}^\alpha\right) = 0\\
&\underset{1}{\overset{4}{P}}_\alpha \left(\underset{1}{v}^\alpha - \underset{3}{v}^\alpha\right) = 0; \quad \underset{1}{\overset{4}{P}}_\alpha \left(\underset{2}{v}^\alpha - \underset{4}{v}^\alpha\right) = \Psi_\alpha \left(\underset{1}{v}^\alpha + \underset{3}{v}^\alpha\right)\\
&\underset{2}{\overset{4}{P}}_\alpha \left(\underset{2}{v}^\alpha - \underset{3}{v}^\alpha\right) = 0; \quad \underset{2}{\overset{4}{P}}_\alpha \left(\underset{2}{v}^\alpha - \underset{4}{v}^\alpha\right) = \Psi_\alpha \left(\underset{2}{v}^\alpha + \underset{4}{v}^\alpha\right)\\
&\underset{3}{\overset{1}{P}}_\alpha \left(\underset{1}{v}^\alpha + \underset{3}{v}^\alpha\right) = \Psi_\alpha \left(\underset{1}{v}^\alpha - \underset{3}{v}^\alpha\right); \quad \underset{4}{\overset{1}{P}}_\alpha \left(\underset{1}{v}^\alpha + \underset{3}{v}^\alpha\right) = 0\\
&\underset{3}{\overset{1}{P}}_\alpha \left(\underset{2}{v}^\alpha + \underset{4}{v}^\alpha\right) = 0; \quad \underset{4}{\overset{1}{P}}_\alpha \left(\underset{2}{v}^\alpha + \underset{4}{v}^\alpha\right) = 0\\
&\underset{3}{\overset{2}{P}}_\alpha \left(\underset{1}{v}^\alpha + \underset{3}{v}^\alpha\right) = 0; \quad \underset{3}{\overset{2}{P}}_\alpha \left(\underset{2}{v}^\alpha + \underset{4}{v}^\alpha\right) = \Psi_\alpha \left(\underset{1}{v}^\alpha - \underset{3}{v}^\alpha\right)\\
&\underset{4}{\overset{2}{P}}_\alpha \left(\underset{1}{v}^\alpha + \underset{3}{v}^\alpha\right) = 0; \quad \underset{4}{\overset{2}{P}}_\alpha \left(\underset{2}{v}^\alpha + \underset{4}{v}^\alpha\right) = \Psi_\alpha \left(\underset{2}{v}^\alpha - \underset{4}{v}^\alpha\right).
\end{aligned} \tag{4.5}$$

According to any equality of equation (4.5) by factorizing them we have

$$\begin{aligned}
&\underset{1}{\overset{3}{P}}_1 - \underset{1}{\overset{3}{P}}_3 = \Psi_1 + \Psi_3; \quad \underset{1}{\overset{3}{P}}_2 - \underset{2}{\overset{3}{P}}_4 = 0 \iff \underset{1}{\overset{3}{P}}_2 = \underset{2}{\overset{3}{P}}_4 \iff \Gamma^3_{12} = \Gamma^3_{24}\\
&\underset{2}{\overset{3}{P}}_1 - \underset{2}{\overset{3}{P}}_3 = \Psi_2 + \Psi_4; \quad \underset{2}{\overset{3}{P}}_2 - \underset{2}{\overset{3}{P}}_4 = 0 \iff \underset{2}{\overset{3}{P}}_2 = \underset{2}{\overset{3}{P}}_4 \iff \Gamma^3_{22} = \Gamma^3_{24}\\
&\underset{1}{\overset{4}{P}}_1 - \underset{4}{\overset{4}{P}}_3 = 0; \quad \underset{1}{\overset{4}{P}}_2 - \underset{4}{\overset{4}{P}}_4 = \Psi_1 + \Psi_3\\
&\underset{2}{\overset{4}{P}}_1 - \underset{2}{\overset{4}{P}}_3 = 0; \quad \underset{2}{\overset{2}{P}}_2 - \underset{2}{\overset{4}{P}}_4 = \Psi_2 + \Psi_1\\
&\underset{3}{\overset{1}{P}}_1 - \underset{3}{\overset{1}{P}}_3 = \Psi_1 - \Psi_3; \quad \underset{4}{\overset{1}{P}}_1 + \underset{4}{\overset{1}{P}}_3 = \Psi_2 - \Psi_1
\end{aligned} \tag{4.6}$$

$$\overset{1}{\underset{3}{P}}_2 + \overset{1}{\underset{3}{P}}_4 = 0; \quad \overset{1}{\underset{4}{P}}_2 + \overset{1}{\underset{4}{P}}_4 = 0$$
$$\overset{2}{\underset{3}{P}}_1 + \overset{2}{\underset{3}{P}}_3 = 0; \quad \overset{2}{\underset{3}{P}}_2 + \overset{2}{\underset{3}{P}}_4 = \Psi_1 - \Psi_3$$
$$\overset{2}{\underset{4}{P}}_1 + \overset{2}{\underset{4}{P}}_3 = 0; \quad \overset{2}{\underset{4}{P}}_2 + \overset{2}{\underset{4}{P}}_4 = \Psi_2 - \Psi_1$$

For equation (4.6) are worth these connections $\Gamma^{i}_{\overline{ij}} = 0$, $\Gamma^{\overline{i}}_{ij} = 0$ and $\Gamma^{k}_{ij} = 0$, $\Gamma^{k}_{\overline{ij}} = 0$. This shows that the theorem is proved. By using [13–16] and equation (4.6) we get

$$R^{\alpha}_{ijk} = 0 \quad or \quad R^{\alpha}_{\overline{ijk}} = 0.$$

□

Consequence 4.1. *If compositions $X_2 \times \overline{X}_2$ and $Y_2 \times \overline{Y}_2$ are symmetric of this kind (qC, qC), then space A_4 is affine space.*

Proof. If compositions $X_2 \times \overline{X}_2$ and $Y_2 \times \overline{Y}_2$ are symmetric of this kind (qC, qC), then from the Theorem 4.1 and Theorem 4.2 we get $\Gamma^{\gamma}_{\alpha\beta\sigma} = 0$. So, in this way A_4 is affine space. □

5. Transformations of connections

Let we take the space A_4 with symmetric affine connection $\Gamma^{\gamma}_{\alpha\beta\sigma}$ $(\alpha\beta\gamma = 1, 2, 3, 4)$. Coefficients of connections, in order it to be the space of composition $X_n \times X_m$ must be of type (qC, qC). Space A_4 is called composition of Quasi-Cartesian. According (4.1), composition $X_n \times X_m$ is of the type (qC, qC), then, and only then, when $A^{\nu}_{\alpha\beta} = 0$.

Let us consider connections

$$(5.1) \qquad {}^{1}\Gamma^{\sigma}_{\alpha\beta} = \Gamma^{\sigma}_{\alpha\beta} + B^{\sigma}_{\alpha\beta},$$

where $B^{\sigma}_{\alpha\beta}$ a deformed is tensor. After $\Gamma^{\sigma}_{\alpha\beta}$ is symmetric and the tensor of the curve ${}^{1}\Gamma^{\sigma}_{\alpha\beta}$ is given with $\overset{\sigma}{\underset{\alpha}{T}}_{\beta} = \overset{\sigma}{\underset{\alpha}{B}}_{\beta} - \overset{\sigma}{\underset{\beta}{B}}_{\alpha}$, and ${}^{1}\Gamma^{\sigma}_{\alpha\beta}$ is symmetric and only symmetric if $B^{\sigma}_{\alpha\beta} = B^{\sigma}_{\beta\alpha}$.

We notice from ${}^1\nabla$ and ${}^1R^{\nu}_{\alpha\beta\sigma}$ derivation of covariant and curve of the tensor is in relation with the connections ${}^1\Gamma^{\sigma}_{\alpha\beta}$. According (5.1) we get

$$ {}^1\nabla_\alpha a^\gamma_\sigma = \nabla_\alpha a^\gamma_\sigma + B^\gamma_{\alpha\nu}a^\nu_\sigma - B^\nu_{\alpha\sigma}a^\gamma_\nu. \tag{5.2} $$

Then (2.5) and (5.2) enables

$$ \widetilde{\Psi}_\alpha = \Psi_\alpha + \frac{1}{2}a^\gamma_\sigma\left(B^\gamma_{\sigma\rho}a^\nu_\sigma - B^\nu_{\alpha\sigma}a^\gamma_\nu\right). \tag{5.3} $$

With equation (2.4) and (5.3) we count

$$ \widetilde{\varphi}_\alpha = \varphi_\alpha - \frac{1}{2}a^\gamma_\sigma\left(B^\nu_{\sigma\rho}a^\nu_\sigma - B^\nu_{\alpha\sigma}a^\gamma_\nu\right)\left(\frac{1}{m}\overset{n}{a}{}^\rho_\alpha + \frac{1}{n}\overset{m}{a}{}^\rho_\alpha\right). \tag{5.4} $$

Then, after replacement of the equation (5.2) and (5.4) in

$$ \widetilde{Q}^\gamma_{\alpha\beta} = \overset{n}{a}{}^\beta_\alpha\,{}^1\nabla_\beta a^\gamma_\sigma - \overset{n}{a}{}^\sigma_\beta\,{}^1\nabla_\sigma a^\gamma_\alpha - 2\widetilde{\varphi}_\alpha\left(\overset{n}{a}{}^\sigma_\alpha\overset{m}{a}{}^\gamma_\beta + \overset{m}{a}{}^\sigma_\alpha\overset{n}{a}{}^\gamma_\beta\right), $$

and taking in consideration the equations (2.1), we get the following equation

$$ Q^\gamma_{\alpha\beta} = \overset{n\sigma}{a}_\alpha\nabla_\beta a^\gamma_\sigma - \overset{n}{a}{}^\sigma_\beta\nabla_\sigma a^\gamma_\alpha - 2\varphi_\sigma\left(\overset{n\beta}{a}_\alpha\overset{m}{a}{}^\gamma_\beta + \overset{n}{a}{}^\sigma_\alpha\overset{m}{a}{}^\gamma_\beta\right). $$

We get

$$ \widetilde{Q}^\gamma_{\alpha\beta} = Q^\gamma_{\alpha\beta} + B^\gamma_{\alpha\beta}, $$

where:

$$ \begin{aligned} B^\gamma_{\alpha\beta} &= \overset{n}{a}{}^\sigma_\alpha\left(K^\gamma_{\beta\rho} - K^\rho_{\beta\sigma}a^\nu_\rho\right) - \overset{n}{a}{}^\sigma_\beta\left(K^\nu_{\sigma\rho}a^\rho_\alpha - K^\rho_{\sigma\alpha}a^\nu_\rho\right) + \\ &\quad + a^\rho_\tau\left(K^\tau_{\rho\mu}a^\mu_\gamma - K^\mu_{\rho\gamma}a^\tau_\mu\right)\left(\frac{1}{m}\overset{n}{a}{}^\gamma_\alpha\overset{m}{a}{}^\nu_\beta + \frac{1}{n}\overset{m}{a}{}^\gamma_\alpha\overset{n}{a}{}^\nu_\beta\right). \end{aligned} \tag{5.5} $$

From equation (5.5) it follows $Q^\gamma_{\alpha\beta} = 0$ and $\widetilde{Q}^\gamma_{\alpha\beta} = 0$, then, and only then, if $B^\beta_{\alpha\beta} = 0$. With suited coordinates according (2.2) and (5.5) we get:

$$ \begin{aligned} &B^i_{jk} = B^i_{j\bar{k}} = B^i_{\bar{j}\bar{k}} = B^{\bar{i}}_{jk} = B^{\bar{i}}_{\bar{j}k} = B^{\bar{i}}_{\bar{j}\bar{k}} = 0, \\ &B^i_{\bar{j}\bar{k}} = 2(K^i_{\bar{j}\bar{k}} - \frac{1}{n}K^i_{j\bar{k}}\delta_{\bar{s}^i}), \\ &B^i_{j\bar{k}} = 2(K^i_{\bar{j}k} - \frac{1}{m}B^{\bar{i}}_{\bar{j}k}\delta_{\bar{s}^i}). \end{aligned} $$

With adopted coordinates from (2.6), (2.7) and (4), we get:

$$ {}^1\Gamma^i_{k\bar{s}} = (\varphi_s + \frac{1}{n}K^i_{j\bar{k}})\delta^i_s; \quad {}^1\Gamma^i_{ks} = (\varphi_s + \frac{1}{m}K^{\bar{i}}_{\bar{j}k})\delta^i_s. \tag{5.6} $$

In the case when $Q^{\gamma}_{\alpha\beta} = \widetilde{Q}^{\gamma}_{\alpha\beta} = 0$, by respecting the adopted coordinates (2.7), (2.8) and (5.6), we count on following components of tensors curve $R^{\gamma}_{\alpha\beta\sigma}$ and ${}^{1}R^{\gamma}_{\alpha\beta\sigma}$, where

$$R^{s}_{ij\bar{k}} = \{\delta^{s}_{ij}\partial K^{l}_{lk} + (\delta^{s}_{ij}\varphi_{j} + K^{k}_{ij})K^{i}_{ij} + K^{p}_{\mu\bar{l}}\delta^{s}_{ij}K^{l}_{jk}\} + 2\{\delta^{s}_{ij}S^{\bar{l}}_{j\bar{k}}\varphi_{\bar{i}} + K^{s}_{ij}\varphi_{\bar{k}}\}.$$

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