

ANALYTICAL SOLUTIONS OF COUPLED NONLINEAR SCHRODINGER EQUATIONS FOR TWO NON LINEARLY INTERACTING STOKES WAVE TRAINS OVER INFINITE DEPTH

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ABSTRACT. An attempt to find the exact analytical solutions of the two coupled nonlinear Schrodinger equations of 3rd order occurring from the oblique interaction of two capillary gravity wave trains in the case of crossing sea states in deep water is the main premise of the present paper. The solutions obtained here are due to the nonlinear interaction of two Stokes wave trains in one spatial dimension. Graphs have been plotted to investigate the influence of capillarity on the amplitudes of such wave trains. From 3D figures it has been observed that the capillarity has diminishing influence on the amplitudes of the either wave packet.

1. INTRODUCTION

There has been much of an interest shown towards the nonlinear Schrodinger equation by the scientific community in the past few decades. The experimental validation of the analytical soliton like solutions of the nonlinear Schrodinger equation has triggered the academicians to investigate various aspects as well

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as the nature of analytical, breather like or rogue wave type solutions of the nonlinear evolution equations.

In the discussions of nonlinear evolution of water waves, nonlinear Schrodinger equations are generally applied due to its proper reflection of modulational instability. Roskes [1] first derived the coupled nonlinear evolution equations of third order for two surface gravity waves in deep water and then performed a stability analysis based on these equations. This analysis was then extended by Dhar and Das [2] starting from the nonlinear evolution equation of the fourth order for surface gravity waves in the water of infinite depth. They further extended this paper to include the effect of capillarity [3]. Later on Onorato et al. [4] derived the coupled 3rd order nonlinear evolution equations for two wave systems in infinite depth of water with two different propagation directions and performed stability analysis by considering one perturbation wave number only. Beginning from this two coupled nonlinear Schrodinger equations Shukla et al. [5] derived a nonlinear dispersion relation and performed stability analysis in the two dimensional perturbation plane. The consideration of two nonlinearly interacting wave trains gained much importance due to the fact that it properly reflects the characteristic of rate of growth of modulational instability than the case of a single wave.

In all of the studies mentioned earlier the exact analytical solution of the problem was not covered, which is the motivation of our present study to find exact analytical solutions of the problem. Though there are studies available by means of classical inverse scattering due to Zakharov [6] along with the general case of periodic solutions [7], as well as the breather type solutions due to Peregrine [8] and Akhmediev et al. [9], [10] with the help of Darboux transform methods for the case of a single wave. Chowdury et al. [11] also derived breather type solution for a nonlinear evolution equation of fourth order in the rogue wave limit. Recently, Degasperis et al. [12] derived rogue wave type solutions for the Manakov model of two weakly resonant monochromatic waves employing the same Darboux-Dressing method. In the present paper we have found exact analytical solutions for two Stokes wave systems in one spatial dimension of the nonlinear Schrodinger equation of 3rd order.

2. BASIC ASSUMPTION AND COUPLED NONLINEAR SCHRODINGER EQUATIONS

Let us consider xyz , the Cartesian co-ordinate system, where oxy plane coincides with the free surface of the water whose equation at any time t is $z = \nu(x, y, t)$ and z axis directing vertically upwards. Now we consider two surface capillary gravity wave trains propagating in the (x, y) plane with basic wave numbers $\bar{k}_1 = (k_1, k_2)$ and $\bar{k}_2 = (k_1, -k_2)$. By a standard procedure [13], we obtain the following two coupled (2+1) dimensional nonlinear Schrodinger equations for two non linearly interacting capillary gravity wave trains under the circumstance of crossing seas under infinite depth of water. For the 1st wave packet with basic wave number $\bar{k}_1 = (k_1, k_2)$ the evolution equation is given as,

$$(2.1) \quad \begin{aligned} & i \frac{\partial \nu_{10}}{\partial t_1} + i \epsilon_1 \frac{\partial \nu_{10}}{\partial x_1} + i \epsilon_2 \frac{\partial \nu_{10}}{\partial y_1} + \epsilon_3 \frac{\partial^2 \nu_{10}}{\partial x_1^2} + \epsilon_4 \frac{\partial^2 \nu_{10}}{\partial y_1^2} + \epsilon_5 \frac{\partial^2 \nu_{10}}{\partial x_1 \partial y_1} \\ & = \lambda_1 \nu_{10}^2 \nu_{10}^* + \lambda_2 \nu_{10} \nu_{01} \nu_{01}^*, \end{aligned}$$

where $*$ denotes the conjugate complex.

For the 2nd wave packet with carrier wave number $\bar{k}_2 = (k_1, -k_2)$, we have obtained the evolution equation as,

$$(2.2) \quad \begin{aligned} & i \frac{\partial \nu_{01}}{\partial t_1} + i \epsilon_1 \frac{\partial \nu_{01}}{\partial x_1} - i \epsilon_2 \frac{\partial \nu_{01}}{\partial y_1} + \epsilon_3 \frac{\partial^2 \nu_{01}}{\partial x_1^2} + \epsilon_4 \frac{\partial^2 \nu_{01}}{\partial y_1^2} - \epsilon_5 \frac{\partial^2 \nu_{01}}{\partial x_1 \partial y_1} \\ & = \lambda_1 \nu_{01}^2 \nu_{01}^* + \lambda_2 \nu_{01} \nu_{10} \nu_{10}^*. \end{aligned}$$

The coefficients of these two evolution equations, which are in the third order, are available in the Appendix. For $T = 0$, the above coefficients of equations (2.1) and (2.2) are in agreement with the corresponding coefficients of the equations made by Onorato et al. [4]. Furthermore in the absence of capillarity, those coefficients of equations (2.1) and (2.2) coincide with the corresponding coefficients of Senapati et al. [13] for $U = 0$ and $r = 0$.

These evolution equations (2.1) and (2.2) have been made non dimensional by employing the following transformations with dropping the tildes:

$$\begin{aligned} (\tilde{x}_1, \tilde{y}_1, \tilde{t}_1) &= (k_0 x_1, k_0 y_1, \sqrt{g k_0} t_1), \quad (\tilde{k}_1, \tilde{k}_2) = \left(\frac{k_1}{k_0}, \frac{k_2}{k_0} \right) \\ \tilde{\Omega} &= \frac{\Omega}{\sqrt{g k_0}}, \quad \tilde{T} = \frac{T k_0^2}{g}, \quad (\tilde{\nu}_{10}, \tilde{\nu}_{01}) = (k_0 \nu_{10}, k_0 \nu_{01}). \end{aligned}$$

Here $x_1 = \delta x$, $y_1 = \delta y$, $t_1 = \delta t$, δ being slow ordering parameter, T is the surface tension, g being the gravitational acceleration, Ω is the frequency. The linear

dispersion relation reads as

$$f(\Omega, k_1, k_2) \equiv \Omega^2 - gk_0 - Tk_0^3 = 0,$$

as well as that of the group velocity

$$\tilde{c}_g = \frac{d\Omega}{dk_0} = \frac{g + 3Tk_0^2}{2\Omega},$$

where $k_1 = k_0 \cos \alpha$ and $k_2 = k_0 \sin \alpha$, 2α being the angle of separation of the two wave trains.

3. SOLUTIONS OF COUPLED NONLINEAR SCHRODINGER EQUATIONS

In spatially one dimension the evolution equations (2.1) and (2.2) reduce to,

$$i \frac{\partial \nu_{10}}{\partial t_1} + i\epsilon_1 \frac{\partial \nu_{10}}{\partial x_1} + \epsilon_3 \frac{\partial^2 \nu_{10}}{\partial x_1^2} = \lambda_1 \nu_{10}^2 \nu_{10}^* + \lambda_2 \nu_{10} \nu_{01} \nu_{01}^*,$$

$$i \frac{\partial \nu_{01}}{\partial t_1} + i\epsilon_1 \frac{\partial \nu_{01}}{\partial x_1} + \epsilon_3 \frac{\partial^2 \nu_{01}}{\partial x_1^2} = \lambda_1 \nu_{01}^2 \nu_{01}^* + \lambda_2 \nu_{01} \nu_{10} \nu_{10}^*.$$

The coefficients of these two equations are in agreement with the corresponding coefficients of the third order terms of Dhar and Das [3].

Setting $\xi = x_1 - \gamma_1 t_1$, ($\xi > 0$) and $\tau = t_1$, the aforesaid equations transform into,

$$(3.1) \quad i \frac{\partial \nu_{10}}{\partial \tau} + \epsilon_3 \frac{\partial^2 \nu_{10}}{\partial \xi^2} = \lambda_1 \nu_{10}^2 \nu_{10}^* + \lambda_2 \nu_{10} \nu_{01} \nu_{01}^*,$$

$$(3.2) \quad i \frac{\partial \nu_{01}}{\partial \tau} + \epsilon_3 \frac{\partial^2 \nu_{01}}{\partial \xi^2} = \lambda_1 \nu_{01}^2 \nu_{01}^* + \lambda_2 \nu_{01} \nu_{10} \nu_{10}^*.$$

To solve these equations (3.1) and (3.2), under the assumption of ($\xi > 0$, $\epsilon_3 > 0$), let us employ, $\nu_{10} = u(\xi)e^{i(c_1\tau+c_2)}$ and $\nu_{01} = v(\xi)e^{i(c_3\tau+c_4)}$, where u, v are real functions of ξ and c_1, c_2, c_3, c_4 are +ve real parameters. Employing these transformations in the aforesaid equations (3.1) and (3.2) we arrive at,

$$(3.3) \quad \epsilon_3 \frac{\partial^2 u}{\partial \xi^2} - c_1 u = (\lambda_1 u^2 + \lambda_2 v^2)u,$$

$$\epsilon_3 \frac{\partial^2 v}{\partial \xi^2} - c_3 v = (\lambda_1 v^2 + \lambda_2 u^2)v,$$

which a system of second order nonlinear differential equations. Now, to solve these system of nonlinear ODE we employ $v = ku$ ($k \neq 0$, a real constant) [14] and consequently, we obtain from (3.3),

$$(3.4) \quad \frac{d^2 u}{d\xi^2} = \lambda_3 u^3 + \frac{c_1}{\epsilon_3} u,$$

where $\lambda_3 = \frac{(\lambda_1 + k^2 \lambda_2)}{\epsilon_3}$. On solving equation (3.4) we arrive at

$$u = -\frac{2c_1}{\epsilon_3 \lambda_3} \frac{1}{\sinh\left(\sqrt{\frac{c_1}{\epsilon_3}} \xi + \sqrt{\frac{2c_1}{\epsilon_3 \lambda_3}} c_5\right)}, \text{ clearly } \xi \neq -\sqrt{\frac{2}{\lambda_3}} c_5,$$

and consequently

$$v = -\frac{2c_1 k}{\epsilon_3 \lambda_3} \frac{1}{\sinh\left(\sqrt{\frac{c_1}{\epsilon_3}} \xi + \sqrt{\frac{2c_1}{\epsilon_3 \lambda_3}} c_5\right)}, \text{ clearly } \xi \neq -\sqrt{\frac{2}{\lambda_3}} c_5,$$

$c_5 > 0$ being the integration constant and k can be obtained from the real root of

$$k^2 = \frac{-\lambda_1(\lambda_2 - c_1) \pm \sqrt{\lambda_1^2(\lambda_2 - c_1)^2 + \lambda_2^2(2c_1\epsilon_3\lambda_2 - \lambda_1^2)}}{\lambda_2^2}.$$

Figures 1 and 2 exhibit that as α increases $|\nu_{10}|$ decreases and for a settled value of α , a decrease in $|\nu_{10}|$ has been observed due to the effect of surface tension ($T=0.035$).

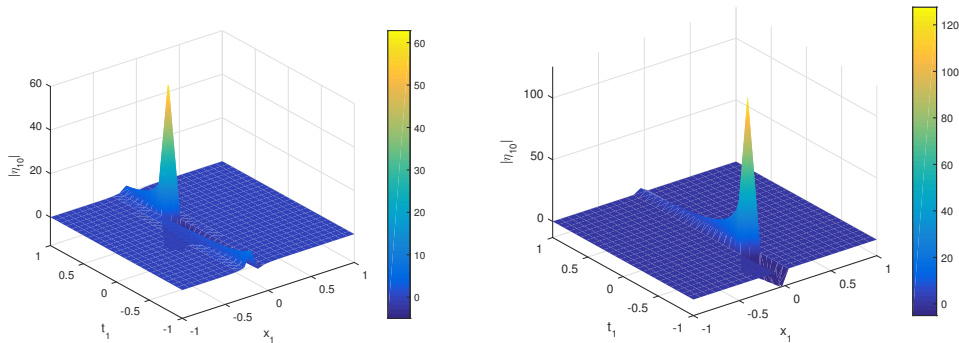


FIGURE 1. $|\nu_{10}|$ as a function of x_1 and t_1 , taking $\alpha = 36^\circ$, $c_1 = 9.2$, $c_5 = 1.5$, $T=0.035$ (left) and $T=0$ (right).

As in Figures 1 and 2, similar types of characteristics have been observed in case of $|\nu_{10}|$ in the Figures 3 and 4 due to the effect of surface tension.

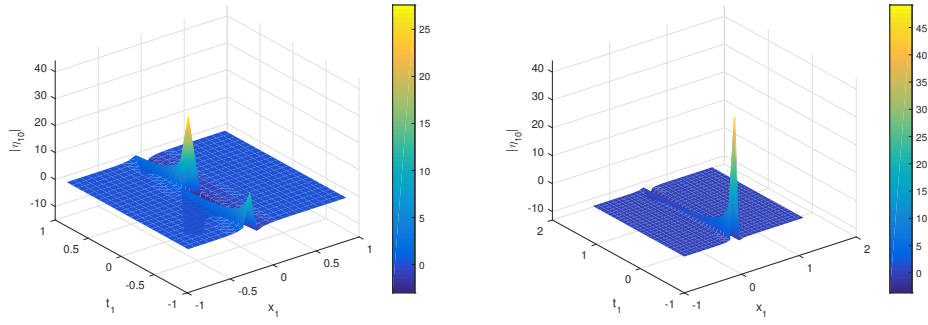


FIGURE 2. $|\nu_{10}|$ as a function of x_1 and t_1 , taking $\alpha = 40^\circ$, $c_1 = 9.2$, $c_5 = 1.5$, $T=0.035$ (left) and $T=0$ (right).

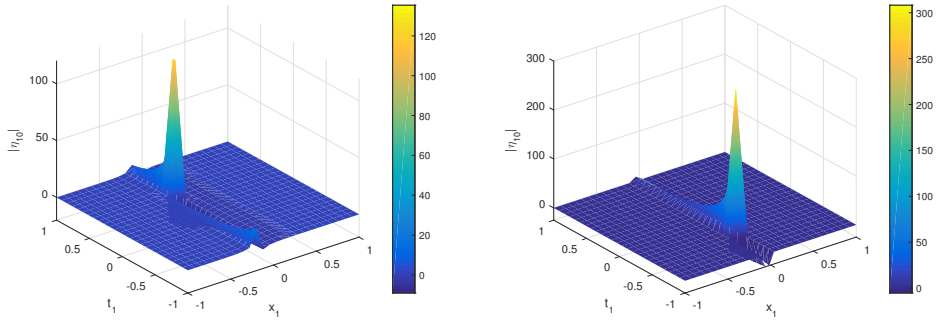


FIGURE 3. $|\nu_{01}|$ as a function of x_1 and t_1 , taking $\alpha = 36^\circ$, $c_1 = 9.2$, $c_5 = 1.5$, $T=0.035$ (left) and $T=0$ (right).

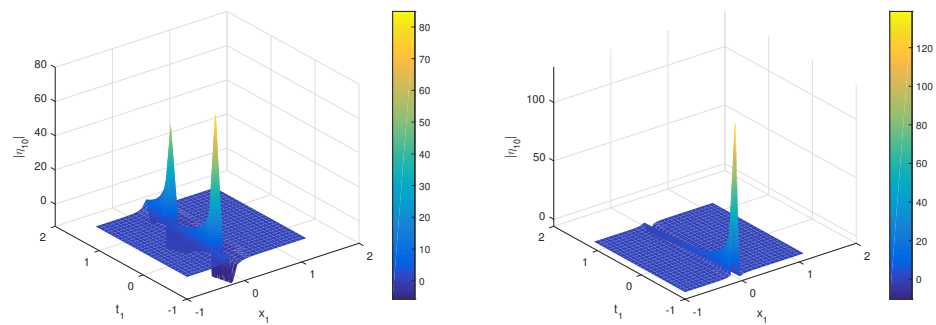


FIGURE 4. $|\nu_{01}|$ as a function of x_1 and t_1 , taking $\alpha = 40^\circ$, $c_1 = 9.2$, $c_5 = 1.5$, $T=0.035$ (left) and $T=0$ (right).

4. CONCLUSION

In this paper, under certain conditions, we have found exact analytical solutions of the two coupled third order nonlinear evolution equations occurring due to the nonlinear interaction of two capillarity gravity Stokes wave trains in case of the cross sea for water of infinite depth. The solutions obtained here are applicable in the case of $\epsilon_3 > 0$, which is dependent on α . So there is certain restriction on α . For $T = 0$ and $T = 0.035$, we found that the present solution is valid when $\alpha > 26.43^\circ$ and $\alpha > 35.26^\circ$ respectively, with the additional condition that $\xi > 0$, the right half of the transformed plane. From the figures it has been observed that the capillarity has a diminishing influence on the amplitudes of both the wave trains. Furthermore, the amplitudes of the wave trains decrease as the angle of separation 2α increases.

APPENDIX

$$\begin{aligned}\epsilon_1 &= \frac{k_1(3T+1)}{2\Omega}, \quad \epsilon_2 = \frac{k_2(3T+1)}{2\Omega}, \quad \epsilon_3 = -\frac{k_1^2}{8\Omega^3} + \frac{k_2^2 + 3T(2k_1^2 + k_2^2)}{4\Omega}, \\ \epsilon_4 &= -\frac{k_2^2}{8\Omega^3} + \frac{k_1^2 + 3T(2k_1^2 + k_2^2)}{4\Omega}, \quad \epsilon_5 = -\frac{k_1k_2}{4\Omega^3} - \frac{k_1k_2(3T+1)}{2\Omega}, \\ \lambda_1 &= \frac{\Omega(3\Omega^2 - 3 - 12T)}{2\Omega^2 - 1 - 4T} + 2\Omega - \frac{T}{2\omega} \left\{ 3k_2^2k_1^2 - \frac{3}{2}(k_1^4 + k_2^4) \right\}, \\ \lambda_2 &= \frac{T}{2\Omega} \{ 2k_2^2k_1^2 + 3(k_1^4 + k_2^4) \} + \Omega(k_1^2 - k_2^2) + 2\Omega k_1^2 - \frac{\Omega^3 k_2^2(k_1^2 - k_2^2 - 2)}{2(1 + 4k_2^2T)} \\ &\quad - \frac{\Omega k_1(k_1^2 + k_2^2 - 2k_1)^2}{k_1 + 4Tk_1^2 - 2\Omega^2}\end{aligned}$$

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