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FIXED POINT THEORY IN A CONVEX GENERALIZED b-METRIC SPACE

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ABSTRACT. In this paper, we generalize Mann's iterative algorithm and prove fixed point results in the framework of a generalization of a b-metric space. A convex structure is imposed on the generalized space and two strong convergence results are provided for two different contraction mappings. The concept of stability is extended to the generalized b- metric space.

1. INTRODUCTION

In 1922, Banach, see [5] proved his famous fixed point theorem that every contraction mapping on a complete metric space has a unique fixed point. Since then there has been numerous extensions to his work, especially in changing the underlying structure of the metric space or introducing new contraction types. Czerwik [2] relaxed the triangular inequality and formally defined a b-metric space. In 1970, Takahasi [6] introduced the concept of convexity in metric spaces and proved fixed point theorems for contraction mappings in such spaces. Chen et. al. [1] discussed fixed point theorems in convex b-metric spaces. Here we discuss such concepts in a convex generalized b-metric space. Fixed point theory is important in non linear analysis and functional analysis. It finds application in systems of non linear differential, integral and algebraic equations.

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2. Preliminaries

Definition 2.1. *[4] Let $X \neq \emptyset$ be a set and $\alpha, \beta \geq 1$ be real numbers. A function $\rho : X \times X \to [0, \infty)$ is called a $\alpha\beta$ b-metric if the following hold, for every $x, y, z \in X$,*

(i) $\rho(x, y) = 0 \iff x = y$;
(ii) $\rho(x, y) = \rho(y, x)$;
(iii) $\rho(x, y) \leq \alpha\rho(x, z) + \beta\rho(z, y)$.

The pair (X, ρ) is called a $\alpha\beta$ b-metric space.

Definition 2.2. *[2] Let $\{x_n\}$ be a sequence in a $\alpha\beta$ b-metric space (X, ρ). Then,*

(i) *The sequence $\{x_n\}$ is said to be convergent in (X, ρ) if there exists $x^\star \in X$ such that $\lim_{n\to\infty} \rho(x_n, x^\star) = 0$.*
(ii) *The sequence $\{x_n\}$ is said to be Cauchy in (X, ρ). if $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$.*
(iii) *(X, ρ) is called a complete $\alpha\beta$ b-metric space if every Cauchy sequence in X is convergent.*

Definition 2.3. *[3] Let $I = [0, 1)$. Define $\rho : X \times X \to [0, \infty)$ and a continuous function $\omega : X \times X \times I \to X$. Then ω is said to be a convex structure on X if the following holds:*

$$\rho(z, \omega(x, y; \mu)) \leq \mu\rho(z, x) + (1 - \mu)\rho(z, y),$$

for each $z \in X$ and $(x, y; \mu) \in X \times X \times I$.

Example 1. *Let $X = [1, 3]$ and define ρ by:*

$$\rho(x, y) = \begin{cases} 3^{|x-y|}, & x \neq y, \\ 0, & x = y, \end{cases}$$

(X, ρ) is a $\alpha\beta$ b-metric space since,

$$\begin{aligned} \rho(x, y) &\leq 3^{|x-z|+|z-y|} \\ &= 3^{\frac{1}{3}|x-z|+\frac{2}{3}|(z-y)|} 3^{\frac{2}{3}|x-z|+\frac{1}{3}|z-y|} \\ &\leq \left(\frac{1}{3}3^{|x-z|} + \frac{2}{3}3^{|z-y|}\right) \sup_{x,y,z\in X} 3^{\frac{2}{3}|x-z|+\frac{1}{3}|z-y|} \\ &= 3\rho(x, z) + 6\rho(z, y). \end{aligned}$$

Furthermore,

$$\rho(1,3) > \rho(1,2) + \rho(2,3),$$

thus (X,ρ) *is not a metric space.*

Define $\omega(x,y;\mu) = \mu x + (1-\mu)y$ *for* $u \in I$*, then*

$$\begin{aligned}
\rho(z,\omega(x,y;\mu)) &= \rho(z,\mu x + (1-\mu)y) \\
&= 3^{|z-\mu x-(1-\mu)y|} \\
&= 3^{|\mu(z-x)+(1-\mu)(z-y)|} \\
&\leq 3^{\mu|(z-x)|+(1-\mu)|(z-y)|} \\
&\leq \mu 3^{|z-x|} + (1-\mu)3^{|z-y|} \\
&= \mu\rho(z,x) + (1-\mu)\rho(z,y)
\end{aligned}$$

3. Main results

Theorem 3.1. *Let* (X,ρ,ω) *be a complete convex* α,β *b-metric space and* $T : X \to X$ *be a contraction mapping, that is there exists* $\lambda \in [0,1)$ *such that* $\rho(Tx,Ty) \leq \lambda\rho(x,y)$*, for all* $x,y \in X$*. Choose* $x_0 \in X$ *such that* $\rho(x_0,Tx_0) < \infty$ *and define* $x_n = \omega(x_{n-1},Tx_{n-1};\mu_{n-1})$*, where*

$$0 < \mu_{n-1} < \frac{\frac{1}{\beta^3} - \lambda}{\frac{\alpha}{\beta} - \lambda} \quad, \lambda < \frac{1}{\beta^3},$$

for each $n \in \mathbb{N}$*, then* T *has a unique fixed point in* X*.*

Proof.

$$\rho(x_n,x_{n+1}) = \rho(x_n,\omega(_n,Tx_n;\mu_n)) \leq (1-\mu_n)\rho(x_n,Tx_n)$$

$$\begin{aligned}
\rho(x_n,Tx_n) &\leq \alpha\rho(x_n,Tx_{n-1}) + \beta\rho(Tx_{n-1},Tx_n) \\
&\leq \alpha\rho(\omega(x_{n-1},Tx_{n-1};\mu_{n-1}),Tx_{n-1}) + \beta\lambda\rho(x_{n-1},x_n) \\
&\leq \alpha\mu_{n-1}\rho(x_{n-1},Tx_{n-1}) + \beta\lambda(1-\mu_{n-1})\rho(x_{n-1},Tx_{n-1}) \\
&= [\alpha\mu_{n-1} + \beta\lambda(1-\mu_{n-1})]\,\rho(x_{n-1},Tx_{n-1}) \\
&< \frac{1}{\beta^2}\rho(x_{n-1},Tx_{n-1}) \qquad (3.1) \\
&< \rho(x_{n-1},Tx_{n-1}) \qquad (3.2)
\end{aligned}$$

Hence $\{\rho(x_n, Tx_n)\}$ is a decreasing sequence of non-negative reals for sequence $\{x_n\}$. Hence, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} \rho(x_n, Tx_n) = \gamma$. If $\gamma > 0$ then letting $n \to \infty$ in (3.2) we have $\gamma < \gamma$, a contraction. Hence $\gamma = 0$ and from (3.1) it follows that $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$. We now show that $\{x_n\}$ is a Cauchy sequence. Let $m > n$ then

$$\begin{aligned}
\rho(x_m, x_n) &\leq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_m) \\
&\leq \alpha\rho(x_n, x_{n+1}) + \beta\left[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_m)\right] \\
&\leq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_m) \\
&\leq \alpha\rho(x_n, x_{n+1}) + \beta\alpha\rho(x_{n+1}, x_{n+2}) + \beta^2\rho(x_{n+2}, x_m) \\
&+ \cdots + \beta^{m-n-1}\rho(x_{m-1}, x_m) \\
&< \alpha \sum_{k=0}^{m-n-1} \beta^k \rho(x_{n+k}, x_{n+k+1}) \qquad (3.3)
\end{aligned}$$

Now from (3.1) and (3.2) it follows that

$$\begin{aligned}
\rho(x_n, x_{n+1}) = \rho(x_n, \omega(_n, Tx_n; \mu_n)) &\leq (1-\mu_n)\rho(x_n, Tx_n) \\
&< \rho(x_n, Tx_n) \\
&< \frac{1}{\beta^2}\rho(x_{n-1}, Tx_{n-1}) \\
&< \frac{1}{\beta^4}\rho(x_{n-2}, Tx_{n-2}) \\
&< \frac{1}{\beta^{2k}}\rho(x_{n-k}, Tx_{n-k}).
\end{aligned}$$

Hence

$$\rho(x_{n+k}, x_{n+k+1}) < \frac{1}{\beta^{2k}}\rho(x_n, Tx_n). \qquad (3.4)$$

Substituting (3.4) in (3.3), we obtain

$$\begin{aligned}
\rho(x_m, x_n) &< \alpha \sum_{k=0}^{m-n-1} \frac{1}{\beta^k}\rho(x_n, Tx_n) \\
&< \alpha\rho(x_n, Tx_n) \sum_{k=0}^{\infty} \left(\frac{1}{\beta}\right)^k \\
&= \frac{\alpha\beta}{\beta - 1}\rho(x_n, Tx_n).
\end{aligned}$$

Hence $\lim_{m,n\to\infty} \rho(x_m, x_n) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of X there exists $x^\star \in X$ such that $\lim_{n\to\infty} \rho(x_n, x^\star) = 0$. We now verify that $x^\star$ is a fixed point of T:

$$\begin{aligned}
\rho(x^\star, Tx^\star) &\leq \alpha\rho(x^\star, x_n) + \beta\rho(, x_n, Tx^\star) \\
&\leq \alpha\rho(x^\star, x_n) + \beta\left[\alpha\rho(x_n, Tx_n) + \beta\rho(Tx_n, Tx^\star)\right] \\
&\leq \alpha\rho(x^\star, x_n) + \beta\alpha\rho(x_n, Tx_n) + \beta^2\lambda\rho(x_n, x^\star) \\
&= \left(\alpha + \beta^2\lambda\right)\rho(x^\star, x_n) + \beta\alpha\rho(x_n, Tx_n). \qquad (3.5)
\end{aligned}$$

Let $n \to \infty$ in (3.5), we conclude that $\rho(x^\star, Tx^\star) \to 0$, hence $Tx^\star = x^\star$. If $x^{\star\star}$ is another fixed point of T, then

$$\rho(x^\star, x^{\star\star}) \leq \rho(Tx^\star, Tx^{\star\star}) \leq \lambda\rho(x^\star, x^{\star\star}).$$

Hence $\rho(x^\star, x^{\star\star}) = 0$, otherwise $\lambda \geq 1$, is a contradiction and the fixed point is unique. □

Theorem 3.2. *Let (X, ρ, ω) be a complete convex α, β b-metric space and $T : X \to X$ be defined by $\rho(Tx, Ty) \leq \lambda\left[\rho(x, Tx) + \rho(y, Ty)\right]$, for all $x, y \in X$ and for $0 < \lambda < \frac{1}{\beta^4}$. Choose $x_0 \in X$ such that $\rho(x_0, Tx_0) < \infty$ and define $x_n = \omega(x_{n-1}, Tx_{n-1}; \mu_{n-1})$, where*

$$0 < \mu_{n-1} < \frac{1}{\alpha}\left(\frac{1}{\beta^2} - \frac{1}{\beta^3} - \frac{1}{\beta^5}\right), \quad 1 + \beta^2 < \beta^3,$$

for each $n \in \mathbb{N}$, then T has a unique fixed point in X.

Proof.

$$\rho(x_n, x_{n+1}) = \rho(x_n, \omega(_n, Tx_n; \mu_n)) \leq (1 - \mu_n)\rho(x_n, Tx_n)$$

$$\begin{aligned}
\rho(x_n, Tx_n) &\leq \alpha\rho(x_n, Tx_{n-1}) + \beta\rho(Tx_{n-1}, Tx_n) \\
&\leq \alpha\rho(x_n, Tx_{n-1}) + \beta\lambda\left[\rho(x_{n-1}, Tx_{n-1}) + \rho(x_n, Tx_n)\right] \\
&= \alpha\rho(\omega(x_{n-1}, Tx_{n-1}; \mu_{n-1}), Tx_{n-1}) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) \\
&+ \beta\lambda\rho(x_n, Tx_n) \\
&= \alpha\mu_{n-1}\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x_{n-1}, Tx_{n-1}) + \beta\lambda\rho(x_n, Tx_n).
\end{aligned}$$

We observe that $0 < 1 - \beta\lambda$, and hence

$$\begin{aligned}\rho(x_n, Tx_n) &\leq \frac{\alpha\mu_{n-1} + \beta\lambda}{1 - \beta\lambda}\rho(x_{n-1}, Tx_{n-1}) \\ &\leq \frac{\rho(x_{n-1}, Tx_{n-1})}{\beta^2},\end{aligned}$$

as proved in Theorem 3.1. Hence $\rho(x_n, Tx_n)$ is a decreasing sequence that converges to zero and hence is a Cauchy sequence. If $\lim_{n\to\infty}\rho(x_n, x^\star) = 0$, then

$$\begin{aligned}\rho(x^\star, Tx^\star) &\leq \alpha\rho(x^\star, x_n) + \beta\rho(x_n, Tx^\star) \\ &\leq \alpha\rho(x^\star, x_n) + \beta[\alpha\rho(x_n, Tx_n) + \beta\rho(Tx_n, Tx^\star)] \\ &\leq \alpha\rho(x^\star, x_n) + \alpha\beta\rho(x_n, Tx_n) + \beta^2\lambda[\rho(x_n, Tx_n) + \rho(x^\star, Tx^\star)].\end{aligned}$$

Then it follows that

$$\begin{aligned}\left(1 - \beta^2\lambda\right)\rho(x^\star, Tx^\star) &\leq \alpha\rho(x^\star, x_n) + \left(\alpha\beta^2\lambda\right)\rho(x_n, Tx_n) \\ &\leq \alpha\rho(x^\star, x_n) + \left(\alpha\beta^2\lambda\right)\frac{\rho(x_0, Tx_0)}{\beta^{2n}}.\end{aligned}$$

Letting $n \to \infty$, we obtain $\rho(x^\star, Tx^\star) = 0$, so $x^\star$ is a fixed point of T. If $x^{\star\star}$ is another fixed point of T, then

$$\rho(x^\star, x^{\star\star}) = \rho(Tx^\star, Tx^{\star\star}) \leq \lambda[\rho(x^\star, Tx^\star) + \rho(x^{\star\star}, Tx^{\star\star})] = 0,$$

proving that the fixed point is unique. □

Lemma 3.1. *Let $\{y_n\}, \{z_n\}$ be non-negative sequences satisfying $y_{n+1} \leq z_n + hy_n$ for all $n \in \mathbb{N}$, $0 \leq h < 1$, $\lim_{n\to\infty} z_n = 0$, then $\lim_{n\to\infty} y_n = 0$.*

Definition 3.1. *Let T be a self map on a complete $\alpha\beta$ b-metric space (X, ρ). The iterative procedure $x_{n+1} = f(T, x_n)$ is weakly T-stable if $\{x_n\}$ converges to a fixed point $x^\star$ of T and if $\{y_n\}$ is a sequence in X such that $\lim_{n\to\infty}\rho(y_{n+1}, f(T, y_n)) = 0$ and $\{\rho(y_n, Ty_n)\}$ is bounded then $\lim_{n\to\infty} y_n = x^\star$.*

Theorem 3.3. *Under the assumptions of Theorem 3.1, if in addition $\lim_{n\to\infty}\mu_n = 0$, then Mann's iteration is weakly T-stable.*

Proof. From Theorem 3.1, $x^\star$ is a fixed point of T in X. If $\{y_n\}$ is a sequence such that $\lim_{n\to\infty} \rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) = 0$ and $\{\rho(y_n, Ty_n)\}$ is bounded, then

$$\begin{aligned}\rho(y_{n+1}, x^\star) &\le \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\rho(\omega(y_n, Ty_n; \mu_n), x^\star)\\ &\le \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta[\alpha\rho(\omega(y_n, Ty_n; \mu_n), Ty_n)\\ &\quad + \beta\rho(Ty_n, Tx^\star)]\\ &\le \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\alpha\mu_n\rho(y_n, Ty_n) + \beta^2\lambda\rho(y_n, x^\star)\\ &= z_n + \beta^2\lambda\rho(y_n, x^\star).\end{aligned}$$

Since $\beta^2\lambda < 1$ and $\{\rho(y_n, Ty_n)\}$ is bounded, $\lim_{n\to\infty} z_n = 0$ and hence by Lemma 3.1, $\lim_{n\to\infty} \rho(y_n, x^\star) = 0$. □

Theorem 3.4. *Under the assumptions of Theorem 3.2, if in addition* $\lim_{n\to\infty} \mu_n = 0$*, and if* α, β, λ *satisfy additionally* $\frac{\alpha\beta^2}{1-\lambda\beta} < 1$ *then Mann's iteration is weakly* T*-stable.*

Proof. From Theorem 3.2 $x^\star$ is a fixed point of T in X. If $\{y_n\}$ is a sequence such that $\lim_{n\to\infty} \rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) = 0$ and $\{\rho(y_n, Ty_n)\}$ is bounded, then

$$\begin{aligned}\rho(y_{n+1}, x^\star) &\le \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\rho(\omega(y_n, Ty_n; \mu_n), x^\star)\\ &\le \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\alpha\rho(\omega(y_n, Ty_n; \mu_n), Ty_n)\\ &\quad + \beta^2\rho(Ty_n, x^\star).\end{aligned}$$

Now,

$$\begin{aligned}\rho(Ty_n, x^\star) = \rho(Ty_n, Tx^\star) &\le \lambda\rho(y_n, Ty_n)\\ &\le \lambda\alpha\rho(y_n, x^\star) + \lambda\beta\rho(x^\star, Ty_n).\end{aligned}$$

From which we get

$$\rho(Ty_n, x^\star) \le \frac{\lambda\alpha}{1-\lambda\beta}\rho(y_n, x^\star),$$

$$\begin{aligned}\rho(y_{n+1}, x^\star) &\le \alpha\rho(y_{n+1}, \omega(y_n, Ty_n; \mu_n)) + \beta\alpha\mu_n\rho(y_n, Ty_n) + \frac{\lambda\alpha\beta^2}{1-\lambda\beta}\rho(y_n, x^\star)\\ &= z_n + \frac{\lambda\alpha\beta^2}{1-\lambda\beta}\rho(y_n, x^\star).\end{aligned}$$

Since $\frac{\lambda\alpha\beta^2}{1-\lambda\beta} < 1$ and $\{\rho(y_n, Ty_n)\}$ is bounded, $\lim_{n\to\infty} z_n = 0$ and hence by Lemma 3.1, $\lim_{n\to\infty} \rho(y_n, x^\star) = 0$. □

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