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ON THE DYNAMICS OF THE SINGULARLY PERTURBED RICCATI DIFFERENCE EQUATION WITH CONTINUOUS ARGUMENT

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ABSTRACT. In this paper, we study the dynamic properties of the singularly perturbation of the Riccati difference equation with continuous arguments.

1. INTRODUCTION

The difference equation with continuous argument

$$x(t) = f(x(t-1)),\ t \in [0, T]. \tag{1.1}$$

is one of the principal mathematical instruments of modern nonlinear dynamics. Let $\epsilon \in (0, 1]$, the equation

$$\epsilon \frac{dx}{dt} + x(t) = f(x(t-1)),\ t \in I = [0, T]$$

is the singular perturbation of the difference equation with continuous argument (1.1). Here we study the dynamic properties of the Riccati difference equation with continuous argument

$$x(t) = 1 - \rho x^2(t-1),\ t \in I,\ x(t) = x_0,\ t \leq 0. \tag{1.2}$$

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and its singularly perturbation

$$\epsilon\frac{dx}{dt}+x(t)=1-\rho x^2(t-1),\ t\in I,\ x(t)=x_0,\ t\le 0 \tag{1.3}$$

and compare the results when $\epsilon \to 1$ with the Riccati differential difference equation

$$\frac{dx}{dt}+x(t)=1-\rho x^2(t-1),\ x(t)=x_0,\ t\le 0.$$

2. THE SINGULAR PERTURBED EQUATION

Now, for the equation (1.2) we have the results (see [1,2]).

Theorem 2.1. *The system (1.2) has two fixed points namely,* $x^*_{1,2}=\frac{-1\pm\sqrt{1+4\rho}}{2\rho}$ *such that* $\Rightarrow x^*_{1,2}=\frac{-1\pm\sqrt{1+4\rho}}{2\rho}$ *is stable if* $|(1\pm\sqrt{1+4\rho})|<1$.

Theorem 2.2. *When* $-(1\pm\sqrt{1+4\rho})$ *passes through the critical value* $-(1\pm\sqrt{1+4\rho})=\sqrt{1+\xi^2}$, *there is a Hopf bifurcation from the equilibrium* $(x_{1,2})^*=(\frac{-1}{2\rho})(1\pm\sqrt{1+4\rho})$ *to a periodic orbit.*

Theorem 2.3. *The fixed points* $(x_{1,2})^*=(\frac{-1}{2\rho})(1\pm\sqrt{1+4\rho})$ *are stable if* $-1<-(1\pm\sqrt{1+4\rho})<\sqrt{1+\xi^2}$ *and unstable if* $-1>-(1\pm\sqrt{1+4\rho})$, $-(1\pm\sqrt{1+4\rho})>\sqrt{1+\xi^2}$.

2.1. Local stability and existence of bifurcation.

In this section, we will consider the local stability of fixed points of the delay equation (1.3). The system has two fixed points which are the solution of the follow equation $x+1-\rho x^2=0$ which has two fixed points, namely,

$$x^*_{1,2}=\frac{-1}{2\rho}(-1\pm\sqrt{1+4\rho}).$$

The linearized equation is

$$\frac{dy}{dt}=\frac{-1}{\epsilon}y(t)+\frac{-1}{\epsilon}(1\pm\sqrt{1+4\rho})y(t-1),$$

where,

$$y(t)=x(t)-(\frac{-1}{2\rho})(1\pm\sqrt{1+4\rho}).$$

The characteristic equation is of the form

$$\lambda+\frac{1}{\epsilon}-\frac{1}{\epsilon}(1\pm\sqrt{1+4\rho})e^{-\lambda}=0. \tag{2.1}$$

Lemma 2.1. *All roots of the characteristic equation* $\lambda + c + be^{-\lambda} = 0$, *where c and b are real, have negative real parts if and only if*

$$c > -1,\ c + b > 0,\ b < \sqrt{c^2 + \xi^2},$$

where ξ *is the root of*

$$\xi = -c\tan\xi,\ 0 < \xi < \pi.\ If\ c \neq 0, \xi = \frac{\pi}{2}, if c = 0.$$

Applying lemma 2.1 to to equation (2.1) with $c = \frac{1}{\epsilon}$, and $b = \frac{-1}{\epsilon}(1 \pm \sqrt{1+4\rho})$. We have the following theorem.

Theorem 2.4. *The fixed points* $(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho})$ *are stable if*

$$\frac{-1}{\epsilon} < \frac{-1}{\epsilon}(1 \pm \sqrt{1+4\rho}) < \sqrt{\frac{1}{\epsilon^2} + \xi^2},$$

and unstable if

$$\frac{-1}{\epsilon} > \frac{-1}{\epsilon}(1 \pm \sqrt{1+4\rho}),\ \frac{-1}{\epsilon}(1 \pm \sqrt{1+4\rho}) > \sqrt{\frac{1}{\epsilon^2} + \xi^2}.$$

2.2. **Hopf bifurcation.**

Here we discuss the Hopf bifurcation. We have the following theorem.

Theorem 2.5. *When* $-(1 \pm \sqrt{1+4\rho})$ *passes through the critical value* $-(1 \pm \sqrt{1+4\rho}) = \sqrt{\frac{1}{\epsilon^2} + \xi^2}$, *there is a Hopf bifurcation from the equilibrium* $(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho})$ *to a periodic orbit.*

Proof. Let $(1 + \sqrt{1+4\rho}) = K$, then, assume that $\lambda = i\omega_0$, $\omega_0 \in R^+$ is a pure imaginary solution of equation (2.1) for some parameter value $K = K_*$. This leads to the following equation

$$i\omega_0 + \frac{1}{\epsilon} - \frac{K_*}{\epsilon}e^{-i\omega_0} = 0,\ \frac{1}{\epsilon} = \frac{K_*}{\epsilon}\cos(\omega_0),\ \omega_0 = \frac{K_*}{\epsilon}\sin(\omega_0),$$

$$\omega_0^2 + \frac{1}{\epsilon^2} = \frac{K_*^2}{\epsilon^2}[\cos(\omega_0)^2 + \sin(\omega_0)^2] = \frac{K_*^2}{\epsilon^2}$$

and

$$K_* = \pm\epsilon\sqrt{\frac{1}{\epsilon^2} + \omega_0^2},\ \omega_0 = \frac{-1}{\epsilon}\tan(\omega_0).$$

Next, we have

$$K_* = -\epsilon\sqrt{\frac{1}{\epsilon^2 + \omega_0^2}},\ \text{is the critical value of } K,$$

where, ω_0 is the root of $\omega_0 = \frac{-1}{\epsilon}\tan(\omega_0)$, $0 < \omega_0 < \pi$.

The condition $\frac{d(Re(\lambda))}{dK}|_{K=K_*}$ is the last condition for occurrence of a Hopf bifurcation. To show that this condition is satisfied, let $\lambda = Z(K) + i\omega(K)$ and using (2.1), we obtain $Z + i\omega + \frac{1}{\epsilon} - \frac{K}{\epsilon}e^{-z-i\omega} = 0$,

$$Z + \frac{1}{\epsilon} - \frac{K}{\epsilon}e^{-z}\cos(\omega) = 0, \tag{2.2}$$

$$\omega + \frac{K}{\epsilon}e^{-z}\sin(\omega) = 0. \tag{2.3}$$

Differentiate (2.2) and (2.3) with respect to K, we obtain

$$\epsilon\frac{dZ}{dK} - e^{-z}\cos(\omega) + Ke^{-z}\cos(\omega)\frac{dz}{dk} + Ke^{-z}\sin(\omega)\frac{d\omega}{dK} = 0, \tag{2.4}$$

$$\epsilon\frac{d\omega}{dK} + e^{-z}\sin(\omega) + Ke^{-z}\cos(\omega)\frac{d\omega}{dK} - Ke^{-z}\sin(\omega)\frac{dZ}{dK} = 0. \tag{2.5}$$

Solving equation (2.4) and equation (2.5) for $\frac{dZ}{dK}$, we obtain

$$\begin{aligned}\frac{d(Re(\lambda))}{dK}|_{k=k_*} &= \frac{d(Re(\lambda))}{dK}|_{z=0,\omega=\omega_0,k=k_*}\\ &= \frac{\epsilon\cos(\omega_0 + K_*}{(\epsilon + K_*\cos(\omega_0))^2 + (K_*\sin(\omega_0))^2}\\ &= \frac{\epsilon K_*\cos(\omega_0) + K_*^2}{K_*^2[(\epsilon + K_*\cos(\omega_0))^2 + (K_*\sin(\omega_0))^2]}\\ &= \frac{\epsilon + K_*^2}{K_*[(\epsilon + K_*\cos(\omega_0))^2 + (K_*\sin(\omega_0))^2]} \neq 0.\end{aligned}$$

Similarly, we can prove that there is a Hopf bifurcation from the equilibrium $(x_2)^* = (\frac{-1}{2\rho})(1 - \sqrt{1+4\rho})$ to a periodic orbit. □

2.3. **The discretized system.**

In this section, the discretized analogue of the system is obtained via the method of steps as follows . By applying the method of steps then the equation

$$\epsilon\frac{dx}{dt} = -x(t) + (1 - \rho x^2(t-1)),$$

has the solution

$$x_{n+1}(t) = e^{\frac{-(t-n)}{\epsilon}}x_n + (1 - \rho x_n^2)(1 - e^{\frac{-(t-n)}{\epsilon}}),\ t \in (n, n+1].$$

Let $t \to n+1$, then

$$x_{n+1} = x_n e^{\frac{-1}{\epsilon}} + (1 - \rho x_n^2)(1 - e^{\frac{-1}{\epsilon}}). \tag{2.6}$$

2.4. **Stability and bifurcation of the discretized system.**

The system (2.6) has two fixed points namely, $x^*_{1,2} = \frac{-1\pm\sqrt{1+4\rho}}{2\rho}$. Now

$$f'(x) = e^{\frac{-1}{\epsilon}} - 2\rho x(1 - e^{\frac{-1}{\epsilon}}),$$

then $x^*_{1,2} = \frac{-1\pm\sqrt{1+4\rho}}{2\rho}$ is stable if

$$|e^{\frac{-1}{\epsilon}} + (1 \pm \sqrt{1+4\rho})(1 - e^{\frac{-1}{\epsilon}})| < 1.$$

We have two cases for the perturbation parameter ϵ. 1-As $\epsilon \to 0$, $x^*_{1,2} = \frac{-1\pm\sqrt{1+4\rho}}{2\rho}$ is stable if $|(1 \pm \sqrt{1+4\rho})| < 1$.

This is the same results of (1.2). 2-As $\epsilon \to 1$, $x^*_{1,2} = \frac{-1\pm\sqrt{1+4\rho}}{2\rho}$ is stable if $|e^{-1} + (1 - e^{-1})(1 \pm \sqrt{1+4\rho})| < 1$.

This is the same results of (2.5).

3. NUMERICAL SIMULATIONS

We confirm all the previous analytical findings with the help of numerical simulations performed via Matlab. In all numerical simulations the initial condition is taken as $(x_0, y_0) = (0.4, 0.4)$ and the bifurcation parameter is taken as ρ.

We have the following examples

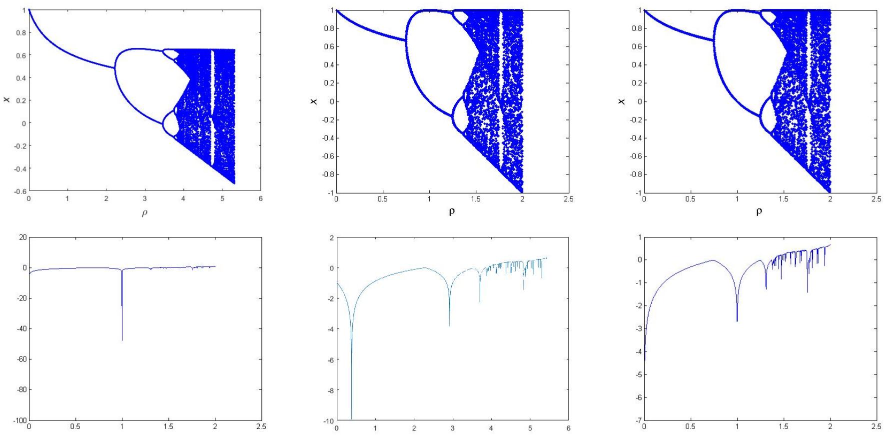

FIGURE 1. Bifurcation diagram as a function of ρ and corresponding Lyapunov exponent of system (1.3).

4. Conclusion

In this work, we discussed stability, bifurcation and chaos of the singularly perturbed differential difference Riccati equation. Local stability and bifurcation analysis of the discretized system. We find that the singularly perturbed Riccati differential difference equation behaves as the Riccati difference equation with continuous argument when the perturbation parameter $\epsilon \to 0$ and behaves as the Riccati differential difference equation when the perturbation parameter $\epsilon \to 1$.

References

[1] A. M. A. El-Sayed, M. E. Nasr: *Discontinuous dynamical system represents the Logistic retarded functional equation with two different delays*, Malaya Journal of Matematik, **1**(1)(2013), 50–56.

[2] M. Jiang, Y. Shen, J. Jian, X. Liao: *Stability, bifurcation and a new chaos in the logistic differential equation with delay*, Physics Letters A, **350** (2006), 221–227.

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