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IDEALS AND FILTERS ON IMPLICATION ALGEBRAS

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ABSTRACT. The concept of Ideals and filters on implication algebra is introduced. Using the idea of upper sets we investigate basic ideas of ideals and filters in an implication algebra. Finally, Self distributive implication algebra, basic concepts on implication algebras are explained and important theorems have been proved.

1. INTRODUCTION

The theory of classes of abstract algebras of BCK-algebras and BCI-algebras were introduced in [5]. J. Negger and et. al. introduced the concept of B-algebras in [4], and J. C. Abbott explored a concept of orthoimplication algebra in [1]and Ivan Chajda initiated the ideas of implication algebra in [3].

In this paper, the concept of implication ideals, Subalgebras, normal subsets, upper sets and filters in implication algebras are introduced, and important theorems and corollaries have been proved. Also we use $\Rightarrow$ as a binary operation but not as a logical connective.

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2. Preliminaries

Definition 2.1. *[2] A B-algebra* $(X, *, 0)$ *is a non-empty set X with a constant 0 and binary operation "* $*$ *" satisfying the following axioms:*

(1) $a * a = 0$.
(2) $a * 0 = a$.
(3) $(a * b) * c = a * (c * (0 * b)), forall \ \ b, c \in X$.

Definition 2.2. *[2] A non- empty subset N of a B- algebra X is called a B-subalgebra of X if* $a * b \in N$, *for all* $a, b \in N$.

Definition 2.3. *[2] A non-empty subset N of a B- algebra X is said to be normal if* $(x * a) * (y * b) \in N$, *whenever* $x * y \in N$ *and* $a * b \in N$.

Proposition 2.1. *[4] If* $(X, *, 0)$ *is a commutative B-algebra, then* $(0*a)*(0*b) = b * a$*,for any,* $a, b \in X$.

Definition 2.4. *[4] A B-algebra* $(X, *, 0)$ *is said to be commutative if* $a * (0 * b) = b * (0 * a)$*,for any* $a, b \in X$.

Proposition 2.2. *[4] If* $(X, *, 0)$ *is a commutative B-algebra, then* $(0*a)*(a*b) = b * a^2$*,for any* $a, b \in X$.

3. Results

3.1. Ideals and filters in an implication algebras.

Definition 3.1. *An algebra* $(B, \Rightarrow, 1)$ *of type* $(2, 0)$ *is called Implication algebra if the follwing condition holds:*

(1) $a \Rightarrow a = 1$, $\forall a \in B$.
(2) $a \Rightarrow 1 = 1$, $\forall a \in B$.
(3) $1 \Rightarrow a = a$, $\forall a \in B$.
(4) $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$, $\forall \ a, b, c \in B$.

Definition 3.2. *Let B be an implication algebra. Define a relation"* $\leq$ *" on B by* $a \leq b$*, if, and only if,* $a \Rightarrow b = 1$.

Proposition 3.1. *If* $(B, \Rightarrow, 1)$ *is an implication algebra, then for any* $a, b \in B$,

$$a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}.$$

Theorem 3.1. *If* $(B, \Rightarrow, 1)$ *is an implication algebra, then* $a \Rightarrow (b \Rightarrow a) = 1$*, for any* $a, b \in B$*.*

Proof. Assume $(B, \Rightarrow, 1)$ be an implication algebra and $a, b \in B$. Then we have
$1 = b \Rightarrow 1 = b \Rightarrow (a \Rightarrow a)$, since $a \Rightarrow a = 1$.
= $a \Rightarrow (b \Rightarrow a)$ (exchange rule).
Hence $a \Rightarrow (b \Rightarrow a) = 1$. □

Example 1. *Let* $B = \{1, a, b, c, d\}$ *be a set defined by the table below:*

⇒	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	c
c	1	a	b	1	b
d	1	1	a	1	1

Hence $(B, \Rightarrow, 1)$ *is an implication algebra.*

Definition 3.3. *Let* $(B, \Rightarrow, 1)$ *be an implication algebra. Then a subset I of B is called an ideal of B if it satisfies :*

(1) $0 \in I$.
(2) $a \Rightarrow b \in I$ *and* $b \in I$ *implies* $a \in I$*,* $\forall a, b \in B$*.*

Definition 3.4. *Let* $(B, \Rightarrow, 1)$ *be an implication algebra. Then a non-empty subset S of an implication algebra B is called a subalgebra of B if* $a, b \in S$*, then* $a \Rightarrow b \in S$*.*

Theorem 3.2. *Let* $(B, \Rightarrow, 1)$ *be an implication algebra and* $\emptyset \neq S \subseteq B$*. Then the following are equivalent:*

(1) *S is a subalgebra of B.*
(2) $a \Rightarrow (1 \Rightarrow b), 1 \Rightarrow b \in S$*,* $\forall a, b \in S$*.*

Proof.
1 ⇒ 2: Assume 1 holds. Since $S \neq \emptyset$, there exists an element $a \in S$ and $1 = a \Rightarrow a \in S$. Again S is closed under " ⇒ ", $1 \Rightarrow b \in S$, for some $b \in S$. Thus, $a \Rightarrow (1 \Rightarrow b) \in S$.

2 ⇒ 1: Assume 2 holds. Since $a \Rightarrow b = a \Rightarrow (1 \Rightarrow (1 \Rightarrow b))$, for any $a, b \in S$. Hence, $a \Rightarrow b \in S$, $\forall a, b \in S$. Thus S, is a subalgebra of B. □

Definition 3.5. *A non-empty subset N of B is said to be normal subset of B if* $(c \Rightarrow a) \Rightarrow (d \Rightarrow b) \in N$*, for any* $a \Rightarrow b, c \Rightarrow d \in N$*.*

Example 2. *Let* $B = \{0, 1, 2, 3\}$ *with 3 as greatest element defined by the table below:*

$\Rightarrow$	0	1	2	3
0	3	1	2	3
1	0	3	2	3
2	0	1	3	3
3	0	1	2	3

Then $(B, \Rightarrow, 0)$ *is an implication algebra with greatest element 3.*

Let $N = \{0, 3\}$ is normal of B. Since $(0 \Rightarrow 3) \Rightarrow (3 \Rightarrow 0) = 3 \Rightarrow 0 = 0 \in N$ and $(3 \Rightarrow 0) \Rightarrow (0 \Rightarrow 3) = 0 \Rightarrow 3 = 3 \in N$.

Theorem 3.3. *Every normal subset N of an implication algebra B is a subalgebra of B.*

Definition 3.6. *An implication algebra* $(B, \Rightarrow, 1)$ *is a 1-commutative implication algebra, if it satisfy* $a \Rightarrow b = b \Rightarrow (1 \Rightarrow a)$*,*$\forall a, b \in B$*.*

Example 3. *Let* $B = \{a, b, c, 1\}$ *with* $a \leq b \leq c \leq 1$ *defined by the table below:*

$\Rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	1
c	1	a	b	1

Then $(B, \Rightarrow, 1)$ *is an implication algebra.*

The set $K = \{1, a\}$ *is a subalgebra of B. Since* $a \Rightarrow 1 = 1 \in K$*. But K is not an ideal as* $b \Rightarrow 1 = 1 \in K$ *and* $b \notin K$*. Also* $T = \{b, c\}$ *is not a subalgebra B. Since* $b \Rightarrow c = 1 \notin T$*.*

Lemma 3.1. *Let S be a subalgebra of a 1-commutative implication algebra, and let* $a, b \in B$*. If* $a \Rightarrow b \in S$*, then* $b \Rightarrow a \in S$*.*

Proof. Let $a \Rightarrow b \in S$. Then

$$b \Rightarrow a = a \Rightarrow (1 \Rightarrow b)$$
$$= 1 \Rightarrow (a \Rightarrow b)(exchangerule)$$
$$= a \Rightarrow b \in S$$

Hence $b \Rightarrow a \in S$. □

Lemma 3.2. *Let* $(B, \Rightarrow, 1)$ *be an implication algebra. Then the following holds:*

(1) $(a \Rightarrow b) \Rightarrow a = a$

(2) $(a \Rightarrow b) \Rightarrow b = (b \Rightarrow a) \Rightarrow a, \forall, a, b \in B$.

Definition 3.7. *For an implication algebra B, for any* $a, b \in B$*, the following holds:*

(1) $a \vee b = (a \Rightarrow b) \Rightarrow b$.

(2) $a \wedge b = b \Rightarrow (b \Rightarrow a)$.

Definition 3.8. *Let* $(B, \Rightarrow, 1)$ *be an implication algebra and let F be a non-empty subset of B. Then F is said to be a filter of an implication algebra B if*

(1) $1 \in F$.

(2) $a \Rightarrow b \in F$ *and* $a \in F$ *imply* $b \in F$.

In Example 3 $F_1 = \{1, a, b\}$ is a filter of B. As $1 \in F_1$,and $a \Rightarrow b = a \in F_1$ and $b \in F_1$. But $F_2 = \{1, a\}$ is not afilter of B. Because $1 \in F_2$ holds and $a \Rightarrow b = a \in F_2$ and $a \in F_2$. But $b \notin F_2$.

Definition 3.9. *Let B be an implication algebra and let* $a, b \in B$*. Define* $A(a, b) = \{c \in B | a \Rightarrow (b \Rightarrow c) = 1\}$*. Then* $A(a, b)$ *is called an upper set of a and b.*

Remark 3.1. *Let B be an implication algebra. Then*

$1 \in A(a, b) \Leftrightarrow \{1 \in B | a \Rightarrow (b \Rightarrow 1) = 1\} = \{1\}$.

$a \in A(a, b) \Leftrightarrow \{a \in B | a \Rightarrow (b \Rightarrow a) = 1\} = \{a \in B | 1 = 1\} = B$.

$b \in A(a, b) \Leftrightarrow \{b \in B | a \Rightarrow (b \Rightarrow b) = 1\} = \{b \in B | 1 = 1\} = B$.

Hence $1, a, b \in A(a, b)$*,*$a, b \in B$ *and*

$$A(1, a) = \{b \in B | 1 \Rightarrow (a \Rightarrow b) = 1\} = \{b \in B | a \Rightarrow b = 1\} = \{1, a\} = F_2.$$

But F_2 *is not a filter as* $b \notin F_2$.

Hence $A(1, a)$ *is not a filter.*

Theorem 3.4. *Let B be an implication algebra. If* $b \in B$ *satisfy* $b \Rightarrow c = 1$*,* $\forall c \in B$*, then* $A(a, b) = B = A(b, a)$*,*$\forall a \in B$.

Proof. Let $a, b.c \in B$ and $b \Rightarrow c = 1$ for an implication algebra B.

$$\begin{aligned} A(a, b) &= \{c \in B | a \Rightarrow (b \Rightarrow c) = 1\} \text{ by definition of } A(a, b). \\ &= \{c \in B | a \Rightarrow 1 = 1\}, \text{ since } b \Rightarrow c = 1. \qquad (3.1) \\ &= \{c \in B | 1 = 1\} = B \end{aligned}$$

Again

$$\begin{aligned} A(b,a) &= \{c \in B | b \Rightarrow (a \Rightarrow c) = 1\} \\ &= \{c \in B | a \Rightarrow (b \Rightarrow c) = 1\} \text{ (exchange rule)} \\ &= \{c \in B | a \Rightarrow 1 = 1\} = \{c \in B | 1 = 1\} = B \end{aligned}$$

Hence $A(a,b) = B = A(b,a)$, for all $a, b \in B$. □

Definition 3.10. *An implication algebra* $(B, \Rightarrow, 1)$ *is said to be self distributive if* $a \Rightarrow (b \Rightarrow c) = (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$*,* $\forall a, b, c \in B$*.*

Example 4. *Let* $B = \{1, a, b, c\}$ *be a set and define "* $\Rightarrow$ *" by the table below:*

$\Rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then $(B, \Rightarrow, 1)$ *is an implication algebra.*

Let $a, b, c \in B$*. Then* $a \Rightarrow (b \Rightarrow c) = a \Rightarrow c = c$ *and* $(a \Rightarrow b) \Rightarrow (a \Rightarrow c) = b \Rightarrow c = c$*. Hence* $a \Rightarrow (b \Rightarrow c) = (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$ *holds for all* $a, b, c \in B$*. Thus* $(B, \Rightarrow, 1)$ *is a self distributive implicative algebra.*

Example 5. *Let* $R = \{1, a, b, c, d, 0\}$ *be a set defined by the table below:*

$\Rightarrow$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(B, \Rightarrow, 1)$ *is an implication algebra. Since* $d \Rightarrow (a \Rightarrow 0) = d \Rightarrow d = 1$ *and* $(d \Rightarrow a) \Rightarrow (d \Rightarrow 0) = 1 \Rightarrow a = a$*.*

Hence $d \Rightarrow (a \Rightarrow 0) \neq (d \Rightarrow a) \Rightarrow (d \Rightarrow 0)$*. Thus* $(B, \Rightarrow, 1)$ *is not self distributive implication algebra.*

Theorem 3.5. *Let* $(B, \Rightarrow, 1)$ *be a self distributive implication algebra. Then the upper set* $A(a, b)$ *is a filter of B, where* $a, b \in B$*.*

Proof. $1 \in A(a,b) = \{1 \in B | a \Rightarrow (b \Rightarrow 1) = 1\} = \{1 \in B | 1 = 1\} = \{1\}$.

Let $c \Rightarrow d \in A(a,b)$ and $c \in A(a,b)$. Then $1 = a \Rightarrow (b \Rightarrow (c \Rightarrow d))$ and $1 = a \Rightarrow (b \Rightarrow c)$. Since B is self -distributive implication algebra:

$$\begin{aligned} 1 &= a \Rightarrow (b \Rightarrow (c \Rightarrow d)) \\ &= a \Rightarrow ((b \Rightarrow c) \Rightarrow (b \Rightarrow d)) \\ &= [a \Rightarrow (b \Rightarrow c)] \Rightarrow [a \Rightarrow (b \Rightarrow d)] \\ &= 1 \Rightarrow [a \Rightarrow (b \Rightarrow d)] \\ &= a \Rightarrow (b \Rightarrow d). \end{aligned}$$

Hence $1 = a \Rightarrow (b \Rightarrow d)$. Thus $d \in A(a,b)$. This proves that $A(a,b)$ is a filter of B. □

Using the idea of upper set $A(a,b)$ we give an equivalent condition of the filter in an implication algebras.

Theorem 3.6. *Let F be anon-empty subset of an implication algebra B. Then F is a filter of B if and only if* $A(a,b) \subseteq F$, $\forall a, b \in F$.

Proof. Assume that F is a filter of B and let $a, b \in F$. If $c \in A(a,b)$, then $a \Rightarrow (b \Rightarrow c) = 1 \in F$. Since $a, b \in F$ by applying (F_2) we receive that $b \Rightarrow c \in F$.

Hence $c \in F$. Since F is afilter. Therefore $A(a,b) \subseteq F$.

Conversely, suppose that $A(a,b) \subseteq F$,$\forall a, b \in F$.

Since $a \Rightarrow (b \Rightarrow 1) = a \Rightarrow 1 = 1$, $1 \in A(a,b) \subseteq F$. So that $1 \in F$.

Assume $a \Rightarrow b, a \in F$. Since $(a \Rightarrow b) \Rightarrow (a \Rightarrow b) = 1$, we have $b \in A(a \Rightarrow b, a) \subseteq F$. Implies that $b \in F$. Hence F is afilter of B. □

Theorem 3.7. *If F is a filter of an implication algebra B, then* $F = \cup_{a,b \in F} A(a,b)$.

Proof. Let F be a filter of B and let $c \in F$. Since $c \Rightarrow (1 \Rightarrow c) = c \Rightarrow c = 1$. We have $c \in A(c,1)$. Hence $F \subseteq \cup_{c \in F} A(c,1) \subseteq \cup_{a,b \in F} A(a,b)$.

If $c \in \cup_{a,b \in F} A(a,b)$, then there exist $x, y \in F$ such that $c \in A(x,y)$. It follows by theorem 3.18 $c \in F$. This implies that $\cup_{a,b \in F} A(a,b) \subseteq F$. Therefore $F = \cup_{a,b \in F} A(a,b)$. □

Corollary 3.1. *If F is a filter of an implication algebra B, then* $F = \cup_{a \in F} A(a,1)$.

4. Conclusion

The concept of implication algebra is introduced,and we also elaborate ideals and filters in an implication algebra. In addition the idea of upper set is investigated and it enables to explain filters in an implication algebra. We also discussed Self -distributive lattice in an implication algebra and different theorems and corollaries have been proved.

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