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BI-INTERIOR IDEALS IN TGSR

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ABSTRACT. In this paper, we will discuss notation of bi-interior ideals as a generalization of quasi-ideal, bi-ideal, interior ideals and bi-interior ideals of TGSR and study the properties of bi-interior ideals of TGSR.

1. INTRODUCTION AND PRELIMINARIES

During 1950-1980, the concept of bi-ideals, quasi-ideals and interior ideals were studied by many mathematicians. In this paper, we introduced the notation of prime bi-interior ideals of TG Semi rings. G. Srinivasa Rao et.al [5–9] studied ternary semi rings. A lot of literature is available related to this work [1–4].

Let $(R,+)$ and $(\Gamma,+)$ be commutative semi groups. Then we call R a ***TG-semi ring*** (TGS), if there is mapping $R\times\Gamma\times R\times\Gamma\times R\to R(imagesof(p,a,q,b,r)$ will be denoted by $paqbr$, $\forall p,q,r\in R$,$a,b,\in\Gamma)\ni$ it satisfies the following axioms for all $p,q,r,s,t\in R$ and $a,b,c,d\in\mathit{\Gamma}$:

(1) $pa(q+r)bs=paqbs+parbs$
(2) $(p+q)arbs=parbs+qarbs$

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(3) $paqb(r+s) = paqbr + paqbs$

(4) $pa(qbrcs)dt = paqb(rcsdt) = (paqbr)csdt$

A TGS R is said to be commutative TGS, if $paqbr = parbq = qarbp = qapbr = rapbq = raqbp$,$\forall pqr \in R$ and $a, b \in \mathit{\Gamma}$. Let R be a TGS. An element $e \in R$ is said to be $unity\ element$ or $neutralelemnt$ if for each $p, q \in R\ \exists\ a, b \in \mathit{\Gamma}\ \ni paqbe = paebp = eapbp = e$. A TGS R is said to have zero element if there exists an element $0 \in R$ such that $0 + p = p$ and $0apb0 = pa0b0 = 0a0bp = 0\ \forall p \in R$, $a, b \in \Gamma$. If there exists $a, b \in \Gamma\ \ni p = papbp$, then an element $p$is known as an $idempotent$ element. R is said to be an TGS R, if each element in R is an idempotent. A TGS R is called a division TGS if for each non-zero element of R has inverse with respect to multiplication. An element p in TGS R is said to be $regularelement$, id $\exists x, y$ in R and a, b, c, d in Γ such that $p = paxbpcydp$. If every element in TGS R is regular element, then R is called $regularTGSR$.

Definition 1.1. *A non-empty subset S is said to be $ternarysub - \mathit{\Gamma} - semi - ringR$, if S is a sub-semi-group with respect to $+$ of R and $a\alpha b\beta c \in S, \forall a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

Definition 1.2. *A non-empty subset I of a ternary Γ-semi-ring of a ternary Γ-semi-ring R is said to $left(lateral, right)$ ternary Γ-ideal of R, if (1) $a, b \in I \to a + b \in I$; (2) $a, b \in R, i \in I$,$\alpha, \beta \in \Gamma \implies a\alpha b\beta i \in I(a\alpha i\beta b \in I, i\alpha a\beta b \in \Gamma)$. An ideal I is said to be* ternary *Γ-ideal, if it is left, lateral and right Γ ideal of R.*

Example 1. *Consider the set $Z = \{0, \pm1, \pm2, \pm3, \cdots\}$ and $Gamma$ be the set of all even numbers. Then with respect to usual addition and ternary multiplication, Z is ternary $Gamma$ semi ring.*

Example 2. *Let $Q = R$ be the set of all rational numbers and Γ the set of all natural numbers. Define a mapping $R \times \Gamma \times R \times \Gamma \times R \to R$ by usual addition and ternary multiplication defined by $(p, a, q, b, r) = paqbr$, $\forall\ p, q, r \in R$, $a, b \in \Gamma$ then R is a ternary Γ semi ring.*

Definition 1.3. *Let $\phi \neq S \subseteq R$, where R is a TGS. The set S is said to be a TG-subsemi ring of R, if $(S, +)$ is a ternary sub semi group $(TSSG)$ of $(R, +)$ and $S\Gamma S\Gamma S \subseteq S$.*

Definition 1.4. *Let R be a TGS and $\phi \neq S \subseteq R$. The set S is said to be a quasi-ideal (QI) of R, if S is a TG-sub semi ring $(TGSSR)$ of R and $(S\Gamma R\Gamma R)\ \cap (R\Gamma S\Gamma R + R\Gamma R\Gamma S\Gamma R\Gamma R) \cap (R\Gamma R\Gamma S)\ \subseteq S$.*

Definition 1.5. *Let R be a TGS and $\phi \neq S \subseteq R$. The set S is said to be a bi-ideal (BI) of R, if S is a $(TGSSR)$ of R and $S\Gamma R\Gamma S\Gamma R\Gamma S \subseteq S$.*

Definition 1.6. *Let R be a TGS and $\phi \neq S \subseteq R$. The set S is said to be a interior-ideal (II) of R, if S is a $(TGSSR)$ of R and $R\Gamma R\Gamma S\Gamma R\Gamma R \subseteq S$.*

Definition 1.7. *Let R be a TGS and $\phi \neq S \subseteq R$. The set S is said to be a rt. $(medial, lt.)$ ideal of R, if S is a $(TGSSR)$ of R and $S\Gamma R\Gamma R \subseteq S(R\Gamma S\Gamma R \subseteq S, R\Gamma R\Gamma S \subseteq S)$.*

Definition 1.8. *Let R be a TGS and $\phi \neq S \subseteq R$. The set S is said to be an ideal of R, if S is a $(TGSSR)$ of R and $S\Gamma R\Gamma R \subseteq S, R\Gamma S\Gamma R \subseteq S, R\Gamma R\Gamma S \subseteq S$.*

Definition 1.9. *Let R be a TGS and $\phi \neq S \subseteq R$. The set S is said to be a k-ideal of R, if S is a $(TGSSR)$ of R and $S\Gamma R\Gamma R \subseteq S, R\Gamma S\Gamma R \subseteq S, R\Gamma R\Gamma S \subseteq S$ and $p \in R$, $p + q \in S$, $q \in S$ then $p \in S$.*

Definition 1.10. *Let R be a TGS and $\phi \neq S \subseteq R$. The set P is said to be a bi-interior-ideal (BII) of R, if P is a $(TGSSR)$ of R and $(R\Gamma R\Gamma P\Gamma R\Gamma R) \cap (P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P$.*

Definition 1.11. *A TGSSR R is said to be $left(lateral, right) simple TGSR$, if R has no proper $left(lateral, right)$ ideal of R. A TGSSR R is said to be $simple TGSR$, if R has no proper ideals. A TGSSR R is said to be a $bi-quasi-simple$ TGSSR, if R has no proper bi-quasi-ideals of R.*

Example 3. *Consider the Tsemiring R= Γ=$M_{2\times2}(W)$ where $W = 0, 1, 2, 3, ...$ Then R is a TG-semi ring with $P\alpha Q\beta S$ is the ordinary ternary multiplication of matrices, $\forall P, \alpha, Q, \beta, S \in R$.*

$$U = \left\{ \begin{pmatrix} x & p \\ 0 & q \end{pmatrix} : p, q \in W \right\}$$

is a bi-ideal of R. Also

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & p \end{pmatrix} : p \in W \right\}$$

is a bi-ideal of R.

2. Bi ideals, interior ideals, bi interior ideals of TGSR

Throughout this paper R is a commutative TGSR with unity element.

Definition 2.1. *A non-empty subset P of a TGSR R is said to be bi-interior ideal of R, if P is a ternary Γ sub semi ring of R and $(R\Gamma R\Gamma P\Gamma R\Gamma R) \cap (P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P$.*

Definition 2.2. *A TGSR R is called bi interior simple TGSR if R has no bi interior ideal other than R itself.*

Theorem 2.1. *Let R be a TGSR. Then the following are hold:*

(1) *Every left $(right, lateral)$ ideal is a BII of R.*
(2) *Every QI is a BII of R.*
(3) *If A, B and C are bi-interior ideals of R, then $A\Gamma B\Gamma C, B\Gamma C\Gamma A, C\Gamma A\Gamma B$ are BIIs of R.*
(4) *Every ideal is a BII of R.*
(5) *If P is a BII of R then $P\Gamma R\Gamma R, R\Gamma P\Gamma R and R\Gamma R\Gamma p$ are BIIs of R.*

Theorem 2.2. *Every BI of a TGSR R is a BII of R.*

Theorem 2.3. *Every interior ideal of a TGSR R is a BII of R.*

Theorem 2.4. *Let R be a simple TGSR. Every BII of R is a BI of R.*

Proof. Given R is a simple TGSR. Suppose P be a BII of R then $(R\Gamma R\Gamma P\Gamma R\Gamma R) \cap (P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P$. Since $(R\Gamma R\Gamma P\Gamma R\Gamma R)$ is and R is a simple TGSR, we have $(R\Gamma R\Gamma P\Gamma R\Gamma R) = R$. Since $(R\Gamma R\Gamma P\Gamma R\Gamma R) \cap (P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P \Rightarrow ((P\Gamma R\Gamma P\Gamma R\Gamma P) \cap R \subseteq P \Rightarrow ((P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P$. Hence P is a BI of R. ☐

Theorem 2.5. *Let R be a TGSR. Then R is a bi-interior simple TGSR $\Leftrightarrow (R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma) \cap (a\Gamma R\Gamma a\Gamma R\Gamma a) = R, \forall a$ in R*

Proof. Given R is a TGSR. Suppose R is a bi-interior simple TGSR a in R. Since R is a BII of R, we have $(R\Gamma R\Gamma a\Gamma R\Gamma R)\cap(a\Gamma R\Gamma a\Gamma R\Gamma a) \subseteq R$. Let a be a in $R \Rightarrow a \in R\Gamma R\Gamma a\Gamma R\Gamma R$ and $a \in a\Gamma R\Gamma R\Gamma a \Rightarrow a \in (R\Gamma R\Gamma a\Gamma R\Gamma R)\cap a\Gamma R\Gamma a\Gamma R\Gamma a$. Hence $(R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma) \cap (a\Gamma R\Gamma a\Gamma R\Gamma a) = R$.

Conversely suppose that $(R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma)\cap(a\Gamma R\Gamma a\Gamma R\Gamma a) = R, \forall a$ in R. Let P be a BII of the TGSR R and $a \in P$. Then $R = (R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma)\cap(a\Gamma R\Gamma a\Gamma R\Gamma a) \subseteq (R\Gamma R\Gamma P\Gamma R\Gamma R)\cap P\Gamma R\Gamma P\Gamma R\Gamma P\subseteq P$. Therefore $R = P$. Thus R is a bi-interior simple TGSR. ☐

Theorem 2.6. *If D is a minimal left ideal, A is a minimal right ideal and C is a lateral ideal of a TGSR R, then $P = A\Gamma C\Gamma D$ is a minimal BII of R.*

Proof. Clearly $P = A\Gamma C\Gamma D$ is a BII of S. It is enough if we show P is a minimal BII of R. Let E be a BII of the TGSR S such that $E \subseteq P$.

$R\Gamma R\Gamma E \subseteq R\Gamma R\Gamma P = R\Gamma R\Gamma(A\Gamma C\Gamma D) \subseteq D$, since D is a right ideal of R. Similarly it is easy to prove that $E\Gamma R\Gamma R \subseteq A$ and $R\Gamma C\Gamma R \subseteq C$. Therefore $E\Gamma R\Gamma R = A$, $R\Gamma C\Gamma R = C$ and $R\Gamma R\Gamma E = D$. Hence $P = A\Gamma C\Gamma D = (E\Gamma R\Gamma R)\Gamma(R\Gamma C\Gamma R)\Gamma(R\Gamma R\Gamma E) \subseteq A\Gamma R\Gamma R\Gamma A \subseteq R\Gamma R\Gamma A = R\Gamma R\Gamma E\Gamma R\Gamma R$ and $P = A\Gamma C\Gamma D \subseteq A\Gamma C\Gamma(R\Gamma R\Gamma E) \subseteq R\Gamma R\Gamma E \subseteq E\Gamma R\Gamma R\Gamma E$. Hence $P \subseteq (E\Gamma R\Gamma R\Gamma E) \cap (R\Gamma R\Gamma E\Gamma R\Gamma R\Gamma) \subseteq E$. Thus $P = E$. Therefore P is a minimal BII of R. □

Theorem 2.7. *The intersection of a BII P of a TGSR R and a TGSSR Q of R is a BII of R.*

Theorem 2.8. *Let A, C and D be TGSSRs of a TGSR R and $P = A\Gamma D\Gamma C$. If A is a left ideal, then P is BII of R.*

Proof. Suppose A, C and D be TGSSRs of a TGSR R, $P = A\Gamma D\Gamma C$ and A is a left ideal of TGSR R.

Consider $P\Gamma R\Gamma P\Gamma R\Gamma P = (A\Gamma D\Gamma C)\Gamma R\Gamma \; (A\Gamma D\Gamma C)\Gamma R\Gamma(A\Gamma D\Gamma C) \subseteq (A\Gamma D\Gamma C)\Gamma(A\Gamma D\Gamma C)\Gamma(A\Gamma D\Gamma C) \subseteq A\Gamma D\Gamma C = P\Gamma \; (P\Gamma R\Gamma P\Gamma R\Gamma P) \cap R\Gamma R\Gamma P\Gamma R\Gamma R \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P \subseteq P$. Hence P is a BII of R. □

Remark 2.1. *Let A, C and D be TGSSRs of a TGSR R and $P = A\Gamma D\Gamma C$. If C is a right ideal, then P is a BII ideal of R.*

Remark 2.2. *Let A, C and D be TGSSRs of a TGSR R and $P = A\Gamma D\Gamma C$. If D is a lateral ideal, then P is a BII of R.*

Theorem 2.9. *Let R be a TGSSR and T be a TGSSR of R. Every TGSSR of T containing $(T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R)$ is a BII of R.*

Proof. Let P be a TGSSR of T containing $(T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R)$. Now we show that P is a BII of R. Consider $(P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq (T\Gamma R\Gamma T\Gamma R\Gamma T) \subseteq (T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P$.

Hence $(T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P$. Thus P is a BII of R. □

Definition 2.3. *Let R be a TGS. An element $p \in R$ is said to be an $regularelement$ if there exists $x, y \in Randa, b, c \in \Gamma$ such that $p = paxbpcydp$. Every element in TGS is an regular element then R is a known as a Regular TGS.*

Theorem 2.10. *Let R be a regular TGSR. Then every BII of R is an ideal of R.*

Proof. Given R is a regular element. Let us suppose P be BII of R. Now we show that P is an ideal of R. Since P is an II of R, we have $(T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P$. Consider $P\Gamma R\Gamma R \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P$ and $P\Gamma R\Gamma R \subseteq R\Gamma R\Gamma P\Gamma R\Gamma R \Rightarrow P\Gamma R\Gamma R \subseteq T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P$. Similarly, it is easy prove that $R\Gamma R\Gamma P \subseteq P\Gamma R\Gamma R \subseteq T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P$ and $R\Gamma P\Gamma R \subseteq P\Gamma R\Gamma R \subseteq T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P$. Hence P is an ideal of R. □

Theorem 2.11. *Let R be a TGSR. Prove that the following statements are equivalent:*

(1) *R is a bi-interior simple TGSR.*
(2) *$R\Gamma R\Gamma a = R, \forall a \in R$.*
(3) *$< a >= R, \forall a \in R$ and where $< a >$ is the smallest bi-interior ideal generated by a.*

Proof. Given R is a TGSR. To show $(1) \Rightarrow (2)$: Suppose R is a bi-interior simple TGSR and $a \in R$ and $P = R\Gamma R\Gamma a \Rightarrow P$ is a left ideal of R. By theorem 3.4, P is a BII of R. Clearly $P \subseteq R$ and let $x \in R \Rightarrow x = x\alpha x\beta a \in R\Gamma R\Gamma a \Rightarrow R \subseteq P$ therefore $P = R$. Hence $R\Gamma R\Gamma a = R, \forall a \in R$. To show $(2) \Rightarrow (3)$: Suppose $R\Gamma R\Gamma a = R, \forall a \in R$. Consider $R\Gamma R\Gamma a \subseteq < a > \subseteq R$ and $R \subseteq < a > \subseteq R$. Therefore $< a >= R$. To show $(3) \Rightarrow (1)$: Suppose $< a >$ is the smallest BII generated by a, $< a >= R, \forall a \in R$. Let P be a BII and $a \in P$ then $< a > \subseteq P \subseteq R \Rightarrow R \subseteq P \subseteq R$. Therefore, $P = R$. Hence R is a bi-interior simple TGSR. □

Theorem 2.12. *If P is a BII of a TGSSR R, T is a TGSSR of R and $T \subseteq P$ such that $P\Gamma T\Gamma T$ is a ternary sub-semi-group of the ternary semi-group $(R, .)$, then $P\Gamma T\Gamma T$ is a BII of R.*

Proof. Let P be a BII of TGSR R, T be a TGSSR of R and $T \subseteq$ such that $P\Gamma T\Gamma T$ is a ternary sub-semigroup of the ternary semi-group $(R, \cdot)$. Now we show that $P\Gamma T\Gamma T$ is a BII of R. Clearly $(P\Gamma T\Gamma T)\Gamma P \subseteq R\Gamma P\Gamma R \subseteq P\Gamma R\Gamma R \subseteq T\Gamma R\Gamma T\Gamma R\Gamma T) \cup (R\Gamma R\Gamma T\Gamma R\Gamma R) \subseteq P \Rightarrow P\Gamma T\Gamma T\Gamma P\Gamma T\Gamma T \subseteq P\Gamma T\Gamma T$.

Hence $P\Gamma T\Gamma T$ is a TGSSR of R.

Also, $R\Gamma R\Gamma P\Gamma T\Gamma T\Gamma R\Gamma R \subseteq R\Gamma R\Gamma T\Gamma R\Gamma R$ and $(P\Gamma T\Gamma T)\Gamma R\Gamma R\Gamma(P\Gamma T\Gamma T) \subseteq P\Gamma R\Gamma R\Gamma R\Gamma P \Rightarrow (R\Gamma R\Gamma P\Gamma T\Gamma T\Gamma R\Gamma R)\cap$ $(P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq (R\Gamma R\Gamma P) \cap (P\Gamma R\Gamma R) \subseteq P$.

Hence $P\Gamma T\Gamma T$ is a BII of the TGSR of R. □

Theorem 2.13. *Let P be a BI of a TGSR R and Q be an interior ideal of R. Then $P \cap Q$ is a BII of R.*

Proof. Suppose P be a BI of a TGSSR R and Q be an interior ideal of R. Now we show that $P\cap Q$ is a BII of R. By the known theorem, $P\cap Q$ is a TGSSR of R. Also, $(P\cap Q)\Gamma R\Gamma(P\cap Q)\Gamma R\Gamma(P\cap Q) \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P \subseteq P$ and $R\Gamma R\Gamma(P\cap Q)\Gamma R\Gamma R \subseteq R\Gamma R\Gamma Q\Gamma R\Gamma R \subseteq Q$.

Here $(P \cap Q)\Gamma R\Gamma(P \cap Q)\Gamma R\Gamma(P \cap Q)(R\Gamma R\Gamma(P \cap Q)\Gamma R\Gamma R) \subseteq (P \cap Q)$.

Hence $P \cap Q$ is a BII of R. □

Theorem 2.14. *Let R be a TGSR. If $R = R\Gamma R\Gamma a, \forall a \in R$. Then every BII of R is a QI of R.*

Theorem 2.15. *If P is a minimal BII of a TGSSR R, then any two non-zero elements of P generate the same right (left, lateral) ideal of R.*

Theorem 2.16. *Let R be a regular TGSR. Then P is a BII of R $\Leftrightarrow$ $(P\Gamma R\Gamma P\Gamma R\Gamma P)\cap$ $(R\Gamma R\Gamma P\Gamma R\Gamma R) = P, \forall BIIsPofR$.*

Proof. Given R is a regular TGSR. Suppose P be a BII of TGSR R and $a \in P$. Now we show that $(P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) = P$. Since P is a BII of TGSR R, we have $(P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq P$. Also, $R\Gamma R\Gamma a \subseteq R\Gamma R\Gamma P$. Let $a \in P$ then a is a regular element, because R is a regular TGSR $\Rightarrow \exists x, y \in R, \alpha, \beta, \gamma \in \Gamma \ni a = a\alpha x\beta y\gamma a \in P\Gamma R\Gamma P\Gamma R\Gamma P \Rightarrow a \in P\Gamma R\Gamma P\Gamma R\Gamma P) \cap$ $(R\Gamma R\Gamma P\Gamma R\Gamma R) \Rightarrow P \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P)\cap(R\Gamma R\Gamma P\Gamma R\Gamma R)$. Hence $P\Gamma R\Gamma P\Gamma R\Gamma P)\cap$ $(R\Gamma R\Gamma P\Gamma R\Gamma R) = P$.

Conversely, assume that $P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) = P$, for all $BIIsPofR$. Clearly $P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq P$, we have P is a BII of R. □

Theorem 2.17. *Let R be a TGSR. If P is a BII of R and P is a regular TGSSR of R, then any BII of P is a BII of R.*

Proof. Given R is a TGSR. Let P be a BII of R and it is a regular TGSSR of R. Suppose Q be a BII of P. Now we show that Q is a BII of R. By the theorem 3.21, $(Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) = Q$. Since $Q \subseteq P$ and $P \subseteq R$, we have $(P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq (Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) = Q \subseteq P \Rightarrow (P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq P$. $\Rightarrow (Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) = Q \subseteq P$ and $(Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) \subseteq P \Rightarrow \{(Q\Gamma R\Gamma Q\Gamma R\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P)\} \cap \{(P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R)\} \subseteq (Q \cap P) \Rightarrow (Q\Gamma R\Gamma Q\Gamma R\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) \subseteq (Q \cap P) \subseteq Q \Rightarrow Q$ is a BII of R. Hence the theorem. □

Theorem 2.18. *Let R be a TGSR. Then R is a bi-interior simple TGSR, if, and only if,*

$$(R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma)\ (a\Gamma R\Gamma a\Gamma R\Gamma a) = R, \forall a \in R.$$

Theorem 2.19. *Let R be a TGSR and P be a BII of R. Then P is a minimal BII of $R \Leftrightarrow P$ is a bi-interior simple TGSr of R.*

Proof. Given R is a TGSR and P is a BII of R. Let Q be a BII of P. Now we show that P is a bi-interior simple TGSR of R. Since Q is a BIs of P, we have $(Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) \subseteq Q \Rightarrow (Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P)$ is a BII of R. Since P is a minimal BII of R, we have $(Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) = P \Rightarrow P = (Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) \subseteq Q \Rightarrow P = Q$. Hence P is a minimal BII of R.

Conversely suppose that P is a bi-interior simple TGSR of R. Now we show that P is a minimal BII of R. Let Q be a BII of R and $Q \subseteq P$. It is enough to show $P = Q$. Here $(Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) \subseteq (Q\Gamma R\Gamma Q\Gamma R\Gamma Q) \cap R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq Q$, because Q is a BII of R. $(Q\Gamma R\Gamma Q\Gamma R\Gamma Q) \cap (R\Gamma R\Gamma Q\Gamma R\Gamma R) \subseteq (P\Gamma R\Gamma Q\Gamma R\Gamma Q) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq P \Rightarrow (Q\Gamma P\Gamma Q\Gamma P\Gamma Q) \cap (P\Gamma P\Gamma Q\Gamma P\Gamma P) \subseteq (P \cap Q) \subseteq Q \Rightarrow P = Q$. Hence P is a minimal BII of R. □

Theorem 2.20. *The intersection of BIIs $\{P_i : i \in \Delta\}$ of a TGSR is a BII of R.*

Proof. Let $P = \cap_{i\in\Delta} P_i$, where P_i is a BII of R. Now we show that P is a BII of R. By the known theorem, P is a TGSSR R. Since P_i is a BII of R, we have $(R\Gamma R\Gamma P_i\Gamma R\Gamma R) \cap (P_i\Gamma R\Gamma P_i\Gamma R\Gamma P_i) \subseteq P_i$, for each P_i and $i \in \Delta \Rightarrow (R\Gamma R\Gamma \cap P_i\Gamma R\Gamma R) \cap (\cap P_i\Gamma R\Gamma \cap P_i\Gamma R\Gamma \cap P_i) \subseteq \cap P_i \Rightarrow ((P\Gamma R\Gamma P\Gamma R\Gamma P) \cap R \subseteq P \Rightarrow ((P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P$. Hence P is a BII of R. □

Definition 2.4. *An element ita of a TGSR R is said to be α-idempotent element, if $a = a\alpha a\alpha a$. A ternary TGSR R is said to be $\alpha - idempotent$ TGSR, if every element of R is $\alpha - idempotent$. An element a of a TGSR R is said to be $(\alpha, \beta) - idempotent$ element, if $a = a\alpha a\beta a$.*

Theorem 2.21. *Let P be a BII of a TGSR R, e be $(\alpha, \beta) - idempotent$ element and $e\Gamma e\Gamma P \subseteq P$. Then $e\Gamma e\Gamma P$ is a BII of R.*

Proof. Given R is a BII of a TGSR R. Suppose $a \in P \cap (e\Gamma R\Gamma R) \Rightarrow a \in P$ and $a = e\alpha y\beta z$, where $\alpha, \beta \in \Gamma$ and $y, z \in R$. Consider $a = e\alpha y\beta z = (e\gamma e\delta e)\alpha y\beta z = (e\gamma e\delta)e\alpha y\beta z = e\gamma e\delta a \in e\Gamma e\Gamma P$. Therefore $P \cap (e\Gamma R\Gamma R \subseteq e\Gamma e\Gamma P \subseteq P)$ and $e\Gamma e\Gamma P \subseteq e\Gamma R\Gamma R$ then $e\Gamma e\Gamma P = e\Gamma R\Gamma R$. Hence $e\Gamma R\Gamma R$ is a BII of R. ☐

Theorem 2.22. *Let R be a TGSR and e be a α-idempotent. Then $e\Gamma R\Gamma R, R\Gamma e\Gamma R$ and $R\Gamma R\Gamma e$ are BIIs of R.*

Theorem 2.23. *Let e and f be a α-idempotent and a β-idempotent of TGSSR R respectively. Then, $e\Gamma R\Gamma R\Gamma R\Gamma f$ is a BII of R.*

Theorem 2.24. *Let P be a TGSSR of a regular TGSR R. Then P can be expressed as $P = K\Gamma M\Gamma L$, where K is a right ideal, M is a lateral ideal and L is a left ideal of $R \Leftrightarrow P$ is a BII of R.*

Proof. Given P is a TGSSR of a regular TGSR R. Suppose $P = K\Gamma M\Gamma L$, where K is a right ideal, M is a lateral ideal and L is a left ideal of R. Now we show that P is a BII of R. Consider $P\Gamma R\Gamma P\Gamma R\Gamma P = (K\Gamma M\Gamma L)\Gamma R\Gamma R\Gamma(K\Gamma M\Gamma L) \subseteq (K\Gamma M\Gamma L) = P$.

Consider $(R\Gamma R\Gamma P\Gamma T\Gamma T\Gamma R\Gamma R) \cap (P\Gamma R\Gamma P\Gamma R\Gamma P) \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P \subseteq K\Gamma M\Gamma L = P$. Hence P is a BII of R.

Conversely, suppose that P is a BII of R. Now we show that P can be expressed as $P = K\Gamma M\Gamma L$, where K is a right ideal, M is a lateral ideal and L is a left ideal of R. Since P is a BII of R by the known Theorem 3.22, $(R\Gamma R\Gamma P\Gamma T\Gamma T\Gamma R\Gamma R) \cap (P\Gamma R\Gamma P\Gamma R\Gamma P) = P$. Let us take $K = e\Gamma R\Gamma R, L = R\Gamma R\Gamma e$ and $M = R\Gamma e\Gamma R$, where e is the identity element of $R \Rightarrow K = P\Gamma R\Gamma R$ is a right ideal of R, $L = R\Gamma R\Gamma P$ is a left ideal of R and $M = R\Gamma P\Gamma R$ is a lateral ideal of R. Consider $(P\Gamma R\Gamma R) \cap (R\Gamma P\Gamma R) \cap (R\Gamma R\Gamma P) \subseteq (P\Gamma R\Gamma P\Gamma R\Gamma P) \cap ((P\Gamma R\Gamma P\Gamma R\Gamma P) = P \Rightarrow (P\Gamma R\Gamma R) \cap (R\Gamma P\Gamma R) \cap (R\Gamma R\Gamma P) \subseteq P \Rightarrow K \cap M \cap L \subseteq P$. Also, $P \subseteq P\Gamma R\Gamma R = R, P \subseteq R\Gamma P\Gamma R = P, P \subseteq R\Gamma R\Gamma P = P \Rightarrow P \subseteq$

$K \cap M \cap L \Rightarrow P = K \cap M \cap L = K\Gamma M\Gamma L$, because R is a regular TGSR. Hence P can be expressed as $P = K\Gamma M\Gamma L$, where K is a right ideal, M is a lateral ideal and L is a left ideal of R. □

Theorem 2.25. *Let R be a TGSR. Then R is regular TGSR $\Leftrightarrow P \cap I \cap L \subseteq P\Gamma I\Gamma L$, for any BII P lateral ideal I and ideal L of R.*

Proof. Given R is a TGSR. Suppose R is a regular TGSR. Now we show that $P \cap I \cap L \subseteq P\Gamma I\Gamma L$, for any BII P lateral ideal I and ideal L of R. Let $x \in P \cap I \cap L \Rightarrow a \in R$ and since R is a regular TGSR, we have $a = a\alpha x\beta a\gamma y\delta a$, where $x, y \in R$ and $alpha, \beta, \gamma, \delta \in \Gamma \Rightarrow a \in a\Gamma R\Gamma R\Gamma a \subseteq a\Gamma R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma a\Gamma R\Gamma R\Gamma a \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P$.

Also $a \in a\mathit{\Gamma} R\mathit{\Gamma} R\mathit{\Gamma} a \subseteq a \in a\mathit{\Gamma} R\mathit{\Gamma} R\mathit{\Gamma} a \subseteq a\mathit{\Gamma} R\mathit{\Gamma} R\mathit{\Gamma} a\mathit{\Gamma} R\mathit{\Gamma} R\mathit{\Gamma} a\mathit{\Gamma} R\mathit{\Gamma} R\mathit{\Gamma} a \subseteq R\mathit{\Gamma} R\mathit{\Gamma} P\mathit{\Gamma} R\mathit{\Gamma} R \Rightarrow a \in (P\mathit{\Gamma} R\mathit{\Gamma} P\mathit{\Gamma} R\mathit{\Gamma} P) \cap (R\mathit{\Gamma} R\mathit{\Gamma} P\mathit{\Gamma} R\mathit{\Gamma} R) = P \Rightarrow P \cap I \cap L \subseteq P$.

Conversely, assume that $P \cap I \cap L \subseteq P\Gamma I\Gamma L$, for any BII P, ideal I and left ideal L of R. Now we show that R is a regular TGSR. Let K be a right ideal, M be a lateral ideal and L be a left ideal of R. Then by our assumption, $K \cap M \cap L \subseteq K \cap R \cap L \subseteq K\Gamma R\Gamma L \subseteq K\Gamma M\Gamma L$, we have $K\Gamma M\Gamma L \subseteq K, K\Gamma M\Gamma L \subseteq M$ and $K\Gamma M\Gamma L \subseteq L \Rightarrow K\Gamma M\Gamma L \subseteq K \cap M \cap L$.

Hence, $K\Gamma M\Gamma L = K \cap M \cap L$.

Therefore, R is a regular TGSR. □

Theorem 2.26. *If TGSR R is a left (lateral, right) simple TGSR, then every BII of R is a right (lateral, left) ideal of R.*

Proof. Let P be a BII of the left simple TGSR R. Then $R\Gamma R\Gamma P$ is a left ideal of R and $R\Gamma R\Gamma P \subseteq R$ and clearly $R \subseteq R\Gamma R\Gamma P$. Then $R = R\Gamma R\Gamma P$. $R\Gamma R\Gamma P\Gamma R\Gamma R = R\Gamma R\Gamma R \subseteq R$ and $P\Gamma R\Gamma P\Gamma R\Gamma P \subseteq P\Gamma R\Gamma R$; $(P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) = R \cap (P\Gamma R\Gamma R)$.

Also, $P\Gamma R\Gamma R \subseteq P\Gamma R\Gamma P\Gamma R\Gamma P \subseteq (P\Gamma R\Gamma P\Gamma R\Gamma P) \cap (R\Gamma R\Gamma P\Gamma R\Gamma R) \subseteq P$. Hence, every BII is a right ideal of R.

Similarly, we can prove for the right simple TGSR R.

The proof is completed. □

Theorem 2.27. *Let P be a TGSSR of a TGSR R. If P is a BII of R, then P is a left bi-quasi ideal of R.*

Theorem 2.28. *Let P be a TGSSR of a TGSR R. If P is a BII of R, then P is a right (lateral) of R.*

Theorem 2.29. *Let P be a BII of R and Q be a non-empty subset of P such that $P\Gamma Q\Gamma Q$ is a TGSSR of R, then $P\Gamma Q\Gamma Q$ is a BII of R.*

Definition 2.5. *An element a of a ternary semi ring R is said to be invertible in R, if there exists an element $b \in R$ (called the ternary semi ring inverse of a) such that $abt = bat = tab = tba = atb = bta = t, \forall t \in R$. An element a of a ternary gamma semi ring R is said to be invertible in R, if there exists an element $b \in R$ (called the ternary gamma semi ring inverse of a) such that $a\alpha b\beta t = b\alpha a\beta t = t\alpha a\beta b = t\alpha b\beta a = a\alpha t\beta b = b\alpha t\beta a = t, \forall t \in R, \alpha, \beta \in \Gamma$.*

Definition 2.6. *A ternary gamma semi ring R with $\mid R \mid\geq 2$ is said to be a ternary division gamma semi ring, if every non-zero element of R is invertible.*

Theorem 2.30. *Every ternary division gamma semi ring is a regular ternary gamma semi ring.*

Definition 2.7. *A commutative ternary division gamma semi ring R is said to be a ternary gamma semi field i.e., a commutative ternary semi ring R with $\mid R \mid\geq 2$, is a ternary gamma semi field, if for every non-zero element a of R, there exists an element $b \in R, \alpha, \beta \in \Gamma$ such that $a\alpha b\beta x = x, \forall x \in R$.*

Remark 2.3. *A ternary gamma semi field (TGSF) R has always an identity.*

Theorem 2.31. *If R is a field TGSR, then R is a bi-interior simple TGSR.*

Proof. Let P be a proper BII of the TGSF R, $t \in P$ and $0 \neq a \in P$. Since R is a TGSF, $\exists b \in R$ and $\alpha, \beta \in \Gamma \ni a\alpha b\beta t = b\alpha a\beta t = t\alpha a\beta b = t\alpha b\beta a = a\alpha t\beta b = b\alpha t\beta a = t, \forall t \in R \Rightarrow \gamma, \delta \in \Gamma \ni t = a\gamma b\delta t = a\gamma b\delta(a\alpha b\beta t) \Rightarrow t \in P\Gamma R\Gamma R \Rightarrow R \subseteq p\Gamma R\Gamma R$ and clearly $P\Gamma R\Gamma R \subseteq R \Rightarrow R = P\Gamma R\Gamma R$.

Similarly, it is easy to prove that $R = R\Gamma R\Gamma P = R\Gamma P\Gamma R$. Consider $R = P\Gamma R\Gamma R = R\Gamma P\Gamma R = R\Gamma R\Gamma P \subseteq P \Rightarrow R \subseteq P$ and since $P \subseteq R$, we have $P = R$. Hence TGSF R is a bi-interior simple TGSR.

Hence the theorem. □

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