ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.3, 1197–1213
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.3.9

ON ATOM-BOND CONNECTIVITY STATUS INDEX OF GRAPHS

D. S. Revankar[1], Priyanka S. Hande, and S. P. Hande

ABSTRACT. The atom-bond connectivity (ABC) status index of a graph is defined by V. R. Kulli as $ABCS(G)= \sum_{uv\in E(G)} \sqrt{(\sigma_u + \sigma_v - 2)/\sigma_u\sigma_v}$, where σ_u is a status of a vertex $u \in V(G)$ and is defined as the sum of its distance from every other vertex in $V(G)$. In this paper we have obtained the bounds for the atom-bond connectivity status index. Also obtained atom-bond connectivity status index of some graphs.

1. INTRODUCTION

A topological index is a molecular structure descriptor having many applications in rationalizing the stability of linear and branched alkanes as well as the strain energy of cycloalkanes. It is a numeric numerical quantity calculated mathematically of molecule obtained from its structural graph. Estrada et.al. [12] has modified the Randić connectivity index [11] and proposed a new topological index named atom–bond connectivity (ABC) index. The atom–bond connectivity (ABC) index is widely studied [2, 4–8, 10, 12] and for a connected

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 05C05, 05C12, 05C35, 05C90.
Key words and phrases. Atom-bond connectivity status index, Distance, Status, diameter.
Submitted: 30.12.2020; *Accepted:* 14.01.2021; *Published:* 15.03.2021.

graph G it is defined as,

$$ABC\,(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

Where d_u is the degree of vertex $u \in V(G)$.

Status [9] of a vertex $u \in V(G)$ is denoted by σ_u and is defined by the sum of its distance from every other vertex in $V(G)$.

Harmonic status index [3] is defined by H.S. Ramane et. al. as

$$HS\,(G) = \sum_{uv\in E(G)} \frac{2}{\sigma_u + \sigma_v}.$$

Here σ_u is the status of vertex u of G, $E(G)$ is the edge set. V. R. Kulli defined atom-bond connectivity status index [2] of G as,

$$ABCS\,(G) = \sum_{uv\in E(G)} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u \sigma_v}}.$$

2. Preliminary results

Theorem 2.1. *[2] For a complete graph K_n with n vertices,*

$$ABCS\,(K_n) = \frac{n}{\sqrt{2}}\sqrt{(n-2)}.$$

Theorem 2.2. *[2] For a complete bipartite graph $K_{p,q}$ with $p+q$ vertices and pq edges,*

$$ABCS\,(K_{p,\,q}) = pq \times \sqrt{\frac{3\,(p+q) - 6}{2\,(p^2+q^2) - 6\,(P+q) + (5pq+4)}}.$$

Theorem 2.3. *[2] For a cycle C_n with n vertices and n edges,*

$$ABCS\,(C_n) = \begin{cases} \frac{2\left(\sqrt{2(n^2-4}\right)}{n}, & \text{if n is even} \\ \frac{2n\sqrt{2(n^2-5)}}{n^2-1}, & \text{if n is odd} \end{cases}.$$

Theorem 2.4. *[2] For a wheel graph W_n with $n+1$ vertices and $2n$ edges,*

$$ABCS\,(W_n) = \frac{2n\sqrt{n-2}}{(2n-3)} + \sqrt{\frac{2n\,(3n-2)}{(2n-3)}}.$$

Theorem 2.5. *[2] For a friendship graph F_n with $2n+1$ vertices and $3n$ edges,*

$$ABCS\left(F_n\right)=\frac{n\sqrt{8n-6}}{(4n-2)}+\sqrt{\frac{n\left(3n-5\right)}{(2n-1)}}.$$

3. Obtained bounds for the atom-bond connectivity status index

Theorem 3.1. *If G is a connected graph having n vertices and let D be the diameter of G then,*

$$\sum_{uv\in E(G)}\sqrt{\frac{2D\left(n-1\right)-\left(D-1\right)\left[d\left(u\right)+d\left(v\right)\right]-2}{D^2\left(n-1\right)^2-D\left(n-1\right)\left[d\left(u\right)+d\left(v\right)\right]\left(D-1\right)+d\left(u\right).d\left(v\right)\left(D-1\right)^2}}$$
$$\leq ABCS\left(G\right)\leq\sum_{uv\in E(G)}\sqrt{\frac{4n-6-\left[d\left(u\right)+d\left(v\right)\right]}{\left(2n-2-d\left(u\right)\right).\ \left(2n-2-d\left(v\right)\right)}}.$$

Equality holds if and only if $diam(G)\leq 2$.

Proof.

Lower Bound: For a vertex $u\in V\left(G\right)$ of a graph $G, d(u)$ vertices are at distance 1 from u. Then the remaining vertices are $\left[n-1-d\left(u\right)\right]$ which are of at most diameter *D* from *u*, and

$$\sigma\left(u\right)\leq d\left(u\right)+D\left(n-1-d\left(u\right)\right)=D\left(n-1\right)-\left(D-1\right)d\left(u\right)$$
$$\left[\sigma\left(u\right)+\sigma\left(v\right)\right]\leq 2D\left(n-1\right)-\left(D-1\right)\left[d\left(u\right)+d\left(v\right)\right]$$
$$\sigma\left(u\right).\sigma\left(v\right)\leq\left[D\left(n-1\right)-\left(D-1\right)d\left(u\right)\right]\cdot\left[D\left(n-1\right)-\left(D-1\right)d\left(v\right)\right].$$

Therefore,

$$ABCS\left(G\right)=\sum_{uv\in E(G)}\sqrt{\frac{\sigma_u+\sigma_v-2}{\sigma_u\sigma_v}}$$
$$\geq\sum_{uv\in E(G)}\sqrt{\frac{2D\left(n-1\right)-\left(D-1\right)\left[d\left(u\right)+d\left(v\right)\right]-2}{D^2\left(n-1\right)^2-D\left(n-1\right)\left[d\left(u\right)+d\left(v\right)\right]\left(D-1\right)+d\left(u\right).d\left(v\right)\left(D-1\right)^2}}.$$

Upper Bound: Out of n vertices for $u\in V(G), d(u)$ vertices are at distance 1 from u and the remaining $\left[n-1-d\left(u\right)\right]$ vertices are at the distance 2.

$$\sigma\left(u\right)\geq d\left(u\right)+2\left(n-1-d\left(u\right)\right)=2n-2-d\left(u\right)$$
$$\sigma\left(v\right)\geq d\left(v\right)+2\left(n-1-d\left(v\right)\right)=2n-2-d\left(v\right)$$

Therefore,

$$ABCS\,(G) \leq \sum_{uv\in E(G)} \sqrt{\frac{(4n-4) - [d\,(u) + d\,(v)] - 2}{(2n-2-d\,(u))\,.\,\,(2n-2-d\,(v))}}.$$

Hence,
(3.1)

$$\sum_{uv\in E(G)} \sqrt{\frac{2D\,(n-1) - (D-1)\,[d\,(u) + d\,(v)] - 2}{D^2\,(n-1)^2 - D\,(n-1)\,[d\,(u) + d\,(v)]\,(D-1) + d\,(u)\,.d\,(v)\,(D-1)^2}}$$

$$\leq ABCS\,(G) \leq \sum_{uv\in E(G)} \sqrt{\frac{4n-6-[d\,(u) + d\,(v)]}{(2n-2-d\,(u))\,.\,\,(2n-2-d\,(v))}}.$$

Equality holds when the diameter D is 1 or 2.

Conversely, let $ABCS\,(G) = \sum_{uv\in E(G)} \sqrt{\frac{4n-6-[d(u)+d(v)]}{(2n-2-d(u)).\,(2n-2-d(v))}}$. Suppose $D \geq 3$ therefore there exist at least one pair vertices *u* and *v* such that $d\,(u,v) \geq 3$. Therefore, $\sigma\,(u) \geq d\,(u) + 3 + 2\,(n-2-d\,(u)) = 2n-1-d\,(u)$. Hence,

$$ABCS\,(G) \leq \sum_{uv\in E(G)} \sqrt{\frac{4n-2-[d\,(u) + d\,(v)]}{(2n-1-d\,(u))\,.\,\,(2n-1-d\,(v))}}$$

$$< \sum_{uv\in E(G)} \sqrt{\frac{4n-6-[d\,(u) + d\,(v)]}{(2n-2-d\,(u))\,.\,\,(2n-2-d\,(v))}}.$$

This is a contradiction. Therefore $diam(G) \leq 2$. □

Corollary 3.1. *Let G be a connected graph having n vertices and m edges and let D be the diameter of G. Let δ be the minimum and Δ be the maximum degree of the vertices of G, then*

$$m \cdot \sqrt{\frac{2D\,(n-1) - (D-1)\,.\,2\delta - 2}{D^2\,(n-1)^2 - 2D\delta\,(n-1)\,(D-1) + \delta^2\,(D-1)^2}}$$

$$\leq ABCS(G) \leq \sqrt{\frac{4n-6-2\Delta}{(2n-2-2\Delta)^2}}.$$

Proof. For any vertex $u \in V(G)$, $d(u) \geq \delta$ and $d(u) \leq \Delta$. Therefore substituting $[d(u) + d(v)] \geq 2\delta$ on LHS and $[d(u) + d(v)] \leq 2\Delta$ on the RHS of equation 3.1 we obtain the result. □

Corollary 3.2. *For a connected regular graph G of degree r having n vertices and m edges and $diam(G) = D$, then,*

$$m \cdot \sqrt{\frac{2D\left(n-1\right)-2r\left(D-1\right).\ -2}{D^2\left(n-1\right)^2-2Dr\left(n-1\right)\left(D-1\right)+r^2\left(D-1\right)^2}}$$
$$\leq ABCS(G) \leq \sqrt{\frac{4n-6-2r}{\left(2n-2-2r\right)^2}}.$$

Equality holds if and only if $diam(G) \leq 2$.

4. Atom-bond connectivity status index of some graphs

Here we have obtained $ABCS$ index of some graphs

Proposition 4.1. *Let $W_{(n+1)}$ is a wheel graph with $n \geq 3$. Then,*

$$ABCS\left(W_{(n+1)}\right) = n \times \left(\sqrt{\frac{(3n-5)}{n(2n-3)}} + \sqrt{\frac{(4n-8)}{(2n-3)^2}}\right).$$

Proof. We give alternate proof of Theorem 2.4. Partitioning the edge set of $W_{(n+1)}$ in to two sets E_1 and E_2 where, $E_1 = \{uv/d\left(u\right) = n$ and $d\left(v\right) = 3\}$ and E_2=$\{uv/d\left(u\right) = 3$ and $d\left(v\right) = 3\}$. Also, $diam\left(W_{n+1}\right)$ = 2,

$$ABCS\left(W_{n+1}\right) = \sum_{uv\in E_1(G)} \sqrt{\frac{4\left(n+1\right)-6-\left(n+3\right)}{\left[2\left(n+1\right)-2-n\right]\left[2\left(n+1\right)-2-3\right]}}$$
$$+ \sum_{uv\in E_2(G)} \sqrt{\frac{4\left(n+1\right)-6-\left(3+3\right)}{\left[2\left(n+1\right)-2-n\right]\left[\left[2\left(n+1\right)-2-3\right]\right.}}.$$

Thus, $ABCS\left(W_{\ n+1}\ \right) = n \times \left(\sqrt{\frac{3n-5}{n(2n-3)}} + \sqrt{\frac{4n-8}{(2n-3)^2}}\right)$. □

Proposition 4.2. *Let F_n, $n \geq 2$ be a Friendship graph. Then,*

$$ABCS\left(F_n\ \right) = \left(2n \times \sqrt{\frac{(3n-2)}{2n\left(2n-1\right)}}\right) + \left(n \times \sqrt{\frac{4n-3}{2\left(2n-1\right)^2}}\right).$$

Proof. We give alternate proof of Theorem 2.5.

Partitioning the edge set of F_n in to two sets E_1 and E_2 where, $E_1 = \{uv/d\left(u\right)$ $= 2n$ and $d\left(v\right) = 2\}$ and $E_2 = \{uv/d\left(u\right) = 2$ and $d\left(v\right) = 2\}$. Also, $|E_1| = 2n$

and $|E_2| = n$. Also, $diam\,(F_n) = 2$ and F_n has $2n+1$ vertices. Therefore, by the equality part of Theorem 3.1

$$\begin{aligned} ABCS\,(F_n) = & \sum_{uv\in E_1(G)} \sqrt{\frac{4\,(2n+1)-6-(2n+2)}{[2\,(2n+1)-2-2n]\,[2\,(2n+1)-2-2]}} \\ & + \sum_{uv\in E_2(G)} \sqrt{\frac{4\,(2n+1)-6-(2+2)}{[2\,(2n+1)-2-2]\,[2\,(2n+1)-2-2]}} \\ = & \sum_{uv\in E_1(G)} \sqrt{\frac{6n-4}{(2n)\,(4n-2)}} + \sum_{uv\in E_2(G)} \sqrt{\frac{8n-6}{(4n-2)^2}} \\ = & \sum_{uv\in E_1(G)} \sqrt{\frac{2\,(3n-2)}{4\,(n)\,(2n-1)}} + \sum_{uv\in E_2(G)} \sqrt{\frac{2\,(4n-3)}{4\,(2n-1)^2}}. \end{aligned}$$

Therefore $ABCS\,(F_n)$= $2n \times \sqrt{\frac{(3n-2)}{2n(2n-1)}}$+$n \times \sqrt{\frac{4n-3}{2(2n-1)^2}}$. □

Proposition 4.3. *For a path on n vertices,*

$$ABCS\,(P_n) = \sum_{i=1}^{n-1} \sqrt{\frac{(n-i)^2 + i^2 - 2}{\left[\frac{n^2+n}{2} + i\,(i-n-1)\right]\left[\frac{n^2+n}{2} + (i+1)\,(i-n)\right]}}.$$

Proof. Let v_1, v_2, v_3,..., v_n be the vertices, where v_i is adjacent to v_{i+1}, i = 1, 2, 3,..., $(n-1)$. Therefore, $\sigma\,(v_i)$=$(i-1)+(i-2)+\cdots+1+1+2+\cdots+(n-i)$ = $\left[\frac{n^2+n}{2} + i\,(i-n-1)\right]$ and $[\sigma\,(u)+\sigma\,(v)-2]$ = $(n-i)^2 + i^2 - 2$.
Hence the result follows. □

5. Atom-bond connectivity status index of subdivision graph of some graph

Definition 5.1. *If $G = (V, E)$ be a connected graph on n vertices and m edges then the subdivision graph of G is denoted by $S(G)$ and defined as a graph resulting from introducing a vertex of degree two for every edge.*

Theorem 5.1. *Let K_n is a complete graph on n vertices. Then,*

$$ABCS\,[S\,(K_n)] = 2m \times \sqrt{\frac{7n^2-9n-4}{n^2\,(6n^2-15n+9)}}.$$

Proof. Partitioning the vertex set of $S(K_n)$ into two vertex set.

Let $U=\{u_1, u_2, u_3, \ldots, u_n\}$ with $|U| = n$ be the vertex set of K_n and let $V = \{v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of subdivision vertices with $|V| = m$. For any edge E in $S(K_n)$, $E = \{uv/u \in U \ and \ v \in V\}$. Therefore, every vertex $u_i \in U$ is at a distance 2 from every vertex $u_j \in U$ in $S(K_n)$. As such there are $(n-1)$ vertices at a distance 2 from u_i.

Also $(n-1)$ subdivision vertices are at distance 1 from u_i and the remaining $[m-(n-1)]$ vertices are at distance 3 from u_i.

Therefore,

$$\begin{aligned}\sigma(u_i) &= 2(n-1) + (n-1) + 3[m-(n-1)]\\ &= 3(n-1) + 3\left[\frac{n(n-1)}{2} - (n-1)\right].\end{aligned}$$

Hence, $\sigma(u_i)=3\left[\frac{n(n-1)}{2}\right]$.

Similarly, for every vertex $v_i \in V$ there are two vertices in U at distance 1 and the remaining $(n-2)$ vertices of U at a distance 3.

Also, $(2n-4)$ subdivision vertices are at distance 2 and $[(m-1) - 2d(u) - 1]$ number of vertices are at distance 4.

$$\begin{aligned}\sigma(v_i) &= 2 + 2(2n-4) + 3(n-2) + 4[(m-1) - 2(d(u)-1)]\\ &= 7n - 12 + 4[(nC_2 - 1) - 2((n-1)-1)] = 2n^2 - 3n = n(2n-3).\end{aligned}$$

Therefore,

$$\begin{aligned}ABCS[S(K_n)] &= \sum_{uv\in E(S(K_n))} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u\ \sigma_v}}\\ &= \sum_{uv\in E(S(K_n))} \sqrt{\frac{7n^2-9n-4}{n^2(6n^2-15n+9)}}.\end{aligned}$$

Since there are $2m$ edges in $S(K_n)$, $ABCS[S(K_n)]= 2m \times \sqrt{\frac{7n^2-9n-4}{n^2(6n^2-15n+9)}}$. □

Example 1. *From the figure 1 in $S(K_4)$, $\sigma(v_i)$=18, i=1,2,3,4. Let s_j, j = 1, 2, 3, 4, 5, 6 be the subdivision vertices,then $\sigma(s_j)$=20. Then,*

$$ABCS[S(K_4)] = \sum_{uv\in E(G)} \sqrt{\frac{18+20-2}{20\times 18}} = 12 \times \sqrt{\frac{36}{360}} = 3.7947.$$

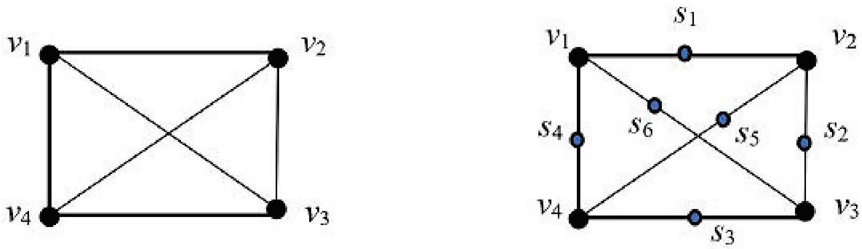

FIGURE 1. K_4 *and* $S\,(K_4)$

By the formula for $m = 6$ *and* $n = 4$,

$$ABCS[S(K_4)] = 2m \times \sqrt{\frac{7n^2 - 9n - 4}{n^2\,(6n^2 - 15n + 9)}}$$

$$= 12 \times \sqrt{\frac{7(16) - 9(4) - 4}{4^2\,[6(16) - 15(4) + 9]}} = 3.7947.$$

Theorem 5.2. *For a complete bipartite graph* $K_{p,q}$ *on* n *vertices,*

$$ABCS\,[S\,(K_{p,q}\,)] = m \times \left[\sqrt{\frac{7m + n + 4p - 10}{(3m + 4p - 4)\,(4m + n - 4)}} + \sqrt{\frac{7m + n + 4q - 10}{(3m + 4q - 4)\,(4m + n - 4)}}\right].$$

Proof. Partitioning the vertex set of subdivision graph of $K_{p,q}$ in to three vertex set $U = \{u_1, u_2, u_3, \dots, u_p\}$; $V = \{v_1, v_2, v_3, \dots, v_q\}$; $W = \{w_1, w_2, w_3, \dots, w_m\}$. Here $n = p + q$ and $m = pq$. For any edge in $S\,(K_{p,q}\,)$, partitioning the edge set, $E = \{uv/u \in U or\ V$ and $v \in W\}$. Let $E_1 = \{uv/u \in U$ and $v \in W\}$ and $E_2 = \{uv/u \in V$ and $v \in W\}$. Every vertex $u \in E_1$ is at a distance 1 from q subdivision vertices, at a distance 2 from q vertices of V, At a distance 4 from $(p - 1)$ vertices of U, at a distance 3 from $(p - 1)$ subdivision vertices and at a distance 3 from $(p - 1)(q - 1)$ subdivision vertices.

Therefore,

$$\sigma\,(u) = q + 2q + 3\,(p - 1) = 4\,(p - 1) + 3\,(p - 1)\,(q - 1)$$
$$\sigma\,(u) = 3pq + 4p - 4 = 3m + 4p - 4.$$

Similarly, every vertex $u \in E_2$ is at a distance 1 from p subdivision vertices, at a distance 2 from p vertices of U, at a distance 4 from $(q-1)$ vertices of U, at a distance 3 from $p\,(q-1)$ subdivision vertices.

Therefore, $\sigma\,(u){=}p{+}2p{+}3p\,(q-1){+}4\,(q-1)\;\sigma\,(u){=}3pq{+}4q{-}4{=}3m{+}4q{-}4$. For every vertex $v \in E_1$ or E_2, two vertices are at a distance 1, $(p-1)$ and $(q-1)$ vertices of U and V are at a distance 3, $(p-1)$ and $(q-1)$ vertices are at distance 2 and $(p-1)\,(q-1)$ vertices at distance 4. Therefore, $\sigma\,(v){=}2+3\,(p+q-2)\;+\;2\,[(p-1)+(q-1)]\;+\;4\,(p-1)\,(q-1)$.
$\sigma\,(v) = 4m + n - 4$.

By the definition of Atom bond connectivity status index of a graph G,

$$ABCS\,[S\,(K_{p,q}\,)] = \sum_{uv\in E_1}\sqrt{\frac{7m+n+4p-10}{(3pq+4p-4)\,(4m+n-4)}} + \sum_{uv\in E_2}\sqrt{\frac{7m+n+4q-10}{(3pq+4q-4)\,(4m+n-4)}}.$$

Hence,

$$ABCS\,[S\,(K_{p,q})] = m\times\left[\sqrt{\frac{7m+n+4p-10}{(3m+4p-4)\,(4m+n-4)}} + \sqrt{\frac{7m+n+4q-10}{(3m+4q-4)\,(4m+n-4)}}\right].$$

□

Example 2. *From the figure 2 in,* $S\,(K_{2,3})$,$\sigma(u_1) = \sigma(u_2) =$ *22,* $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) =$ *26. Let* w_i*, i=1, 2, 3, 4, 5, 6 be the subdivision vertices. Then,* $\sigma(w_i) =$

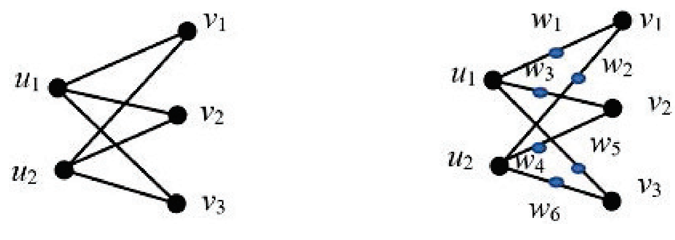

FIGURE 2. $K_{2,3}$ and $S\,(K_{2,3})$

25 for i=1, 2, 3, 4, 5, 6. Now,

$$ABCS\left[S\left(K_{2,3}\right)\right] = \sum_{uv\in E_1}\sqrt{\frac{\sigma_u+\sigma_v-2}{\sigma_u\sigma_v}} + \sum_{uv\in E_2}\sqrt{\frac{\sigma_u+\sigma_v-2}{\sigma_u\sigma_v}}$$
$$= \sum_{uv\in E_1}\sqrt{\frac{22+25-2}{(22)\,(25)}} + \sum_{uv\in E_2}\sqrt{\frac{26+25-2}{(26)\,(25)}} = 3.3635.$$

By the Formula for $m=6, n=5, p=2, q=3$,

$$ABCS\left[S\left(K_{2,3}\right)\right]$$
$$= m \times \left[\sqrt{\frac{7m+n+4p-10}{(3m+4p-4)\,(4m+n-4)}} + \sqrt{\frac{7m+n+4q-10}{(3m+4q-4)\,(4m+n-4)}}\right]$$
$$= 6 \times \left[\sqrt{\frac{7(6)+5+4(2)-10}{[3(6)+4(2)-4][4(6)+5-4]}} + \sqrt{\frac{7(6)+5+4(3)-10}{[3(6)+4(3)-4][4(6)+5-4]}}\right]$$
$$= 3.3635.$$

Theorem 5.3. *If P_n is a path graph on n vertices, then*

$$ABCS\left[S\left(P_n\right)\right] = \sum_{i=1}^{2n-2}\sqrt{\frac{2n(2n-1)+i(i-2n)+(i+1)[(i+1)-2n]-2}{[n(2n-1)+i(i-2n)][n(2n-1)+(i+1)[(i+1)-2n]]}}.$$

Proof. The subdivision graph of P_n has $n+n-1=2n-1$ vertices. Let v_1, v_2, v_3,...,v_{2n-1} be the vertices, where v_i is adjacent to v_{i+1}, i = 1, 2, 3,...,$2n-2$. Therefore,

$$\sigma\left(v_i\right) = \left[\frac{(2n-1)^2+(2n-1)}{2} + i\left(i-(2n-1)-1\right)\right]$$
$$= n\left(2n-1\right) + i\left(i-2n\right)$$
$$\sigma\left(v_{i+1}\right) = n\left(2n-1\right) + (i+1)\left[(i+1)-2n\right]$$
$$\left[\sigma\left(u\right)+\sigma\left(v\right)-2\right] = 2n\left(2n-1\right)+i\left(i-2n\right)+(i+1)\left[(i+1)-2n\right]-2$$
$$\left[\sigma\left(u\right).\sigma\left(v\right)\right] = \left[n\left(2n-1\right)+i\left(i-2n\right)\right]\left[n\left(2n-1\right)+(i+1)\left[(i+1)-2n\right]\right].$$

Hence,

$$ABCS\left[S\left(P_n\right)\right] = \sum_{i=1}^{2n-2}\sqrt{\frac{2n(2n-1)+i(i-2n)+(i+1)[(i+1)-2n]-2}{[n(2n-1)+i(i-2n)][n(2n-1)+(i+1)[(i+1)-2n]]}}.$$

□

Example 3. *From the figure3 in,* $S(P_4)$*.If* v_1, v_2, v_3 *are the subdivision vertices then,* $\sigma(u_1) = 21,\ \sigma(v_1) = 16,\ \sigma(u_2) = 13,\ \sigma(v_2) = 12,\ \sigma(u_3) = 13,\ \sigma(v_3) = 16,\ \sigma(v_4) = 21$*, and also*

$$ABCS\left[S\left(P_4\right)\right] = \sum_{uv\in E[S(P_4)]} \sqrt{\frac{\sigma_u + \sigma_v - 2}{\sigma_u\sigma_v}}$$

$$= \sqrt{\frac{21+16-2}{(21)(16)}} + \sqrt{\frac{13+16-2}{(13)(16)}} + \sqrt{\frac{13+12-2}{(13)(12)}}$$

$$+ \sqrt{\frac{13+12-2}{(13)(12)}} + \sqrt{\frac{13+16-2}{(13)(16)}} + \sqrt{\frac{21+16-2}{(21)(16)}}$$

$$= 0.3227 + 0.3602 + 0.3839 + 0.3839 + 0.3602 + 0.3227 = 2.1336.$$

FIGURE 3. P_4 and $S(P_4)$

By the formula given in Theorem 5.3,

$$ABCS\left[S\left(P_4\right)\right] = \sum_{i=1}^{6} \sqrt{\frac{56 + i(i-8) + (i+1)[(i+1)-8] - 2}{[28 + i(i-8)][28 + (i+1)[(i+1)-8]]}} = 2.1336.$$

Theorem 5.4. *For a cycle* C_n*,* $n \geq 3$ *on* n *vertices,*

$$ABCS\left[S\left(C_n\right)\right] = \frac{2}{n}\left(\sqrt{2n^2-2}\right).$$

Proof. The subdivision graph of C_n has $2n$ vertices. For any vertex u of $S(C_n)$, $\sigma(u) = 2\left[1 + 2 + \cdots + \frac{n-1}{2}\right] + \frac{n}{2} = \frac{(2n)^2}{4} = n^2$. Therefore, $ABCS(C_n) = 2n \times \sqrt{\frac{2n^2-2}{n^4}} = \frac{2}{n}\left(\sqrt{2n^2-2}\right)$ □

FIGURE 4. C_4 and $S(C_4)$

Example 4. *Let v_i, i = 1, 2, 3, 4 be the subdivision vertices then from the above figure 4 in $S(C_4)$, $\sigma(u_i)$=$\sigma(v_i)$=16, i =1, 2, 3, 4. Then, $ABCS[S(P_4)] = \sum_{uv\in E[S(C_4)]}\sqrt{\frac{\sigma_u+\sigma_v-2}{\sigma_u\sigma_v}} = 8\sqrt{\frac{16+16-2}{(16)(16)}}$ =2.7386.*

By the formula, $ABCS[S(C_n)] = \frac{2}{n}\left(\sqrt{2n^2-2}\right)$ =$\frac{2}{4}\left(\sqrt{32-2}\right)$=2.7386.

6. Atom-bond connectivity status index of graphs formed by using the complete graph

In this section we have obtained the atom- bond connectivity status index of some graphs, which are defined in [1].

Proposition 6.1. *For a complete graph K_n with $n \geq 3$, let e_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq n-2$, be the distinct edges all being incident with a single vertex. The graph $Ka_n(k)$ is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from K_n. Then,*

$$ABCS(Ka_n(k)) = [n-k-1] \times \sqrt{\frac{2n+k-4}{n(n-1)}} + \left[\frac{k(k-1)}{2}\right] \times \sqrt{\frac{2n-2}{n^2}} + [(n-k-1)k] \times \sqrt{\frac{2n-3}{n(n-1)}} + \left[\frac{(n-k-1)(n-k-2)}{2}\right] \times \sqrt{\frac{2n-4}{(n-1)^2}}.$$

Proof. By the equality part of Theorem 3.1,

$$ABCS(G) = \sum_{uv\in E(G)} \sqrt{\frac{4n-6-[d(u)+d(v)]}{(2n-2-d(u))(2n-2-d(v))}}.$$

The edge set $E(Ka_n(k))$ can be partitioned into four sets E_1, E_2, E_3 and E_4, where E_1=$\{uv/d(u) = n-1-k$ and $d(u) = n-1\}$, E_2=$\{uv/d(u) = n-2$ and $d(u) = n-2\}$, $E_3 = \{uv/d(u) = n-2$ and $d(u) = n-1\}$, E_4=$\{uv/d(u) = n-1$ and $d(u) = n-1\}$, with $|E_1| = n-k-1$, $|E_2|$ =$(k-1)/2$, $|E_3|$ =$(n-k-1)k$, $|E_4|$ =$(n-k-1)(n-k-2)/2$. Also $diam((Ka_n(k)) = 2$.

Therefore,

$$ABCS\left(Ka_n\left(k\right)\right) = \sum_{uv\in E(G)} \sqrt{\frac{4n-6-[n-1-k+n-1]}{\left(2n-2-\left(n-1-k\right)\right)\left(2n-2-\left(n-1\right)\right)}}$$
$$+\sum_{uv\in E_2(G)} \sqrt{\frac{4n-6-[n-2+n-2]}{\left(2n-2-\left(n-2\right)\right)\left(2n-2-\left(n-2\right)\right)}}$$
$$+\sum_{uv\in E_3(G)} \sqrt{\frac{4n-6-[n-2+n-1]}{\left(2n-2-\left(n-2\right)\right)\left(2n-2-\left(n-1\right)\right)}}$$
$$+\sum_{uv\in E_4(G)} \sqrt{\frac{4n-6-[n-1+n-1]}{\left(2n-2-\left(n-1\right)\right)\left(2n-2-\left(n-1\right)\right)}}.$$

Therefore

$$ABCS(Ka_n(k)) = \sum_{uv\in E_1(G)} \sqrt{\frac{2n+k-4}{n(n-1)}} + \sum_{uv\in E_2(G)} \sqrt{\frac{2n-2}{n^2}} + \sum_{uv\in E_3(G)} \sqrt{\frac{2n-3}{n\left(n-1\right)}} + \sum_{uv\in E_4(G)} \sqrt{\frac{2n-4}{(n-1)^2}}.$$

Hence,

$$ABCS(Ka_n(k)) = [n-k-1] \times \sqrt{\frac{2n+k-4}{n(n-1)}} + \left[\times \frac{k\left(k-1\right)}{2}\right] \times \sqrt{\frac{2n-2}{n^2}} + [(n-k-1)k] \times \sqrt{\frac{2n-3}{n(n-1)}} + \left[\frac{\left(n-k-1\right)\left(n-k-2\right)}{2}\right] \times \sqrt{\frac{2n-4}{\left(n-1\right)^2}}.$$

□

Proposition 6.2. *For a complete graph K_n with $n \geq 3$, let f_i, $i = 1, 2, \ldots, k$, $1 \leq k \leq \lfloor n/2 \rfloor$, be independent edges. The graph $Kb_n(k)$ is obtained by deleting f_i,*

$i = 1, 2, \ldots, k$ *edges from* K_n. *Then,*

$$ABCS(Kb_n(k)) = [2k(n-2k)] \times \sqrt{\frac{2n-3}{n(n-1)}} + \left[\frac{(n-2k)(n-2k-1)}{2}\right] \times \sqrt{\frac{2n-4}{(n-1)^2}} + \left[\left(\frac{2k(2k-1)}{2}\right) - k\right] \times \sqrt{\frac{2n-2}{n^2}}.$$

Proof. The edge set $E(Kb_n(k))$ can be partitioned into three sets E_1, E_2 and E_3, where $E_1 = \{uv/d(u) = n-2 \ and \ d(v) = n-1\}$, $E_2 = \{uv/d(u) = n-1$ and $d(v) = n-1\}$, $E_3 = \{uv/ \ d(u) = n-2$ and $d(v) = n-2\}$. It is easy to check that $|E_1| = 2k(n-2k)$, $|E_2| = ((n-2k)(n-2k-1)/2)$ and $|E_3| = (2k(2k-1)/2) - k$. Also $diam((Kb_n(k)) = 2$.

By the equality part of Theorem 3.1,

$$ABCS\left(G\right) = \sum_{uv \in E(G)} \sqrt{\frac{4n-6-[d\left(u\right)+d\left(v\right)]}{\left(2n-2-d\left(u\right)\right)\left(2n-2-d\left(v\right)\right)}}$$

$$ABCS(Kb_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{4n-6-[n-2+n-1]}{(2n-2-(n-2))(2n-2-(n-1))}} + \sum_{uv \in E_2(G)} \sqrt{\frac{4n-6-[n-1+n-1]}{(2n-2-(n-1))(2n-2-(n-1))}} + \sum_{uv \in E_3(G)} \sqrt{\frac{4n-6-[n-2+n-2]}{(2n-2-(n-2))(2n-2-(n-2))}}.$$

Therefore,

$$ABCS(Kb_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{2n-3}{n(n-1)}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2n-4}{(n-1)^2}} + \sum_{uv \in E_3(G)} \sqrt{\frac{2n-2}{n^2}}.$$

Hence,

$$ABCS(Kb_n(k)) = [2k(n-2k)] \times \sqrt{\frac{2n-3}{n(n-1)}} + \left[\frac{(n-2k)(n-2k-1)}{2}\right] \times \sqrt{\frac{2n-4}{(n-1)^2}} + \left[\left(\frac{2k(2k-1)}{2}\right) - k\right] \times \sqrt{\frac{2n-2}{n^2}}.$$

□

Proposition 6.3. *For a complete graph K_n, $n \geq 3$, let V_k be a k-element subset of the vertex set $2 \leq k \leq n-1$. The graph $Kc_n(k)$ is obtained by deleting from all the edges connecting pairs of vertices from V_k. Then,*

$$ABCS(Kc_n(k)) = [(n-k)k] \times \sqrt{\frac{(2n+k-5)}{(n-2+k)(n-1)}} + \left[\frac{(n-k)(n-k-1)}{2}\right] \times \sqrt{\frac{(2n-4)}{(n-1)^2}}.$$

Proof. The edge set $E(Kcn(k))$ can be partitioned into two sets E_1 and E_2, where $E_1 = \{uv/d(u) = n-k \text{ and } d(v) = n-1\}$ and $E_2 = \{uv/d(u) = n-1 \text{ and } d(v) = n-1\}$. Also $|E_1| = (n-k)k$, $|E_2| = (n-k)(n-k-1)/2.$ and $diam((Kb_n(k)) = 2$.

By the equality part of Theorem 3.1,

$$ABCS(Kc_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{4n-6-[n-k+n-1]}{(2n-2-(n-k))(2n-2-(n-1))}} + \sum_{uv \in E_2(G)} \sqrt{\frac{4n-6-[n-1+n-1]}{(2n-2-(n-1))(2n-2-(n-1))}}.$$

Therefore,

$$ABCS(Kc_n(k)) = \sum_{uv \in E_1(G)} \sqrt{\frac{2n+k-5}{(n-2+k)(n-1}} + \sum_{uv \in E_2(G)} \sqrt{\frac{2n-4}{(n-1)^2}}.$$

Hence the result follows. □

Proposition 6.4. *For a complete graph K_n with $n \geq 5$, let $3 \leq k \leq n$. The graph $Kd_n(k)$ is obtained by deleting from K_n, the edges belonging to a k-membered cycle. Then*

$$ABCS(Kd_n(k)) = \left[\frac{k(k-1)}{2} - k\right] \times \sqrt{\frac{2n}{(n+1)^2}} + [(n-k)k] \times \sqrt{\frac{2n-2}{(n-1)(n-1)}} + \left[\frac{(n-k)(n-k-1)}{2} - k\right] \times \sqrt{\frac{2n-4}{(n-1)^2}}.$$

Proof. The edge set $E(Kd_n(k))$ can be partitioned into three sets E_1, E_2 and E_3, where $E_1 = \{uv/d(u) = n-3 = d(v)\}$, $E_2 = \{uv/d(u) = n-3 \text{ and } d(v) = n-1\}$, E_3

$= \{uv/\ d(u) = n-1 = d(v)\}$. It is easy to check that and $|E_1| = (k(k-1)/2) - k$, $|E_2| = (n-k)k$ and $|E_3| = ((n-k)(n-k-1)/2)$. Also $diam((Kd_n(k)) = 2$.

By the equality part of Theorem 3.1,

$$ABCS(Kd_n(k)) = \sum_{uv\in E_1(G))} \sqrt{\frac{2n}{(n+1)^2}} + \sum_{uv\in E_2(G)} \sqrt{\frac{2n-2}{(n+1)(n-1)}} + \sum_{uv\in E_3(G)} \sqrt{\frac{2n-4}{(n-1)^2}}.$$

Hence the result follows. □

7. Conclusion

In this paper we have obtained bounds for the atom-bond connectivity status index of graph in terms of degree and diameter. Gave alternate proof of atom-bond connectivity status index of some standard graphs. Obtained atom- bond connectivity status index of subdivision graph of some graphs and edge deleted graph obtained from complete graph.

References

[1] I. Gutman, L. Pavlovic: *The energy of some graphs with large number of edges*, Bull. Acad. Serbe Sci. Arts. (Cl. Math. Natur.), **118**,(1999), 35-50.

[2] V. R. Kulli: *Computation of ABC, AG and Augmented Status Indices of Graphs*, International Journal of Mathematics Trends and Technology, **66**(1) (2020), 1-7.

[3] H. S. Ramane, B. Basavanagouda, A. S. Yalnaik: *Harmonic Status Index of Graph*, Bulletin of Mathematical Science and applications, **17** (2016), 24-32.

[4] B. Zhou, R. Xing: *On atom–bond connectivity index*, Z. Naturforsch, **A66** (2011), 61–66.

[5] R. Xing, B. Zhou, Z. Du: *Further results on atom–bond connectivity index of trees*, Discrete Appl. Math. **158** (2010), 1536–1545.

[6] K. C. Das, I. Gutman, B. Furtula: *On atom-bond connectivity index*, Chem. Phys. Lett., **511** (2011), 452-454.

[7] K. C. Das: *Atom-bond connectivity index of graphs*, Discrete Appl. Math., **158** (2010), 1181-1188.

[8] E. Estrada, L. Torres, L. Rodrguez, I. Gutman: *Atom-bond connectivity and the energetic of branched alkanes*, Chem. Phys. Lett., **463** (2008), 422-425.

[9] F. Harary: *Status and Contra status*, Sociometry, **22** (1959), 23-43.

[10] R. XING, B. ZHOU, F. DONG: *On atom-bond connectivity index of connected graph* , Discrete applied mathematics, **159** (2011), 1617-1630.

[11] M. F. RANDIĆ: *On characterization of molecular branching* , J. Am. Chem. Soc., **97** (1975), 6609–6615.

[12] E. ESTRADA, L. TORRES, L. RODRÍGUEZ, I. GUTMAN: *An atom–bond connectivity index: modelling the enthalpy of formation of alkanes*, Indian J. Chem. Sect., **A37** (1998), 849-855.

DEPARTMENT OF MATHEMATICS
KLE DR. M. S. SHESHGIRI COLLEGE OF ENGINEERING AND TECHNOLOGY
BELAGAVI-590008, KARNATAKA, INDIA
Email address: revankards@gmail.com

RESEARCH SCHOLAR, DEPARTMENT OF MATHEMATICS
KLE DR. M. S. SHESHGIRI COLLEGE OF ENGINEERING AND TECHNOLOGY
BELAGAVI-590008, KARNATAKA, INDIA
FACULTY, DEPARTMENT OF MATHEMATICS
KLS GOGTE INSTITUTE OF TECHNOLOGY
BELAGAVI-590008, KARNATAKA, INDIA
Email address: priyanka18hande@gmail.com

DEPARTMENT OF MATHEMATICS
VISHWANATHRAO DESHPANDE INSTITUTE OF TECHNOLOGY
HALIYAL-581329, INDIA
Email address: handesp1313@gmail.com