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MORPHISM OF m-BIPOLAR FUZZY GRAPH

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ABSTRACT. In this article, weak, co-weak, isomorphism and morphism between two m-bipolar fuzzy graphs (m-BPFGs) are defined and studied their various properties.

1. INTRODUCTION

Fuzzy sets are introduced for the parameters to solve problems related to vague and uncertain in real life situations were given by Zadeh [8] in 1965. The limitations of traditional model were overcome by the introduction of bipolar fuzzy set concept in 1994 by Zhang [9]. This was further improved by Chen et al. [3] to m-polar fuzzy set theory.

Free body diagrams using set of nodes connected by lines representing pairs are good problem solving tools in non-deterministic real life situations. Thus, Rosenfeld [6] first initiated the fuzzy graphs by taking fuzzy relations on fuzzy sets in 1975. Akram [1] introduced the notion of bipolar fuzzy graphs and studied some isomorphic properties on it. Rashmanlou et al. [7] studied categorical properties of bipolar fuzzy graphs. Ghorai and Pal [4, 5] introduced generalized m-polar fuzzy graphs and studied some isomorphic properties and density of

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an m-polar fuzzy graph. Bera and pal [2] introduced the concept of m-polar interval-valued fuzzy graph and studied the algebraic properties like density, regularity and irregularity etc. on m-PIVFG.

This paper attempts to develop theory to analyze parameters combining concepts from m-polar fuzzy graphs and bipolar fuzzy graphs as a unique effort. The resultant graph is turned to m-BPFG and studied properties on it.

2. PRELIMINARIES

All the vertices and edges of an m-polar fuzzy graph have m components and those components are fixed. But these components may be bipolar. Using this idea, m-BPFG has been introduced.

Before defining m-bipolar fuzzy graph, we assume the following:

For a given set V, define an equivalence relation $\leftrightarrow$ on $V \times V - \{(k,k) : k \in V\}$ as follows:$(k_1, l_1) \leftrightarrow (k_2, l_2) \Leftrightarrow$ either $(k_1, l_1) = (k_2, l_2)$ or $k_1 = l_2, l_1 = k_2$.

The quotient set got in this way is denoted by $\overleftrightarrow{V^2}$.

Definition 2.1. *An m-bipolar fuzzy set (m-BPFS) S on V is defined by $S(s) = \{\langle [p_1 \circ \psi_S^p(s), p_1 \circ \psi_S^n(s)], [p_2 \circ \psi_S^p(s), p_2 \circ \psi_S^n(s)], \ldots, [p_m \circ \psi_S^p(s), p_m \circ \psi_S^n(s)]\rangle\}$ for all $s \in V$ or shortly $S(s) = \{\langle [p_j \circ \psi_S^p(s), p_j \circ \psi_S^n(s)]_{j=1}^m \, | s \in V\rangle\}$ where the functions $p_j \circ \psi_S^p \colon V \to [0,1]$ and $p_j \circ \psi_S^n \colon V \to [-1,0]$ denote the positive memberships and negative memberships of the element respectively.*

Definition 2.2. *Let S be an m-BPFS on a set V. An m-bipolar fuzzy relation on a set S is an m-BPFS T of $V \times V$, $T(s,t) = \{\langle [p_1 \circ \psi_T^p(s,t), p_1 \circ \psi_T^n(s,t)], [p_2 \circ \psi_T^p(s,t), p_2 \circ \psi_T^n(s,t)], \ldots, [p_m \circ \psi_T^p(s,t), p_m \circ \psi_T^n(s,t)]\}\rangle$ for all $s,t \in V$ or shortly $T(s,t) = \{\langle [p_j \circ \psi_T^p(s,t), p_j \circ \psi_T^n(s,t)]_{j=1}^m \rangle | s,t \in V\}$ such that $p_j \circ \psi_T^p(s,t) \leq min\{p_j \circ \psi_S^p(s), p_j \circ \psi_S^p(t)\}, p_j \circ \psi_T^n(s,t) \geq \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$, for every $j = 1, 2, \ldots, m$ and $s,t \in V$.*

Definition 2.3. *An m-bipolar fuzzy graph (m-BPFG) of a graph $G* = (V,E)$ is a pair $G = (V,S,T)$ where $S = \langle [p_j \circ \psi_S^p, p_j \circ \psi_S^n]_{j=1}^m \rangle, p_j \circ \psi_S^p \colon V \to [0,1]$ and $p_j \circ \psi_S^n \colon V \to [-1,0]$ is an m-BPFS on V; and $T = \langle [p_j \circ \psi_T^p, p_j \circ \psi_T^n]_{j=1}^m \rangle$, $p_j \circ \psi_T^p \colon \overleftrightarrow{V^2} \to [0,1]$ and $p_j \circ \psi_T^n \colon \overleftrightarrow{V^2} \to [-1,0]$ is an m-BPFS in $\overleftrightarrow{V^2}$ such that $p_j \circ \psi_T^p(k,l) \leq min\{p_j \circ \psi_S^p(k), p_j \circ \psi_S^p(l)\}, p_j \circ \psi_T^n(k,l) \geq \max\{p_j \circ \psi_S^n(k), p_j \circ \psi_S^n(l)\}$*

for all $(k,l) \in \overleftrightarrow{V^2}, j = 1,2,\ldots,m$ *and* $p_j \circ \psi_T^p(k,l) = p_j \circ \psi_T^n(k,l) = 0$ *for all* $(k,l) \in \overleftrightarrow{V^2} - E$.

Definition 2.4. *An m-BPFG* $G = (V,S,T)$ *of a graph* $G* = (V,E)$ *is strong if for every* $(s,t) \in E$ *and* $j = 1,2,\ldots,m$ *satisfying* $p_j \circ \psi_T^p(s,t) = min\{p_j \circ \psi_S^p(s), p_j \circ \psi_S^p(t)\}, p_j \circ \psi_T^n(s,t) = \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$.

Definition 2.5. *Let* $G = (V,S,T)$ *be an m-BPFG of* $G^* = (V,E)$.*The complement of* G *is an m-BPFG* $\overline{G} = (V,\overline{S},\overline{T})$ *of* $\overline{G^*} = (V,\overleftrightarrow{V^2})$ *such that* $\overline{S} = S$ *and* $\overline{T}$ *is defined by* $p_j \circ \psi_{\overline{T}}(s,t) = [p_j \circ \psi_{\overline{T}}^p(s,t), p_j \circ \psi_{\overline{T}}^n(s,t)], p_j \circ \psi_{\overline{T}}^p(s,t) = \{p_j \circ \psi_S^p(s) \wedge p_j \circ \psi_S^p(t)\} - p_j \circ \psi_T^p(s,t), p_j \circ \psi_{\overline{T}}^n(s,t) = \{p_j \circ \psi_S^n(s) \vee p_j \circ \psi_S^n(t)\} - p_j \circ \psi_T^n(s,t)$ *for every* $(s,t) \in \overleftrightarrow{V^2}$ *and* $j = 1,2,\ldots,m$.

Definition 2.6. *The degree of a vertex* $s \in V$ *in an m-BPFG* $G = (V,S,T)$ *is defined as* $d_G(s) = \langle [p_j \circ d_G^p(s), p_j \circ d_G^n(s)]_{j=1}^m \rangle = \left\langle \left[\sum_{\substack{s \neq t \\ (s,t)\in E}} p_j \circ \psi_T^p(s,t), \sum_{\substack{s \neq t \\ (s,t)\in E}} p_j \circ \psi_T^n(s,t) \right]_{j=1}^m \right\rangle$. *Every vertex in an m-BPFG* $G = (V,S,T)$ *has the same degree then* $G = (V,S,T)$ *is regular.*

3. ϕ-MORPHISM ON m-BPFGS

In this section, homomorphism, isomorphism, morphism between two m-BPFGs are defined and some of its properties are studied.

Definition 3.1. *Let* $G_1 = (V_1,S_1,T_1)$ *and* $G_2 = (V_2,S_2,T_2)$ *be two m-BPFGs of the graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$ *respectively, for* $j = 1,2,\ldots,m$.

(i) *A homomorphism between* G_1 *and* G_2 *is a mapping* $\phi\colon V_1 \to V_2$ *such that*
  - (a) $p_j \circ \psi_{S_1}^p(\alpha) \leq p_j \circ \psi_{S_2}^p(\phi(\alpha)), p_j \circ \psi_{S_1}^n(\alpha) \geq p_j \circ \psi_{S_2}^n(\phi(\alpha))$ *for all* $\alpha \in V_1$ *and*
  - (b) $p_j \circ \psi_{T_1}^p(\alpha,\beta) \leq p_j \circ \psi_{T_2}^p(\phi(\alpha),\phi(\beta)), p_j \circ \psi_{T_1}^n(\alpha,\beta) \geq p_j \circ \psi_{T_2}^n(\phi(\alpha),\phi(\beta))$ *for all* $(\alpha,\beta) \in \overleftrightarrow{V_1^2}$.

(ii) *A weak isomorphism between* G_1 *and* G_2 *is a bijective mapping* $\phi\colon V_1 \to V_2$ *such that*
  - (a) ϕ *is a homomorphism and*
  - (b) $p_j \circ \psi_{S_1}^p(\alpha) = p_j \circ \psi_{S_2}^p(\phi(\alpha)), p_j \circ \psi_{S_1}^n(\alpha) = p_j \circ \psi_{S_2}^n(\phi(\alpha))$ *for all* $\alpha \in V_1$.

(iii) *A co-weak isomorphism between G_1 and G_2 is a bijective mapping $\phi\colon V_1 \to V_2$ such that*
  (a) *ϕ is a homomorphism and*
  (b) $p_j \circ \psi^p_{T_1}(\alpha, \beta) = p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)), p_j \circ \psi^n_{T_1}(\alpha, \beta) = p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$ *for all* $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$.

(iv) *An isomorphism between G_1 and G_2 is a bijective mapping $\phi\colon V_1 \to V_2$ such that*
  (a) $p_j \circ \psi^p_{S_1}(\alpha) = p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_1}(\alpha) = p_j \circ \psi^n_{S_2}(\phi(\alpha))$ *for all* $\alpha \in V_1$ *and*
  (b) $p_j \circ \psi^p_{T_1}(\alpha, \beta) = p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)), p_j \circ \psi^n_{T_1}(\alpha, \beta) = p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$ *for all* $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$.

Definition 3.2. *Let $G_1 = (V_1, S_1, T_1)$ and $G_2 = (V_2, S_2, T_2)$ be two m-BPFGs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. Then a bijective mapping $\phi\colon V_1 \to V_2$ is called an m-BPF morphism or m-BPF ϕ-morphism if there exists two numbers $z_1 > 0$ and $z_2 > 0$ such that for $j = 1, 2, \ldots, m$.*

(a) $p_j \circ \psi^p_{S_2}(\phi(\alpha)) = z_1 p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^n_{S_2}(\phi(\alpha)) = z_1 p_j \circ \psi^n_{S_1}(\alpha)$ *for all* $\alpha \in V_1$.
(b) $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = z_2 p_j \circ \psi^p_{T_1}(\alpha, \beta), p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) = z_2 p_j \circ \psi^n_{T_1}(\alpha, \beta)$ *for all* $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$.

In such a case, ϕ is called (z_1, z_2) m-BPF ϕ-morphism from G_1 onto G_2. If $z_1 = z_2 = z$, we call ϕ, an m-BPF ϕ-morphism When$z = 1$, we obtain usual m-BPF morphism.

Example 3.1. *A morphism between two m-BPFGs G_1 and G_2 has given below.*

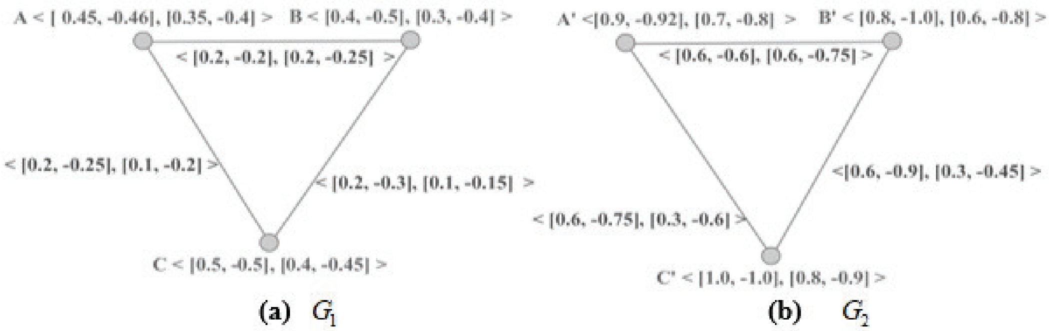

FIGURE 1. ϕ-morphism of m-BPFGs G_1 and G_2

Here, there is a m-BPF ϕ-morphism such that $\phi(A) = A', \phi(B) = B', \phi(C) = C', z_1 = 2, z_2 = 3$.

Theorem 3.1. *The relation ϕ-morphism is an equivalence relation in the collection of m-BPFGs.*

Proof. Let Γ be the collection of all m-BPFGs. Define a relation $\sim$on ΓxΓ as follows: for $G_1, G_2 \in \Gamma$, we say $G_1{\sim}G_2$ if and only if there exists a (z_1, z_2) m-BPF ϕ-morphism from G_1 onto G_2 for some $z_1 \neq 0, z_2 \neq 0$. Now we have to prove that $\sim$is an equivalence relation. First, we see that $\sim$is reflexive by taking identity mapping from G_1 onto G_1 Let $G_1, G_2 \in \Gamma$ and $G_1{\sim}G_2$. Then there exists a (z_1, z_2) m-BPF ϕ-morphism from G_1 onto G_2 for some $z_1 \neq 0, z_2 \neq 0$ Therefore $p_j \circ \psi^p_{S_2}(\phi(\alpha)) = z_1 p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^n_{S_2}(\phi(\alpha)) = z_1 p_j \circ \psi^n_{S_1}(\alpha)$ for all $\alpha \in V_1$ and $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = z_2 p_j \circ \psi^p_{T_1}(\alpha, \beta), p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) = z_2 p_j \circ \psi^n_{T_1}(\alpha, \beta)$ for all $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$. Consider $\phi^{-1} \colon V_2 \to V_1$. Let $m, n \in V_2$. Since ϕ is bijective there exist $\alpha, \beta \in V_1$ such that $m = \phi(\alpha), n = \phi(\beta)$. Then, $p_j \circ \psi^p_{S_1}(\phi^{-1}(m)) = p_j \circ \psi^p_{S_1}(\phi^{-1}(\phi(\alpha))) = p_j \circ \psi^p_{s_1}(\alpha) = \frac{1}{z_1} p_j \circ \psi^p_{S_2}(\phi(\alpha)) = \frac{1}{z_1} p_j \circ \psi^p_{S_2}(m)$, $p_j \circ \psi^p_{T_1}(\phi^{-1}(m), \phi^{-1}(n)) = p_j \circ \psi^p_{T_1}(\phi^{-1}(\phi(\alpha)), \phi^{-1}(\phi(\beta))) = p_j \circ \psi^P_{T_1}(\alpha, \beta) = \frac{1}{z_2} p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = \frac{1}{z_2} p_j \circ \psi^p_{T_2}(m, n)$ for $j = 1, 2, \ldots, m$. Similarly, $p_j \circ \psi^n_{S_1}(\phi^{-1}(m)) = \frac{1}{z_1} p_j \circ \psi^n_{S_2}(m), p_j \circ \psi^n_{T_1}(\phi^{-1}(m), \phi^{-1}(n)) = \frac{1}{z_2} p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$. Thus, there exists $\left(\frac{1}{z_1}, \frac{1}{z_2}\right)$ m-BPF morphism from G_2 to G_1 Therefore, $G_2 \sim G_1$ and hence $\sim$is symmetric. Again, let $G_1, G_2, G_3 \in \Gamma$ be such that $G_1 \sim G_2$ and $G_2 \sim G_3$. Then there exist a (z_1, z_2) m-BPF ϕ morphism from G_1 onto G_2 for some $z_1 \neq 0, z_2 \neq 0$ and a (z_3, z_4) m-BPF q morphism from G_2 onto G_3 for some $z_3 \neq 0, z_4 \neq 0$. Then, $p_j \circ \psi^p_{S_3}(q(\alpha)) = z_3 p_j \circ \psi^p_{S_2}(\alpha), p_j \circ \psi^n_{S_3}(q(\alpha)) = z_3 p_j \circ \psi^n_{S_2}(\alpha)$ for all $\alpha \in V_2$ and $p_j \circ \psi^p_{T_3}(q(\alpha), q(\beta)) = z_4 p_j \circ \psi^p_{T_2}(\alpha, \beta), p_j \circ \psi^n_{T_3}(q(\alpha), q(\beta)) = z_4 p_j \circ \psi^n_{T_2}(\alpha, \beta)$ for all $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$, $j = 1, 2, ..., m$. Let $\delta \colon q \circ \phi \colon V_1 \to V_3$. Then $p_j \circ \psi^p_{S_3}(\delta(\alpha)) = p_j \circ \psi^p_{S_3}((q \circ \phi)(\alpha)) = p_j \circ \psi^p_{S_3}(q(\phi(\alpha))) = z_3 p_j \circ \psi^p_{S_2}(\phi(\alpha)) = z_3 z_1 p_j \circ \psi^p_{S_1}(\alpha)$ and $p_j \circ \psi^p_{T_3}(\delta(\alpha), \delta(\beta)) = p_j \circ \psi^p_{T_3}((q \circ \phi)(\alpha), (q \circ \phi)(\beta)) = p_j \circ \psi^p_{T_3}(q(\phi(\alpha)), q(\phi(\beta))) = z_4 p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = z_4 z_2 p_j \circ \psi^n_{T_1}(\alpha, \beta), j = 1, 2, \ldots, m$. Similarly $p_j \circ \psi^n_{S_3}(\delta(\alpha)) = z_3 z_1 p_j \circ \psi^n_{S_1}(\alpha)$ and $p_j \circ \psi^n_{T_3}(\delta(\alpha), \delta(\beta)) = z_4 z_2 p_j \circ \psi^n_{T_1}(\alpha, \beta)$. Thus, δ is a $(z_3 z_1, z_4 z_2)$m-BPF morphism from G_1 onto G_3. Therefore, $G_1 \sim G_3$ and hence $\sim$is transitive. So, the relation $\sim$is an equivalence relation in the collection of m-BPFGs. □

Theorem 3.2. *Let $G_1 = (V_1, S_1, T_1)$ and $G_2 = (V_2, S_2, T_2)$ be two m-BPFGs and ϕ be a (z_1, z_2) m-BPF morphism from G_1 onto G_2 for some $z_1 \neq 0$ and $z_2 \neq 0$. Then image of strong edge in G_1 is also a strong edge in G_2 if and only if $z_1 = z_2$.*

Proof. Let (α, β) be a strong edge in G_1 such that $(\phi(\alpha), \phi(\beta))$ is also a strong edge in G_2. Now as $G_1 \sim G_2$, we have for each $j = 1, 2, ...m$. $z_2 p_j \circ \psi^p_{T_1}(\alpha, \beta) = p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = p_j \circ \psi^p_{S_2}(\phi(\alpha)) \wedge p_j \circ \psi^p_{S_2}(\phi(\beta)) = z_1 p_j \circ \psi^p_{S_1}(\alpha) \wedge z_1 p_j \circ \psi^p_{S_1}(\beta) = z_1(p_j \circ \psi^p_{S_1}(\alpha) \wedge p_j \circ \psi^p_{S_1}(\beta)) = z_1 p_j \circ \psi^p_{T_1}(\alpha, \beta)$ for each $j = 1, 2, ..., m$. Similarly, $z_2 p_j \circ \psi^n_{T_1}(\alpha, \beta) = z_1 p_j \circ \psi^n_{T_1}(\alpha, \beta)$, therefore $z_1 = z_2$. Conversely suppose that $z_1 = z_2$.and (α, β) is strong edge in G_1 then for each $j = 1, 2, ..., m$ $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = z_2 p_j \circ \psi^p_{T_1}(\alpha, \beta) = z_2(p_j \circ \psi^p_{S_1}(\alpha) \wedge p_j \circ \psi^p_{S_1}(\beta)) = z_2\Big(\frac{1}{z_1} p_j \circ \psi^p_{S_2}(\phi(\alpha)) \wedge \frac{1}{z_1} p_j \circ \psi^p_{S_2}(\phi(\beta))\Big) = (p_j \circ \psi^p_{S_2}(\phi(\alpha)) \wedge p_j \circ \psi^p_{S_2}(\phi(\beta)))$. Similarly, $p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) = (p_j \circ \psi^n_{S_2}(\phi(\alpha)) \vee p_j \circ \psi^n_{S_2}(\phi(\beta)))$. Therefore $(\phi(\alpha), \phi(\beta))$ is strong edge in G_2. □

Theorem 3.3. *Let an m-BPFG $G_1 = (V_1, S_1, T_1)$ be regular. If there is a co-weak isomorphic from $G_1 = (V_1, S_1, T_1)$ to $G_2 = (V_2, S_2, T_2)$ then $G_2 = (V_2, S_2, T_2)$ is also regular.*

Proof. As an m-BPFG G_1 is co-weak isomorphic to G_2, there exists a co-weak isomorphism ϕ: $V_1 \to V_2$. which is bijective such that for $j = 1, 2, ...m$. $p_j \circ \psi^p_{S_1}(\alpha) \leq p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_1}(\alpha) \geq p_j \circ \psi^n_{S_2}(\phi(\alpha))$ for all $\alpha \in V_1$ and $p_j \circ \psi^p_{T_1}(\alpha, \beta) = p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)), p_j \circ \psi^n_{T_1}(\alpha, \beta) = p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$ for all $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$. As G_1 is regular, we have $\sum_{\substack{\alpha \neq \beta \\ (\alpha,\beta) \in E_1}} p_j \circ \psi^p_{T_1}(\alpha, \beta) = constant for all \alpha \in V_1$ Now $\sum_{\substack{\phi(\alpha) \neq \phi(\beta) \\ (\phi(\alpha),\phi(\beta)) \in E_2}} p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = \sum_{\substack{\alpha \neq \beta \\ (\alpha,\beta) \in E_1}} p_j \circ \psi^p_{T_1}(\alpha, \beta) = constant$ for all $\alpha \in V_1$. Similarly, $\sum_{\substack{\phi(\alpha) \neq \phi(\beta) \\ (\phi(\alpha),\phi(\beta)) \in E_2}} p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$ = constant for all $\alpha \in V_1$. Therefore G_2 is regular. □

Theorem 3.4. *Let an m-BPFG $G_1 = (V_1, S_1, T_1)$ be strong. If there is a weak isomorphic from $G_1 = (V_1, S_1, T_1)$ to $G_2 = (V_2, S_2, T_2)$ then G_2 is also strong.*

Proof. As G_1 is weak isomorphic to G_2. Then there exists a weak isomorphism ϕ: $V_1 \to V_2$. which is bijective such that for $j = 1, 2, ...m$. $p_j \circ \psi^p_{S_1}(\alpha) = p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_1}(\alpha) = p_j \circ \psi^n_{S_1}(\phi(\alpha))$ for all $\alpha \in V_1$ and $p_j \circ \psi^p_{T_1}(\alpha, \beta) \leq p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)), p_j \circ \psi^n_{T_1}(\alpha, \beta) \geq p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$ for all $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$. As G_1 is strong, we have $p_j \circ \psi^p_{T_1}(\alpha, \beta) = min(p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^p_{S_1}(\beta))$, for all $(\alpha, \beta) \in E_1$. Now $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) \geq p_j \circ \psi^p_{T_1}(\alpha, \beta) = min(p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^p_{S_1}(\beta)) = min(p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^p_{S_2}(\phi(\beta)))$. By the definition, $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) \leq min(p_j \circ \psi^p_{S_2}(\phi(\alpha), (\phi(\beta))$ for $(\phi(\alpha), \phi(\beta)) \in E_2$. Therefore, $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) =$

$min(p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^p_{S_2}(\phi(\beta)))$. Similarly, $p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) = \max(p_j \circ \psi^n_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_2}(\phi(\beta)))$. Hence, G_2 is strong. □

Theorem 3.5. *Let an m-BPFG $G_2 = (V_2, S_2, T_2)$ be strong regular. If there is a co-weak isomorphic from $G_1 = (V_1, S_1, T_1)$ to $G_2 = (V_2, S_2, T_2)$ then $G_1 = (V_1, S_1, T_1)$ is also strong regular m-BPFG.*

Proof. As an m-BPFG G_1 is co-weak isomorphic to G_2.Then there exists a co-weak isomorphism $\phi : V_1 \to V_2$ which is bijective such that for $j = 1, 2, ..., m$. $p_j \circ \psi^p_{S_1}(\alpha) \le p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_1}(\alpha) \ge p_j \circ \psi^n_{S_2}(\phi(\alpha))$ for all $\alpha \in V_1$ and $p_j \circ \psi^p_{T_1}(\alpha, \beta) = p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)), p_j \circ \psi^n_{T_1}(\alpha, \beta) = p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta))$ for all $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$. $p_j \circ \psi^p_{T_1}(\alpha, \beta) = p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = min(p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^p_{S_2}(\phi(\beta))) \ge min(p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^p_{S_1}(\beta))$. But, by the definition, $p_j \circ \psi^p_{T_1}(\alpha, \beta) \le min(p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^p_{S_1}(\beta))$ for all $(\alpha, \beta) \in \overleftrightarrow{V_1^2}$. So, $p_j \circ \psi^p_{T_1}(\alpha, \beta) = min(p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^p_{S_1}(\beta))$ for all $(\alpha, \beta) \in E_1$. Similarly, $p_j \circ \psi^n_{T_1}(\alpha, \beta) = \max(p_j \circ \psi^n_{S_1}(\alpha), p_j \circ \psi^n_{S_1}(\beta)) \forall (\alpha, \beta) \in E_1$. Therefore G_1 is strong. Also for $\alpha \in V_1, \sum_{\substack{\alpha \ne \beta \\ (\alpha,\beta) \in E_1}} p_j \circ \psi^p_{T_1}(\alpha, \beta) = \sum_{\substack{\phi(\alpha) \ne \phi(\beta) \\ (\phi(\alpha),\phi(\beta)) \in E_2}} p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta))$= constant and

$$\sum_{\substack{\alpha \ne \beta \\ (\alpha,\beta) \in E_1}} p_j \circ \psi^n_{T_1}(\alpha, \beta) = \sum_{\substack{\phi(\alpha) \ne \phi(\beta) \\ (\phi(\alpha),\phi(\beta)) \in E_2}} p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) = constant,$$

since G_2 is regular. Therefore G_1 is regular. □

Theorem 3.6. *Let $G_1 = (V_1, S_1, T_1)$ and $G_2 = (V_2, S_2, T_2)$ be two isomorphic m-BPFGs. Then G_1 is strong regular if and only if G_2 is strong regular.*

Proof. As an m-BPFG G_1 is isomorphic to G_2, there exists an isomorphism $\phi : V_1 \to V_2$ which is bijective such that for $j = 1, 2, \dots, m$, $p_j \circ \psi^p_{S_1}(\alpha) = p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_1}(\alpha) = p_j \circ \psi^n_{S_2}(\phi(\alpha)) \forall \alpha \in V_1$ and $p_j \circ \psi^n_{T_1}(\alpha, \beta) = p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)), p_j \circ \psi^n_{T_1}(\alpha, \beta) = p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) \forall (\alpha, \beta) \in \overleftrightarrow{V_1^2}$. Now G_1 is strong if and only if $p_j \circ \psi^p_{T_1}(\alpha, \beta) = min(p_j \circ \psi^p_{S_1}(\alpha), p_j \circ \psi^p_{S_1}(\beta))$, $p_j \circ \psi^n_{T_1}(\alpha, \beta) = \max(p_j \circ \psi^n_{S_1}(\alpha), p_j \circ \psi^n_{S_1}(\beta)) \forall (\alpha, \beta) \in E_1$ if and only if $p_j \circ \psi^p_{T_2}(\phi(\alpha), \phi(\beta)) = min(p_j \circ \psi^p_{S_2}(\phi(\alpha)), p_j \circ \psi^p_{S_2}(\phi(\beta)))$, $p_j \circ \psi^n_{T_2}(\phi(\alpha), \phi(\beta)) = \max(p_j \circ \psi^n_{S_2}(\phi(\alpha)), p_j \circ \psi^n_{S_2}(\phi(\beta))) \forall (\phi(\alpha), \phi(\beta)) \in E_2$ if and only if G_2 is strong. As G_1 is regular if and only if for all $\alpha \in V_1, \sum_{\substack{\alpha \ne \beta \\ (\alpha,\beta) \in E_1}} p_j \circ \psi^p_{T_1}(\alpha, \beta)$ = constant and $\sum_{\substack{\alpha \ne \beta \\ (\alpha,\beta) \in E_1}} p_j \circ \psi^n_{T_1}(\alpha, \beta)$ = constant if and only if for all $\phi(\alpha) \in V_2, \sum_{\substack{\phi(\alpha) \ne \phi(\beta) \\ (\phi(\alpha),\phi(\beta)) \in E_2}} p_j \circ \psi^p_{T_2}(\phi(\alpha),$

$\phi(\beta))$ = constant and $\sum_{\phi(\alpha)\neq\phi(\beta)} p_j \circ \psi^n_{T_2}(\phi(\alpha),\phi(\beta))$ =constant, if, and only if, G_2 is regular. □

Theorem 3.7. *A strong m-BPFG $G = (V, S, T)$ is strong regular if and only if its complement $\overline{G} = (V, \overline{S}, \overline{T})$ is strong regular.*

Proof. As an m-BPFG $G = (V, S, T)$ is strong m-BPFG, then $\overline{G} = (V, \overline{S}, \overline{T})$ is also strong m-BPFG where $\overline{S} = S$ and $\overline{T}$ is defined by $p_j \circ \psi_{\overline{T}}(\alpha, \beta) = \Big[p_j \circ \psi^p_{\overline{T}}(\alpha,\beta), p_j \circ \psi^n_{\overline{T}}(\alpha,\beta)\Big], p_j \circ \psi^p_{\overline{T}}(\alpha,\beta) = \{p_j \circ \psi^p_S(\alpha) \wedge p_j \circ \psi^p_S(\beta)\} - p_j \circ \psi^p_T(\alpha,\beta), p_j \circ \psi^n_{\overline{T}}(\alpha,\beta) = \{p_j \circ \psi^n_S(\alpha) \vee p_j \circ \psi^n_S(\beta)\} - p_j \circ \psi^n_T(\alpha,\beta)$for every $(\alpha,\beta) \in \overleftrightarrow{V^2}$ and $j = 1, 2, \ldots, m$. Now G is strong regular if and only if $p_j \circ \psi^p_T(\alpha,\beta) = \{p_j \circ \psi^p_S(\alpha) \wedge p_j \circ \psi^p_S(\beta)\}, p_j \circ \psi^n_T(\alpha,\beta) = \{p_j \circ \psi^n_S(\alpha) \vee p_j \circ \psi^n_S(\beta)\}$ if and only if $p_j \circ \psi^p_{\overline{T}}(\alpha,\beta) = \{p_j \circ \psi^p_S(\alpha) \wedge p_j \circ \psi^p_S(\beta)\} - p_j \circ \psi^p_T(\alpha,\beta) = p_j \circ \psi^p_T(\alpha,\beta) - p_j \circ \psi^p_T(\alpha,\beta) = 0$, $p_j \circ \psi^n_{\overline{T}}(\alpha,\beta) = \{p_j \circ \psi^n_S(\alpha) \vee p_j \circ \psi^n_S(\beta)\} - p_j \circ \psi^n_T(\alpha,\beta) = p_j \circ \psi^n_T(\alpha,\beta) - p_j \circ \psi^n_T(\alpha,\beta) = 0$ if and only if $p_j \circ \psi^p_{\overline{T}}(\alpha,\beta) = 0, p_j \circ \psi^n_{\overline{T}}(\alpha,\beta) = 0$ if and only if $\overline{G} = (V, \overline{S}, \overline{T})$ is strong regular. □

Conclusions

We studied the properties of weak, co-weak, isomorphism and morphism between two m-BPFGs in this article. In future we intend to extend our work to study the properties of strongly edge irregular m-BPFGs.

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