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FIXED POINT THEOREMS IN A GENERALIZED S-METRIC SPACE

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ABSTRACT. In this paper, we provide a generalization of an S-metric space by relaxing the triangle inequality. As applications, we provide some fixed point theorems of mappings with common fixed points in the generalized S-metric space.

1. INTRODUCTION

Metric spaces is a very important concept in Mathematics with a wide range of applicability in many fields in applied sciences. Many authors, have given generalizations of metric spaces in several ways. G$\ddot{a}$hler, introduced the concept of 2-metric spaces, [2] and Dhage, [1] introduced the concepts of D-metric spaces. Mustafa al et., introduced a new structure of a generalized metric space which they called G-metric spaces as a generalization of metric spaces, [3]. They developed and introduced new fixed point theory for various mappings in this new space.

Sam al et. established some useful propositions to show that many fixed point theorems on (non-symmetric) G-metric spaces follow directly from results on metric spaces, [5].

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Sedghi et al. introduced D^*-metric spaces, [6] which are modifications of the definition of D-metric spaces introduced by Dhage, [1]. The authors, further introduced the concept of S-metric space and gave some properties with applications as common fixed point theorems for self mappings on complete S-metric spaces, [7].

Definition 1.1. *Let X be a nonempty set. A function $S : X \times X \times X \to [0, \infty)$ is a S-metric on X if for all $x, y, z, w \in X$:*

(i) $S(x, y, z) = 0 \iff x = y = z$

(ii) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$

The pair (X, S) is called an S-metric space, [7].

Example 1. *Let $X = \mathbb{R}^n$ and $\|\cdot\|_X$ be a norm on X, then the function S defined by*

$$S(x, y, z) = \|y + z - 2x\|_X + \|y - z\|_X$$

is an S- metric on X.

Definition 1.2. *Let X be a nonempty set. A function $S_b : X \times X \times X \to [0, \infty)$ and assume that there exists a real number $\alpha \geq 1$, is an S_b-metric on X if for all $x, y, z, w \in X$:*

(i) $S_b(x, y, z) = 0 \iff x = y = z$

(ii) $S_b(x, x, y) = S_b(y, y, x)$ *for all* $x, y \in X$.

(iii) $S_b(x, y, z) \leq \alpha \left[S_b(x, x, w) + S_b(y, y, w) + S_b(z, z, w)\right]$

The pair (X, S_b) is called an S_b-metric space, [9], [8]. If $\alpha = 1$, we have that the S_b-metric is equivalent to the S-metric. It should be noted that the symmetry property follows from the triangle property with $\alpha = 1$.

Definition 1.3. *Let X be a nonempty set. A function $S_{\alpha\beta\gamma} : X \times X \times X \to [0, \infty)$ and assume that there exists real numbers $\alpha, \beta, \gamma \geq 1$ is an $S_{\alpha\beta\gamma}$-metric on X if for all $x, y, z, w \in X$:*

(i) $S_{\alpha\beta\gamma}(x, y, z) = 0 \iff x = y = z$

(ii) $S_{\alpha\beta\gamma}(x, y, z) \leq \alpha S_{\alpha\beta\gamma}(x, x, w) + \beta S_{\alpha\beta\gamma}(y, y, w) + \gamma S_{\alpha\beta\gamma}(z, z, w)$

The pair $(X, S_{\alpha\beta\gamma})$ is called an $S_{\alpha\beta\gamma}$-metric space. If $\alpha = \beta = \gamma = 1$, we obtain that $S = S_{\alpha\beta\gamma}$. If $\alpha = \beta = \gamma$ then we obtain that $S_{\alpha\beta\gamma} = S_b$. Furthermore, if $\alpha, \beta \geq 1$ and $\gamma = 1$ then we have the symmetry property, $S_{\alpha\beta\gamma}(x, x, y) =$

$S_{\alpha\beta\gamma}(y, y, x)$ for all $x, y \in X$. The following example justifies the weakening in the triangle inequality found in Definition 1.3.

Example 2. *Let* $X = (1, 2)$ *and define* $S_{\alpha\beta\gamma}(x, y, z)$ *by*

$$S_{\alpha\beta\gamma}(x, y, z) = \begin{cases} 2^{|x-y|+|y-z|+|z-x|} & x \neq y \neq z \\ 0 & x = y = z. \end{cases}$$

It suffices to verify property (iii) of Definition 1.3. For $x \neq y \neq z$ *we have*

$$\begin{aligned}
& S_{\alpha\beta\gamma}(x, y, z) \\
& = 2^{|x-y|+|y-z|+|z-x|} \\
& \leq 2^{|x-w|+|w-y|+|y-w|+|w-z|+|z-w|+|w-x|} \\
& = 2^{2|x-w|+2|y-w|+2|z-w|} \\
(1.1) \qquad & = 2^{\frac{1}{2}(2|x-w|)+\frac{3}{8}(2|y-w|)+\frac{1}{8}(2|z-w|)} 2^{|x-w|+\frac{5}{4}|y-w|+\frac{7}{4}|z-w|} \\
(1.2) \qquad & \leq \left[\frac{1}{2}\left(2^{2|x-w|}\right) + \frac{3}{8}\left(2^{2|y-w|}\right) + \frac{1}{8}\left(2^{2|z-w|}\right)\right] \sup_{(1,2)} 2^{|x-w|+\frac{5}{4}|y-w|+\frac{7}{4}|z-w|} \\
& = 8S_{\alpha\beta\gamma}(x, x, w) + 6S_{\alpha\beta\gamma}(y, y, w) + 2S_{\alpha\beta\gamma}(z, z, w),
\end{aligned}$$

where we have obtained (1.2) *from* (1.1) *by using Jensen's inequality, [4].*

Definition 1.4. *Let* $(X, S_{\alpha\beta\gamma})$ *a* $S_{\alpha\beta\gamma}$*-metric space. For* $\epsilon > 0$ *and* $x \in X$*, we define the open ball* $B_{S_{\alpha\beta\gamma}}(x, \epsilon) = \{y \in X; S_{\alpha\beta\gamma}(y, y, x) < \epsilon\}$*.*

Definition 1.5. *Let* $(X, S_{\alpha\beta\gamma})$ *be a* $S_{\alpha\beta\gamma}$*-metric space and* $A \subset X$*:*

(i) *If for every* $x \in A$ *there exists* $\epsilon > 0$ *such that* $B_{S_{\alpha\beta\gamma}}(x, \epsilon) \subset A$*, then the subset* A *is open.*

(ii) *Subset* A *is bounded if there exists* $\epsilon > 0$ *such that* $S_{\alpha\beta\gamma}(x, x, y) < \epsilon$ *for all* $x, y \in A$*.*

(iii) *A sequence* $\{x_n\}$ *in* X *converges to* $x \in X \iff$ *for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $S_{\alpha\beta\gamma}(x_n, x_n, x) < \epsilon$ *for all* $n \geq N$*.*

(iv) *A sequence* $\{x_n\}$ *is a Cauchy sequence if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $S_{\alpha\beta\gamma}(x_n, x_n, x_m) < \epsilon$ *for all* $n, m \geq N$*.*

(v) *The* $S_{\alpha\beta\gamma}$*-metric space* $(X, S_{\alpha\beta\gamma})$ *is complete if every Cauchy sequence in* X *is convergent.*

Lemma 1.1. *Let* $(X, S_{\alpha\beta\gamma})$ *be an* $S_{\alpha\beta\gamma}$*-metric space. If a sequence in* X *is convergent then the limit point is unique.*

Proof. Let $\{x_n\}$ be a convergent sequence in X. Then for every $\epsilon > 0$ there exists $x \in X$ and $N_1 \in \mathbb{N}$ such that $S_{\alpha\beta\gamma}(x_n, x_n, x) < \frac{\epsilon}{2(\alpha+\beta)\gamma}$ for all $n \geq N_1$. Assume that there exist $y \in X$ and $N_2 \in \mathbb{N}$ such that $S_{\alpha\beta\gamma}(x_n, x_n, y) < \frac{\epsilon}{2\gamma^2}$ for all $n \geq N_2$. From Definition 1.3, property (ii) it follows that

$$\begin{aligned} S_{\alpha\beta\gamma}(x, x, y) &\leq \alpha S_{\alpha\beta\gamma}(x, x, x_n) + \beta S_{\alpha\beta\gamma}(x, x, x_n) + \gamma S_{\alpha\beta\gamma}(y, y, x_n) \\ &= (\alpha + \beta) S_{\alpha\beta\gamma}(x, x, x_n) + \gamma S_{\alpha\beta\gamma}(y, y, x_n) \\ &\leq (\alpha + \beta)\gamma S_{\alpha\beta\gamma}(x_n, x_n, x) + \gamma^2 S_{\alpha\beta\gamma}(x_n, x_n, y) \\ &< \epsilon \end{aligned}$$

for all $n \geq \max\{N_1, N_2\}$. It follows that $S_{\alpha\beta\gamma}(x, x, y) = 0$ thus we get $x = y$. □

2. Some fixed point results

Definition 2.1. *Let* $(X, S_{\alpha\beta\gamma})$ *be a* $S_{\alpha\beta\gamma}$*-metric space. A mapping* $T : X \to X$ *is a contraction if there exists a constant* $0 \leq \lambda < 1$ *such that*

$$S_{\alpha\beta\gamma}(Tx, Tx, Ty) \leq \lambda S_{\alpha\beta\gamma}(x, x, y)$$

for all $x, y \in X$.

Theorem 2.1. *Let* $(X, S_{\alpha\beta\gamma})$ *be a complete* $S_{\alpha\beta\gamma}$*-metric space and* $T : X \to X$ *be a contraction with* $0 \leq \lambda < \frac{1}{\gamma^2}$. *Then* T *has a unique fixed point* $x \in X$.

Proof. To show uniqueness, we assume that there exists $x, y \in X$ with $Tx = x$ and $Ty = y$. Then

$$\begin{aligned} S_{\alpha\beta\gamma}(x, x, y) &= S_{\alpha\beta\gamma}(Tx, Tx, Ty) \\ &\leq \lambda S_{\alpha\beta\gamma}(x, x, y) \end{aligned}$$

since $\lambda < 1$, we conclude $S_{\alpha\beta\gamma}(x, x, y) = 0$ thus we get $x = y$. To show existence we show that for $x \in X$ that $\{T^n x\}$ is a Cauchy sequence in X. For $n \in \mathbb{N}$, we recursively obtain that

$$\begin{aligned} S_{\alpha\beta\gamma}(T^n x, T^n x, T^{n+1} x) &\leq \lambda S_{\alpha\beta\gamma}(T^{n-1} x, T^{n-1} x, T^n x) \\ &\vdots \\ &\leq \lambda^n S_{\alpha\beta\gamma}(x, x, Tx) \end{aligned} \tag{2.1}$$

For $n, m \in \mathbb{N}$, and from inequality (2.1), we get

$$\begin{aligned}
&S_{\alpha\beta\gamma}(T^n x, T^n x, T^{n+m} x)\\
&\le (\alpha+\beta) S_{\alpha\beta\gamma}(T^n x, T^n, T^{n+1} x) + (\alpha+\beta)\gamma^2 S_{\alpha\beta\gamma}(T^{n+1} x, T^{n+1}, T^{n+2} x)\\
&\quad + \cdots + (\alpha+\beta)\gamma^{2(m-2)} S_{\alpha\beta\gamma}(T^{n+m-2} x, T^{n+m-2}, T^{n+m-1} x)\\
&\quad + (\gamma)^{2(m-1)} S_{\alpha\beta\gamma}(T^{n+m-1} x, T^{n+m-1}, T^{n+m} x)\\
&\le (\alpha+\beta) \sum_{i=0}^{m-1} \gamma^{2i} S_{\alpha\beta\gamma}(T^{n+i} x, T^{n+i} x, T^{n+i+1} x)\\
&\le (\alpha+\beta) \sum_{i=0}^{m-1} \gamma^{2i} \lambda^{n+i} S_{\alpha\beta\gamma}(x, x, Tx)\\
&\le (\alpha+\beta) \lambda^n S_{\alpha\beta\gamma}(x, x, Tx) \frac{1}{1-(\lambda\gamma^2)}.
\end{aligned}$$

It follows that $\{T^n x\}$ is a Cauchy sequence and since X is complete there exists $x_0 \in X$ such that $\lim_{n\to\infty} T^n x = x_0$. Since T is continuous it follows that $x_0 = \lim_{n\to\infty} T^{n+1} x = \lim_{n\to\infty} T T^n x = T(\lim_{n\to\infty} T^n x) = T x_0$. Therefore x_0 is a fixed point of T. Taking $m \to \infty$, we get

$$S(T^n x, T^n x, x_0) \le (\alpha+\beta)\lambda^n S_{\alpha\beta\gamma}(x, x, Tx) \frac{1}{1-(\lambda\gamma^2)}.$$

□

Example 3. *Let $X = [0,1]$ and define $S_{\alpha\beta\gamma}(x, y, z)$ by*

$$S_{\alpha\beta\gamma}(x, y, z) = \left(\frac{1}{4}|x-y| + \frac{1}{4}|y-z| + \frac{1}{2}|z-x|\right)^2.$$

Then, we have that

$$S_{\alpha\beta\gamma}(x, x, w) = \frac{9}{16}|x-w|^2 \tag{2.2}$$

$$S_{\alpha\beta\gamma}(y, y, w) = \frac{9}{16}|y-w|^2 \tag{2.3}$$

$$S_{\alpha\beta\gamma}(z, z, w) = \frac{9}{16}|z-w|^2 \tag{2.4}$$

and by Jensen's inequality, [4], it follows that

$$S_{\alpha\beta\gamma}(x, y, z) \le \frac{1}{4}|x-y|^2 + \frac{1}{4}|y-z|^2 + \frac{1}{2}|z-x|^2. \tag{2.5}$$

As

$$\begin{aligned}|x-y|^2 &\le (|x-w|+|w-y|)^2\\ &= |x-w|^2+|w-y|^2+2|x-w||y-w|\\ &\le 2|x-w|^2+2|w-y|^2\end{aligned}$$

and similar relations hold for $|y-z|^2$ *and* $|z-x|^2$ *we can simplify* (2.5) *as follows*

$$S_{\alpha\beta\gamma}(x,y,z) \le \frac{3}{2}|x-w|^2+|y-w|^2+\frac{3}{2}|z-w|^2.$$

Finally using (2.2)-(2.4) *we conclude that*

$$S_{\alpha\beta\gamma}(x,y,z) \le \frac{24}{9}S_{\alpha\beta\gamma}(x,x,w)+\frac{16}{9}S_{\alpha\beta\gamma}(y,y,w)+\frac{24}{9}S_{\alpha\beta\gamma}(z,z,w).$$

It follows that $(X,S_{\alpha\beta\gamma})$ *is a complete* $S_{\alpha\beta\gamma}$*-metric space. Let* $T:X\to X$ *defined by*

$$Tx=\frac{1}{x+2},$$

then T *is a contraction on* X *as shown below:*

$$\begin{aligned}S_{\alpha\beta\gamma}(Tx,Tx,Ty) &= \frac{9}{16}|Tx-Ty|^2\\ &= \frac{9}{16}\left|\frac{1}{x+2}-\frac{1}{y+2}\right|^2\\ &= \frac{9}{16}\frac{|x-y|^2}{|x+2|^2|y+2|^2}\\ &\le \frac{1}{16}S_{\alpha\beta\gamma}(x,x,y),\end{aligned}$$

where $\frac{1}{16}=\lambda\le\frac{1}{\gamma^2}=\left(\frac{9}{24}\right)^2$. *Thus by Theorem 2.1,* T *has a fixed point* $x^\star=\sqrt{2}-1\in X$.

3. Fixed point results of mappings with common fixed points

Lemma 3.1. *Let* $(X,S_{\alpha\beta\gamma})$ *be an* $S_{\alpha\beta\gamma}$*-metric space and assume that there exists a sequence* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to\infty}S_{\alpha\beta\gamma}(x_n,x_n,y_n)=0$ *whenever* $\{x_n\}$ *is a sequence such that* $\lim_{n\to\infty}S_{\alpha\beta\gamma}(x,x,x_n)=0$ *for some* $x\in X$ *then* $\lim_{n\to\infty}y_n=x$.

Proof. From Definition 1.3, property (ii) we get

$$\begin{aligned}S_{\alpha\beta\gamma}(y_n, y_n, x) &\leq (\alpha+\beta)S_{\alpha\beta\gamma}(y_n, y_n, x_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\\ &\leq (\alpha+\beta)\gamma S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n).\end{aligned}$$

It follows that $\lim_{n\to\infty}\sup\{(\alpha+\beta)S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\} = 0$ since $S_{\alpha\beta\gamma}(\cdot,\cdot,\cdot) \geq 0$ we get

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\inf\{(\alpha+\beta)S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\}\\ &\leq \lim_{n\to\infty}\sup\{(\alpha+\beta)S_{\alpha\beta\gamma}(x_n, x_n, y_n) + \gamma S_{\alpha\beta\gamma}(x, x, x_n)\} = 0.\end{aligned}$$

Hence, we get $\lim_{n\to\infty} S_{\alpha\beta\gamma}(y_n, y_n, x) = 0$ thus we obtain $\lim_{n\to\infty} y_n = x$. □

Definition 3.1. *Let $(X, S_{\alpha\beta\gamma})$ be a $S_{\alpha\beta\gamma}$-metric space. A pair of mappings $\{f, g\}$ are compatible iff $\lim_{n\to\infty} S_{\alpha\beta\gamma}(fgx_n, fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.*

Theorem 3.1. *Assume that f, g, F, G are self maps of a complete $S_{\alpha\beta\gamma}$-metric space $(X, S_{\alpha\beta\gamma})$ with $f(X) \subset F(X)$, $g(X) \subset G(X)$ and the pairs $\{f, G\}$, $\{g, F\}$ are compatible. If*

$$\begin{aligned}&S_{\alpha\beta\gamma}(fx, fy, gz)\\ &\leq \lambda\max\{S_{\alpha\beta\gamma}(Gx, Gy, Fz), S_{\alpha\beta\gamma}(fx, fx, Gx),\\ &\qquad S_{\alpha\beta\gamma}(gz, gz, Fz), S_{\alpha\beta\gamma}(fy, fy, gz)\}\end{aligned}\tag{3.1}$$

for $x, y, z \in X$ with $0 < \lambda < \frac{1}{\gamma^4}$. Then mappings f, g, F, G have a unique common fixed point in X provided that F, G are continuous.

Proof. Let $x_0 \in X$ then $fx_0 = Fx_1$ for some $x_1 \in X$ since $f(X) \subset F(X)$ and $gx_1 = Gx_2$ for some $x_2 \in X$ since $g(X) \subset G(X)$. In general, we get $y_{2n} = fx_{2n} = Fx_{2n+1}$ for some $x_{2n+1} \in X$ and $y_{2n+1} = gx_{2n+1} = Gx_{2n+2}$ for some $x_{2n+2} \in X$. We shall show that the sequence $\{y_n\}$ is a Cauchy sequence in X. For the sequence $\{y_n\}$ using the inequality (3.1), we get

$$\begin{aligned}&S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\\ &\quad= S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gx_{2n+1})\\ &\quad\leq \lambda\max\{S_{\alpha\beta\gamma}(Gx_{2n}, Gx_{2n}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, Gx_{2n}),\end{aligned}$$

$$S_{\alpha\beta\gamma}(gx_{2n+1}, gx_{2n+1}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gx_{2n+1})\}$$
$$\le \lambda \max\{S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}),$$
$$S_{\alpha\beta\gamma}(y_{2n+1}, y_{2n+1}, y_{2n}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\}$$
$$\le \lambda\gamma \max\{S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\} \tag{3.2}$$

If $S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) > S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n})$ then from inequality (3.2) we get

$$\begin{aligned} S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) &\le \lambda\gamma \max\{S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1})\} \\ &< S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) \end{aligned}$$

is a contradiction. Hence, $S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) \le S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n})$, and therefore

$$\begin{aligned} S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n+1}) &\le \lambda\gamma S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &\le \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}). \end{aligned} \tag{3.3}$$

In a similar manner, have that

$$S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n})$$
$$\le \gamma S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})$$
$$= \gamma S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gx_{2n-1})$$
$$\le \lambda\gamma \max\{S_{\alpha\beta\gamma}(Gx_{2n}, G_{2n}, Fx_{2n-1}), S_{\alpha\beta\gamma}(fx_{2n}, f_{2n}, Gx_{2n})\ ,$$
$$S_{\alpha\beta\gamma}(gx_{2n-1}, gx_{2n-1}, Fx_{2n-1}), S_{\alpha\beta\gamma}(fx_{2n}, f_{2n}, gx_{2n-1})\}$$
$$= \gamma\lambda \max\{S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})\ ,$$
$$S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2}), S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1})\}$$
$$= \gamma\lambda \max\{S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}), S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2})\}$$

If $S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}) > S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2})$, then it follows that

$$\begin{aligned} S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) &\le \lambda\gamma S_{\alpha\beta\gamma}(y_{2n}, y_{2n}, y_{2n-1}) \\ &\le \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}), \end{aligned}$$

which is a contradiction. Hence,

$$\begin{aligned} S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n}) &\leq \lambda\gamma S_{\alpha\beta\gamma}(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{2n-2}, y_{2n-2}, y_{2n-1}) \end{aligned} \tag{3.4}$$

Thus, from inequality (3.3) and (3.4) we obtain

$$S_{\alpha\beta\gamma}(y_n, y_n, y_{n-1}) \leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{n-1}, y_{n-1}, y_{n-2}),$$

where $\lambda\gamma^2 < 1$ and $n \geq 2$. It follows that repeated application of inequality (3),

$$\begin{aligned} S_{\alpha\beta\gamma}(y_n, y_n, y_{n-1}) &\leq \lambda\gamma^2 S_{\alpha\beta\gamma}(y_{n-1}, y_{n-1}, y_{n-2}) \\ &\vdots \\ &\leq (\lambda\gamma^2)^{n-1} S_{\alpha\beta\gamma}(y_1, y_1, y_0). \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$S_{\alpha\beta\gamma}(y_n, y_n, y_{n+1}) \leq \gamma S_{\alpha\beta\gamma}(y_{n+1}, y_{n+1}, y_n) \leq \gamma(\lambda\gamma^2)^n S_{\alpha\beta\gamma}(y_1, y_1, y_0).$$

For $n, m \in \mathbb{N}$ we get

$$\begin{aligned} &S_{\alpha\beta\gamma}(y_n, y_n, y_{n+m}) \\ &\leq (\alpha+\beta) S_{\alpha\beta\gamma}(y_n, y_n, y_{n+1}) + (\alpha+\beta)\gamma^2 S_{\alpha\beta\gamma}(y_{n+1}, y_{n+1}, y_{n+2}) + \cdots \\ &\quad + (\alpha+\beta)(\gamma^2)^{m-2} S_{\alpha\beta\gamma}(y_{n+m-2}, y_{n+m-2}, y_{n+m-1}) \\ &\quad + (\gamma^2)^{m-1} S_{\alpha\beta\gamma}(y_{n+m-1}, y_{n+m-1}, y_{n+m}) \\ &\leq (\alpha+\beta) \sum_{i=0}^{m-1} (\gamma)^{2i} S_{\alpha\beta\gamma}(y_{n+i}, y_{n+i}, y_{n+i+1}) \\ &\leq (\alpha+\beta)\gamma(\lambda\gamma^2)^n \sum_{i=0}^{m-1} \left((\gamma)^4\lambda\right)^i S_{\alpha\beta\gamma}(y_1, y_1, y_0) \\ &< (\alpha+\beta)\gamma(\lambda\gamma^2)^n \frac{1}{1-\gamma^4\lambda} S_{\alpha\beta\gamma}(y_1, y_1, y_0), \end{aligned}$$

since $\lambda\gamma^2 < 1$, it follows that $\{y_n\}$ is a Cauchy sequence in a complete $S_{\alpha\beta\gamma}$-metric space, thus there exists $y \in X$ such that $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Fx_{2n+1} = y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Gx_{2n+2}$ We shall now show that y is a common fixed point for mappings f, g, F, G. Since G is continuous, we get $\lim_{n\to\infty} G(Gx_{2n+2}) = Gy$ and $\lim_{n\to\infty} Gfx_{2n} = Gy$ since

f, G are compatible $\lim_{n\to\infty} S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, Gfx_{2n}) = 0$ so by Lemma 3.1 it follows that $\lim_{n\to\infty} fGx_{2n} = Gy$. It follows from inequality (3.1),

$$\begin{aligned} &S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, gx_{2n+1}) \\ &\leq \lambda \max\left\{S_{\alpha\beta\gamma}(GGx_{2n}, GGx_{2n}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, GGx_{2n}),\right. \\ &\quad \left. S_{\alpha\beta\gamma}(gx_{2n+1}, gx_{2n+1}, Fx_{2n+1}), S_{\alpha\beta\gamma}(fGx_{2n}, fGx_{2n}, gx_{2n+1})\right\}. \end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\begin{aligned} &S_{\alpha\beta\gamma}(Gy, Gy, y) \\ &\quad \leq \lambda \max\left\{S_{\alpha\beta\gamma}(Gy, Gy, y), S_{\alpha\beta\gamma}(Gy, Gy, Gy), S_{\alpha\beta\gamma}(y, y, y), S_{\alpha\beta\gamma}(Gy, Gy, y)\right\} \\ &\quad = \lambda S_{\alpha\beta\gamma}(Gy, Gy, y), \end{aligned}$$

since $\lambda < 1$, we get $S_{\alpha\beta\gamma}(Gy, Gy, y) = 0$ thus $Gy = y$. In a similar manner, since F is continuous we get, $\lim_{n\to\infty} FFx_{2n+1} = Fy$, $\lim_{n\to\infty} Fgx_{2n+1} = Fy$ since g and F are compatible, $\lim_{n\to\infty} S_{\alpha\beta\gamma}(gFx_{2n+1}, gFx_{2n+1}, Fgx_{2n+1}) = 0$ and it follows that $\lim_{n\to\infty} gFx_{2n+1} = Fy$. From inequality (3.1),

$$\begin{aligned} &S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gFx_{2n+1}) \\ &\leq \lambda \max\left\{S_{\alpha\beta\gamma}(Gx_{2n}, Gx_{2n}, FFx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, Gx_{2n}),\right. \\ &\quad \left. S_{\alpha\beta\gamma}(gFx_{2n+1}, gFx_{2n+1}, FFx_{2n+1}), S_{\alpha\beta\gamma}(fx_{2n}, fx_{2n}, gFx_{2n+1})\right\} \end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\begin{aligned} &S_{\alpha\beta\gamma}(y, y, Fy) \\ &\quad \leq \lambda \max\left\{S_{\alpha\beta\gamma}(y, y, Fy), S_{\alpha\beta\gamma}(y, y, y),\right. \\ &\left. S_{\alpha\beta\gamma}(Fy, Fy, Fy), S_{\alpha\beta\gamma}(y, y, Fy)\right\} \\ &\quad \leq \lambda S_{\alpha\beta\gamma}(y, y, Fy), \end{aligned}$$

since $\lambda < 1$, it follows that $Fy = y$. Furthermore, we obtain that

$$\begin{aligned} &S_{\alpha\beta\gamma}(fy, fy, gx_{2n+1}) \\ &\leq \lambda \left\{S_{\alpha\beta\gamma}(Gy, Gy, Fx_{2n+1}), S_{\alpha\beta\gamma}(fy, fy, Gy), S_{\alpha\beta\gamma}(gx_{2n+1}, gx_{2n+1}, Fx_{2n+1}),\right. \\ &\left. S_{\alpha\beta\gamma}(fy, fy, gx_{2n+1})\right\}. \end{aligned}$$

Taking the limit $n \to \infty$, and $Gy = Fy = y$ we have

$$S_{\alpha\beta\gamma}(fy, fy, y) \\ \leq \lambda \max\{S_{\alpha\beta\gamma}(Gy, Gy, y), S_{\alpha\beta\gamma}(fy, fy, y), \\ S_{\alpha\beta\gamma}(y, y, y), S_{\alpha\beta\gamma}(fy, fy, y)\} \\ = \lambda S_{\alpha\beta\gamma}(fy, fy, y),$$

since $\lambda < 1$, $fy = y$. Finally, we have $Gy = Fy = fy = y$ and

$$S_{\alpha\beta\gamma}(y, y, gy) = S_{\alpha\beta\gamma}(fy, fy, gy) \\ \leq \lambda \max\{S_{\alpha\beta\gamma}(Gy, Gy, Fy), S_{\alpha\beta\gamma}(fy, fy, Gy), S_{\alpha\beta\gamma}(gy, gy, Fy), S_{\alpha\beta\gamma}(fy, fy, gy)\} \\ = \lambda S_{\alpha\beta\gamma}(y, y, gy).$$

It follows that $gy = y$. Thus we get $Fy = Gy = gy = fy = y$. It remains to show that the common fixed point is unique. Assume that there exists $x \in X$ such that $Fx = Gx = gx = fx = x$ then

$$S_{\alpha\beta\gamma}(x, x, y) = S_{\alpha\beta\gamma}(fx, fx, gy) \\ \leq \lambda \max\{S_{\alpha\beta\gamma}(Gx, Gx, Fy), S_{\alpha\beta\gamma}(fx, fx, Gy), S_{\alpha\beta\gamma}(gy, gy, Fy), S_{\alpha\beta\gamma}(fx, fx, gy)\} \\ = \lambda \max\{S_{\alpha\beta\gamma}(x, x, y), S_{\alpha\beta\gamma}(x, x, x), S_{\alpha\beta\gamma}(x, x, y)\} \\ = \lambda S_{\alpha\beta\gamma}(x, x, y),$$

which implies that $S_{\alpha\beta\gamma}(x, x, y) = 0$ thus $x = y$. □

Corollary 3.1. *Let* $(X, S_{\alpha\beta\gamma})$ *be a complete* $S_{\alpha\beta\gamma}$*- metric space and let* $f, g : X \to X$ *be mappings such that*

$$S_{\alpha\beta\gamma}(fx, fy, gz) \\ \leq \lambda \max\{S_{\alpha\beta\gamma}(x, y, z), S_{\alpha\beta\gamma}(fx, fx, x), \\ S_{\alpha\beta\gamma}(gz, gz, z), S_{\alpha\beta\gamma}(fy, fy, gz)\}$$

for all $x, y, z \in X$ *with* $0 \leq \lambda < 1$ *then there exists a unique fixed point for mappings* f *and* g.

Proof. The proof follows in a similar manner as in Theorem 3.1, by taking mappings F and G as identity mappings. □

4. Conclusion

The results in the paper, demonstrate that the fixed point results can be extended to the generalized space.

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