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ON LAPLACIAN SPECTRA OF SOME CORONA PRODUCT GRAPHS AND APPLICATIONS

Idweep J. Gogoi, Bablee Phukan, Aditya Pegu, and A. Bharali[1]

ABSTRACT. The corona operations on graphs have attracted many researchers because of its applications in various fields. Many variants of the corona operation are defined over the years and their spectral properties have also been studied. In this paper we consider the spectra of some of these corona graphs namely; weighted edge corona product graphs, subdivision double corona, Q-graph double corona and total double corona. In this note, we correct some of the results proposed in the literature and also derive expressions for the number of spanning trees and Kirchhoff index of the above mentioned graphs as applications.

1. INTRODUCTION

We consider simple and connected graphs throughout this paper. For a graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, the Laplacian matrix is defined as $L(G) = D(G) - A(G)$. Here $D(G)$ is the diagonal matrix with diagonal entries as the degrees of the vertices of G

[1]*corresponding author*

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and $A(G)$ is the adjacency matrix. The Laplacian spectrum is the collection of all the eigenvalues of $L(G)$ together with its multiplicities.

Let G be a connected graph with n vertices and m edges. The *subdivision graph* $S(G)$ for graph G is formed by adding a new vertex to every edge of G. The $Q(G)$-*graph* for the graph is the one obtained by adding a new vertex to every edge of G and joining these new vertices by edges if they lie on adjacent edges of G. The *total graph* $T(G)$ has its vertex set as the union of the set of vertices and set of edges of G and two vertices are adjacent if and only if the corresponding elements of G are adjacent.

The corona operations was first introduce by Frucht and Harary [4]. Let G_1 and G_2 be two graphs on disjoint sets of n and m vertices, respectively. The *corona* $G_1 \circ G_2$ *of* G_1 *and* G_2 [16] is the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

A lot of research have been done so far on corona of graphs and its variants, also there spectra are described in [1, 5, 7, 9, 10, 12–15]. Barik & Sahoo (2016), introduced some more variants of corona graphs in [2]. They have also described the Laplacian spectra and eigenvectors for these graphs. We have considered the double corona graphs mentioned in [2] and the definitions are as follows: Let G be a connected graph with n vertices and m edges. Let G_1 and and G_2 be graphs with n_1 and n_2 vertices respectively. The *Subdivision double corona* of G, G_1 and G_2, denoted by $G^{(S)}o\{G_1, G_2\}$, is defined as the graph obtained by taking one copy of $S(G)$, n copies of G_1 and m copies of G_2 and by joining the i^{th} old vertex of $S(G)$ to every vertex of the i^{th} copy of G_1 and the j^{th} new-vertex of $S(G)$ to every vertex of j^{th} copy of G_2. Similarly, if we replace $S(G)$ with $Q(G)(T(G))$, then the resulting graph is the *Q-graph(total) double corona* and denoted by $G^{(Q)}o\{G_1, G_2\}(G^{(T)}o\{G_1, G_2\})$.

Let there be two graphs G_1 with order n_1 and size m_1, G_2 the copy graph with order n_2 and size m_2, respectively. The *edge corona* $G_1 \diamond G_2$ [7] of G_1 and G_2 is generated by making one copy of G_1 and m_1 copies of G_2, joining each vertex of the i^{th} copy of G_2 to the two end vertices of the i^{th} edge of G_1. For the weighted edge corona product graph [3], we assign a unit weight on the initial graph G_1 and weight factor r, $0 < r \leq 1$ on the copy graph G_2.

There are many applications of Laplacian eigenvalues of a graph. Two of such applications are to determine the number of spanning trees and Kirchhoff index of the associated graphs.

Definition 1.1. ***Number of spanning trees:*** *[11]*

The number of spanning trees of a given graph G is the number of subgraphs which contains each vertices of G. Also, all those subgraphs must be trees. It can be expressed in terms of Laplacian eigenvalues as

$$\tau(G) = \frac{1}{\mid N(G) \mid} \prod_{i=2}^{|N(G)|} \mu_i,$$

where $\mid N(G) \mid$ *and* μ_i *are respectively the order and Laplacian eigenvalues of G.*

Using resistive electrical networks Klein and Randić (1993) [8] introduced a novel distance function called resistance distance. Here the graph is viewed as an electrical network and each edge is replaced by a unit resistor.

Definition 1.2. ***Kirchhoff index:*** *[8] If G is a connected(molecular) graph then the Kirchhoff index of G denoted by Kf(G) is the sum of resistance distances between all vertex pairs in G, namely*

$$Kf(G) = \sum_{i \leq j} r_{ij},$$

where r_{ij} *is the resistance distance.*

The Kirchhoff index [6, 17] can expressed as the reciprocal of the Laplacian eigenvalues of G, namely

$$Kf(G) = \mid N(G) \mid \sum_{i=2}^{|N(G)|} \frac{1}{\mu_i},$$

The Laplacian spectra of some double corona graphs which are proposed in [2] given as follows.

Lemma 1.1. *[2] Let* G *be a* r*-regular graph on* n *vertices and* m *edges. Let* G_1 *and* G_2 *be any two graphs on* n_1 *and* n_2 *vertices, respectively. Then the laplacian spectrum of* $G^{(S)} o\{G_1, G_2\}$ *consists of*

(i) *all roots of the equation*

$$\lambda^4-(n_1+n_2+r+4)\lambda^3$$
$$+((n_1+1)(n_2+3)+2(r+1)+n_2r+\lambda_i(G))\lambda^2$$
$$-(r(n_2+1)+2(n_1+\lambda_i(G)+1))\lambda+\lambda_i(G)=0,$$

for $i=1,2,\ldots,n$;

(ii) $\frac{n_2+3\pm\sqrt{(n_2+3)^2-8}}{2}$ *repeated* $m-n$ *times each;*

(iii) $\lambda_i(G_1)+1$ *repeated* n *times, for* $i=2,3,\ldots,n_1$;

(iv) $\lambda_i(G_2)+1$ *repeated* m *times, for* $i=2,3,\ldots,n_2$.

Lemma 1.2. *[2] Let G be a r-regular graph on n vertices and m edges. Let G_1 and G_2 be any two graphs on n_1 and n_2 vertices, respectively. Then the laplacian spectrum of $G^{(Q)}o\{G_1,G_2\}$ consists of*

(i) *all roots of the equation*

$$\lambda^4-(n_1+n_2+r+\lambda_i(G)+4)\lambda^3$$
$$+((n_1+r+1)(n_2+\lambda_i(G)+3)+2(\lambda_i(G)+1)-r)\lambda^2$$
$$-(r(n_2+\lambda_i(G)+3)+(2+\lambda_i(G))(n_1+r+1)-2(2r-\lambda_i(G)))\lambda$$
$$+\lambda_i(G)(r+1)=0,$$

for $i=1,2,\ldots,n$;

(ii) $\frac{n_2+2r+3\pm\sqrt{(n_2+2r+1)^2+4n_2}}{2}$ *repeated* $m-n$ *times each;*

(iii) $\lambda_i(G_1)+1$ *repeated* n *times, for* $i=2,3,\ldots,n_1$;

(iv) $\lambda_i(G_2)+1$ *repeated* m *times, for* $i=2,3,\ldots,n_2$.

Lemma 1.3. *[2] Let G be a r-regular graph on n vertices and m edges. Let G_1 and G_2 be any two graphs on n_1 and n_2 vertices, respectively. Then the laplacian spectrum of $G^{(T)}o\{G_1,G_2\}$ consists of*

(i) *all roots of the equation*

$$\lambda^4-(n_1+n_2+r+2\lambda_i(G)+4)\lambda^3$$
$$+((n_1+r+\lambda_i(G)+1)(n_2+\lambda_i(G)+3)+3\lambda_i(G)-r+2)\lambda^2$$
$$-(r+\lambda_i(G))(n_2+\lambda_i(G)+3)+(2+\lambda_i(G))(n_1+r+\lambda_i(G)+1)$$
$$-2(2r-\lambda_i(G)))\lambda+(\lambda_i(G)+r)(\lambda_i(G)+2)+\lambda_i(G)-2r=0,$$

for $i=1,2,\ldots,n$;

(ii) $\frac{n_2+2r+3\pm\sqrt{(n_2+2r+1)^2+4n_2}}{2}$ *repeated* $m-n$ *times each;*

(iii) $\lambda_i(G_1) + 1$ *repeated* n *times, for* $i = 2, 3, \ldots, n_1$;
(iv) $\lambda_i(G_2) + 1$ *repeated* m *times, for* $i = 2, 3, \ldots, n_2$.

2. LAPLACIAN SPECTRA OF THE WEIGHTED $G_1 \diamond G_2$

Recently, Liu et al. in their paper entitled "On the Generalized Adjacency, Laplacian and Signless Laplacian Spectra of the Weighted Edge Corona Networks" have studied a class of the weighted edge corona networks and tried to obtain the generalized adjacency, Laplacian and signless Laplacian spectra in association with two discrete structures. The spectra obtained in [3] have been found to be prone to some errors. Here we report the corrected versions of some of the results proposed in [3].

In the section 3 of [3], the Theorem 3.1 for the spectra of generalized Laplacian of the weighted edge corona networks is found to be incorrect which can be verified by the following example. Consider the graphs G_1 and G_2 which is shown in the Figure 1 and let the weight factor be 1. Now according to the above theorem we obtain the eigenvalues of $L(G_1 \diamond G_2)$ for the aforementioned graph to be 8, -2, 9.65685425, 9.65685425, -1.65685425, -1.65685425, 11.4031243, -1.40312425, 4, 4, 4, 4, whereas the actual eigenvalues of $L(G_1 \diamond G_2)$ should have been 0, 1.1716, 1.1716, 2, 4, 4, 4, 4, 6, 6.8284, 6.8284, 8. The corrected version of the theorem should be as follows.

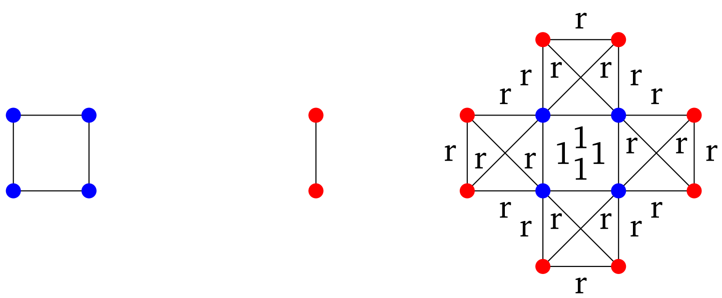

FIGURE 1. G_1, G_2 and $G_1 \diamond G_2$

Theorem 2.1. *Let* G_1 *be a* d_1*-regular graph with order* n_1 *and size* m_1*,* G_2 *be any graph with order* n_2 *and size* m_2*, respectively. Assume that the Laplacian* *spectrum of* G_1 *and* G_2 *are* $l(G_1) = \{0 = \mu_1^{(1)}, \mu_2^{(1)}, \ldots, \mu_{n_1}^{(1)}\}$ *and* $l(G_2) = \{0 = \mu_1^{(2)}, \mu_2^{(2)}, \ldots, \mu_{n_2}^{(2)}\}$. *Then the Laplacian spectra of* $L(G_1 \diamond G_2)$ *are as follows.*

(i) $\frac{2r+rd_1n_2+\mu_i^{(1)} \pm \sqrt{(2r+rd_1n_2+\mu_i^{(1)})^2 - 4(2r\mu_i^{(1)}+r^2n_2\mu_i^{(1)})}}{2} \in l(G_1 \diamond G_2)$ *with multiplicity 1,* $i = 2, 3, \ldots, n_1$.

(ii) 0, $2r + rd_1n_2 \in l(G_1 \diamond G_2)$ *with multiplicity 1.*

(iii) $r(\mu_j^{(2)} + 2) \in l(G_1 \diamond G_2)$ *with multiplicity* m_1, $j = 2, 3, \ldots, n_2$.

(iv) $2r \in l(G_1 \diamond G_2)$ *with multiplicity* $m_1 - n_1$ *(if possible).*

Proof. Consider a row matrix J_{n_2} of order n_2 whose all elements are 1. Now from the definition of weighted edge corona product, we have the Laplacian matrix of weighted $G_1 \diamond G_2$ as below

$$L(G_1 \diamond G_2) = \begin{bmatrix} L(G_1) + rd_1n_2I_{n_1} & -rJ_{n_2} \otimes B(G_1) \\ -r[J_{n_2} \otimes B(G_1)]^T & r[L(G_2) + 2I_{n_2}] \otimes I_{m_1} \end{bmatrix}.$$

Let μ be the Laplacian eigenvalue of $L(G_1 \diamond G_2)$ and the corresponding eigenvector to be $X = [X_1X_2 \cdots X_{n_2+1}]^T, X_1 \in R^{n_1}$ and $X_i \in R^{m_1}$ otherwise.

Now consider the case $\mu \neq 2r$.

Case I. For $X_1 \neq 0$.

According to the definitions of eigenvalues and eigenvectors, we have

$$(L(G_1) + rd_1n_2I_{n_1})X_1 - rB(G_1)(X_2 + X_3 + \cdots + X_{n_2+1}) = \mu X_1. \tag{2.1}$$

Consider the set $E_i = (\overbrace{0_{m_1}0_{m_1}\cdots 0_{m_1}}^{i-1} I_{m_1} \overbrace{0_{m_1}0_{m_1}\cdots 0_{m_1}}^{n_2-i})$ Then

(2.2)
$$\begin{cases} -rB(G_1)^TX_1 + rE_1[(L(G_2) + 2I_{n_2}) \otimes I_{m_1}][X_2 \cdots X_{n_2+1}]^T = \mu X_2, \\ -rB(G_1)^TX_1 + rE_2[(L(G_2) + 2I_{n_2}) \otimes I_{m_1}][X_2 \cdots X_{n_2+1}]^T = \mu X_3, \\ \vdots \\ -rB(G_1)^TX_1 + rE_{n_2}[(L(G_2) + 2I_{n_2}) \otimes I_{m_1}][X_2 \cdots X_{n_2+1}]^T = \mu X_{n_2+1}. \end{cases}$$

By (2.2), we get

$$-rn_2B(G_1)^TX_1 + 2r(X_2 + X_3 + \cdots + X_{n_2+1}) = \mu(X_2 + X_3 + \cdots + X_{n_2+1}).$$

$$\Rightarrow (X_2 + X_3 + \cdots + X_{n_2+1}) = \frac{-rn_2}{\mu - 2r}B(G_1)^TX_1. \tag{2.3}$$

Substituting (2.3) to (2.1), we get

$$
\begin{aligned}
&(L(G_1)+rd_1n_2I_{n_1})X_1 - rB(G_1)\left(\frac{-rn_2}{\mu-2r}\right)B(G_1)^TX_1 = \mu X_1\\
\Rightarrow\ &(L(G_1)+rd_1n_2I_{n_1})X_1 + \frac{2r^2n_2d_1}{\mu-2r}X_1 - \frac{r^2n_2}{\mu-2r}L(G_1)X_1 = \mu X_1 \qquad (2.4)\\
\Rightarrow\ &\left(1-\frac{r^2n_2}{\mu-2r}\right)L(G_1)X_1 = \left(\mu - rd_1n_2 - \frac{2r^2n_2d_1}{\mu-2r}\right)X_1.
\end{aligned}
$$

Let $l(G_1)=\{0=\mu_1^{(1)},\mu_2^{(1)},\ldots,\mu_{n_1}^{(1)}\}$ be the spectrum of $L(G_1)$. From (2.4), we have

$$\mu^2+\mu(-2r-rd_1n_2-\mu_i^{(1)})+(2r+r^2n_2)\mu_i^{(1)}=0$$

Now for $\mu_1^{(1)}=0$, we have $\mu_{1,2}=0,\ 2r+rd_1n_2$.

For $\mu_2^{(1)},\ldots,\mu_{n_1}^{(1)}$ we have,

$$\mu_{1,2}=\frac{2r+rd_1n_2+\mu_i^{(1)}\pm\sqrt{(2r+rd_1n_2+\mu_i^{(1)})^2-4(2r+r^2n_2)\mu_i^{(1)}}}{2}, \qquad (2.5)$$

$i=2,3,\ldots,n_1$.

Case II. For $X_1=0$.

The similar considerations for (2.1) and (2.2) give us

$$
\begin{aligned}
&B(G_1)(X_2+X_3+\cdots+X_{n_2+1})=0,\\
&r[(L(G_2)+2I_{n_2})\otimes I_{m_1}][X_2\cdots X_{n_2+1}]^T=\mu[X_2\cdots X_{n_2+1}]^T.
\end{aligned}
$$

Let the spectrum of $L(G_2)$ is $l(G_2)=\{\mu_1^{(2)},\mu_2^{(2)},\ldots,\mu_{n_2}^{(2)}\}$. Then one can easily get that

$$\mu=r(\mu_j^{(2)}+2), j=2,3,\ldots,n_2. \qquad (2.6)$$

Now, from (2.5) and (2.6) we obtain $(n_2-1)m_1+2n_1$ eigenvalues of $L(G_1\diamond G_2)$. Hence $\mu=2r$ is also an eigenvalue and its multiplicity m_1-n_1. □

3. Some Applications

In this section we derive the formulas of the number of spanning trees and kirchhoff index of the double corona graphs and weighted edge corona product graphs.

3.1. For subdivision graph, Q-graph, and T-graph double corona.

Theorem 3.1. *Let G be a r-regular graph on n vertices and m edges. Let G_1 and G_2 be any two graphs on n_1 and n_2 vertices, respectively. Then*

(i) $\tau(G^{(S)}o\{G_1, G_2\}) = \frac{2^{m-n}\cdot(r(n_2+1)+2(n_1+1))\cdot\prod_{i=2}^{n}\lambda_i(G)\cdot\prod_{i=2}^{n_1}(\lambda_i(G_1)+1)^n\cdot\prod_{i=2}^{n_2}(\lambda_i(G_2)+1)^m}{n(n_1+1)+m(n_2+1)}$.

(ii) $Kf(G^{(S)}o\{G_1, G_2\}) = n(n_1+1)+m(n_2+1)\times\left[\frac{(n_2+3)(m-n)}{2}+\sum_{i=2}^{n_1}\frac{n}{(\lambda_i(G_1)+1)}+\sum_{i=2}^{n_2}\frac{m}{(\lambda_i(G_2)+1)}+\frac{(n_1+1)(n_2+3)+2(r+1)+n_2r}{r(n_2+1)+2(n_1+1)}+\sum_{i=2}^{n}\frac{r(n_2+1)+2(n_1+\lambda_i(G)+1)}{\lambda_i(G)}\right]$.

Proof. The proof goes like this, for tree number, using Definition 1.1 and by the following cases we have,

Case I: For $\lambda_i \neq 0$. By the relation between coefficients and roots of a polynomial we have,

$$x_1x_2x_3x_4 = \lambda_i(G),$$

where $x_1, x_2, x_3 \& x_4$ are roots of equation in Lemma 1.1

case II: For $\lambda_i = 0$,

The fourth order equation of Lemma 1.1 reduces to the cubic equation,

$$\lambda^3-(n_1+n_2+r+4)\lambda^2+((n_1+1)(n_2+3)+2(r+1)+n_2r)\lambda-r(n_2+1)+2(n_1+1)) = 0.$$

Let y_1, y_2, y_3 be its roots, then

$$y_1y_2y_3 = r(n_2+1)+2(n_1+1)),$$

Again,

$$\frac{n_2+3+\sqrt{(n_2+3)^2-8}}{2}\cdot\frac{n_2+3-\sqrt{(n_2+3)^2-8}}{2} = 2.$$

For Kirchhoff index,

By Definition 1.2 and by the following cases, we have

Case I: For $\lambda_i \neq 0$. Using the relation between coefficients and roots of the equation in Lemma 1.1, we have

$$x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 = r(n_2+1) + 2(n_1 + \lambda_i(G) + 1)$$

and

$$\begin{aligned}\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} &= \frac{x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4}{x_1x_2x_3x_4} \\ &= \frac{r(n_2+1) + 2(n_1 + \lambda_i(G) + 1)}{\lambda_i(G)}.\end{aligned}$$

Case II: For $\lambda_i = 0$,
The fourth order equation of Lemma 1.1 reduces to the cubic equation $\lambda^3 - (n_1 + n_2 + r + 4)\lambda^2 + ((n_1+1)(n_2+3) + 2(r+1) + n_2r)\lambda - r(n_2+1) + 2(n_1+1) = 0$. Let y_1, y_2, y_3 be its roots, then $y_1y_2 + y_2y_3 + y_1y_3 = (n_1+1)(n_2+3) + 2(r+1) + n_2r, y_1y_2y_3 = r(n_2+1) + 2(n_1+1)$ and

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} = \frac{y_1y_2 + y_2y_3 + y_1y_3}{y_1y_2y_3} = \frac{(n_1+1)(n_2+3) + 2(r+1) + n_2r}{r(n_2+1) + 2(n_1+1)}.$$

Again

$$\frac{2}{n_2 + 3 + \sqrt{(n_2+3)^3 - 8}} + \frac{2}{n_2 + 3 - \sqrt{(n_2+3)^3 - 8}} = \frac{n_2+3}{2}.$$

Combining the above cases and using Lemma 1.1 we get the required result. □

Similar results corresponding to Q-graph and T-graph can also be obtained as follows, we omit the proofs as they are mutatis mutandis.

Theorem 3.2. *Let G be a r-regular graph on n vertices and m edges. Let G_1 and G_2 be any two graphs on n_1 and n_2 vertices, respectively. Then*

(i) $\tau(G^{(Q)}o\{G_1, G_2\}) = \frac{(rn_2+r+2n_1+2)\cdot\prod_{i=2}^{n}\lambda_i(G)(r+1)\cdot(2r+2)^{m-n}\cdot\prod_{i=2}^{n_1}(\lambda_i(G_1)+1)^n\cdot\prod_{i=2}^{n_2}(\lambda_i(G_2)+1)^m}{n(n_1+1)+m(n_2+1)}$.

(ii) $Kf(G^{(Q)}o\{G_1, G_2\}) = n(n_1+1) + m(n_2+1) \times \Big[\sum_{i=2}^{n}\frac{rn_2+r+2n_1+2+(2r+n_1+3)\lambda_i(G)}{\lambda_i(G)(r+1)} + \frac{n_1n_2+3n_1+rn_2+n_2+2r+5}{rn_2+2n_1+r+2} + \frac{(m-n)(n_2+2r+3)}{2r+2} + \sum_{i=2}^{n_1}\frac{n}{(1+\lambda_i(G_1))} + \sum_{i=2}^{n_2}\frac{m}{(1+\lambda_i(G_2))}\Big]$.

Theorem 3.3. *Let G be a r-regular graph on n vertices and m edges. Let G_1 and G_2 be any two graphs on n_1 and n_2 vertices, respectively. Then*

(i) $\tau(G^{(T)}o\{G_1, G_2\}) = \frac{(rn_2+r+2n_1+2)\cdot\prod_{i=2}^{n}((\lambda_i(G)+r)(\lambda_i(G)+2)+\lambda_i(G)-2r)\cdot(2r+2)^{m-n}}{n(n_1+1)+m(n_2+1)} \cdot \prod_{i=2}^{n_1}(\lambda_i(G_1)+1)^n \cdot \prod_{i=2}^{n_2}(\lambda_i(G_2)+1)^m.$

(ii) $Kf(G^{(T)}o\{G_1, G_2\}) = n(n_1+1) + m(n_2+1) \times \Big[\frac{n_1n_2+3n_1+rn_2+n_2+2r+5}{rn_2+2n_1+r+2} + \sum_{i=2}^{n} \frac{rn_2+(2r+n_2+8+n_1)\lambda_i(G)+2\lambda_i^2(G)+r+2+2n_1}{\lambda_i^2(G)+3\lambda_i(G)+r\lambda_i(G)} + \frac{(m-n)(n_2+2+3)}{2r+2} + \sum_{i=2}^{n_1} \frac{n}{(1+\lambda_i(G_1))} + \sum_{i=2}^{n_2} \frac{m}{(1+\lambda_i(G_2))}\Big].$

3.2. **For weighted edge corona product graphs.** From Theorem 2.1 it is seen that if G_1 is a unicycle graph then $2r$ is not the eigenvalue of the weighted edge corona graphs $G_1 \diamond G_2$ as $m_1 - n_1 = 0$. So, there arises two cases.
Case 1: For the initial graph G_1 which is not unicycle.

In the section 5 of [3], the theorem 5.1 gives the number of spanning trees and Kirchhoff index of weighted edge corona network is also found to be erroneous, which can be verified from following example.

Consider the graph G_1 and G_2 which is shown in the Figure 2 and let the weight factor be 1. Now according to the above theorem we obtain the value of number of spanning trees to be 339738624 and the Kirchhoff index to be 117.33333 whereas their actual values should be 33556455.31 and 77.999276 respectively.

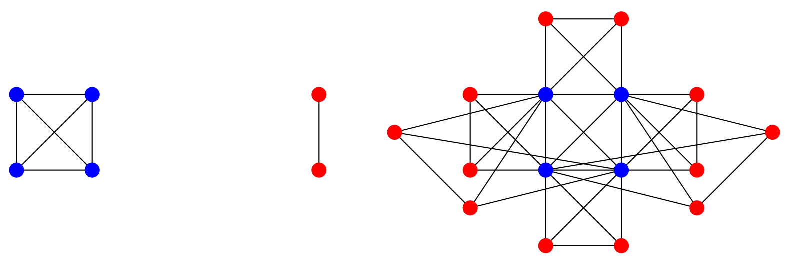

FIGURE 2. G_1, G_2 and $G_1 \diamond G_2$

The correct version of the Theorem 5.1 presented in [3] is as follows.

Theorem 3.4. *Let G_1 be the d_1-regular graph(not unicycle graph) with order n_1 and size m_1, G_2 the any graph with order n_2 and size m_2, respectively. Assume that the Laplacian spectra of G_1 and G_2 are $l(G_1) = \{0 = \mu_1^{(1)}, \mu_2^{(1)}, \ldots, \mu_{n_1}^{(1)}\}$ and $l(G_2) = \{0 = \mu_1^{(2)}, \mu_2^{(2)}, \ldots, \mu_{n_2}^{(2)}\}$. Then one has*

(i) $\tau(G_1 \diamond G_2) = \frac{2r(m_1-n_1)}{n_1+m_1n_2}(2r + rd_1n_2)\prod_{i=2}^{n_1}(2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)})\prod_{j=2}^{n_2}[r(\mu_j^{(2)} + 2)]^{m_1}$.

(ii) $Kf(G_1 \diamond G_2) = (n_1+m_1n_2)\left[\sum_{i=2}^{n_1}\frac{2r+d_1n_2r+\mu_i^{(1)}}{2r\mu_i^{(1)}+n_2r^2\mu_i^{(1)}} + \sum_{j=2}^{n_2}\frac{m_1}{r(\mu_j^{(2)}+2)} + \frac{(m_1-n_1)}{2r} + \frac{1}{(2r+rd_1n_2)}\right]$.

Proof. The order of the weighted edge corona graphs $G_1 \diamond G_2$,

$$| N(G_1 \diamond G_2) |= n_1 + m_1n_2. \tag{3.1}$$

Secondly, let $\chi = \sqrt{(2r + d_1n_2r + \mu_i^{(1)})^2 - 4(2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)})}$. Then

$$\prod_{i=2}^{n_1}\frac{2r + d_1n_2r + \mu_i^{(1)} + \chi}{2}\prod_{i=2}^{n_1}\frac{2r + d_1n_2r + \mu_i^{(1)} - \chi}{2} = \prod_{i=2}^{n_1}(2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)}).$$

This gives

$$\prod_{i=2}^{|N(G)|} = 2r(m_1 - n_1)(2r + rd_1n_2)\prod_{i=2}^{n_1}(2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)})\prod_{j=2}^{n_2}[r(\mu_j^{(2)} + 2)]^{m_1}. \tag{3.2}$$

Combining (3.1) and (3.2), one gets the desired result of the number of spanning trees as below

$$\tau(G_1 \diamond G_2) = \frac{2r(m_1 - n_1)}{n_1 + m_1n_2}(2r + rd_1n_2)\prod_{i=2}^{n_1}(2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)})\prod_{j=2}^{n_2}[r(\mu_j^{(2)} + 2)]^{m_1}.$$

For the Kirchhoff index, one obtains

$$\sum_{i=2}^{n_1}\frac{2}{2r + d_1n_2r + \mu_i^{(1)} + \chi} + \sum_{i=2}^{n_1}\frac{2}{2r + d_1n_2r + \mu_i^{(1)} - \chi} = \sum_{i=2}^{n_1}\frac{2r + d_1n_2r + \mu_i^{(1)}}{2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)}}.$$

This leads

$$\sum_{i=2}^{|N(G)|}\frac{1}{\mu_i} = \sum_{i=2}^{n_1}\frac{2r + d_1n_2r + \mu_i^{(1)}}{2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)}} + \sum_{j=2}^{n_2}\frac{m_1}{r(\mu_j^{(2)} + 2)} + \frac{(m_1 - n_1)}{2r} + \frac{1}{(2r + rd_1n_2)}.$$

Thus one gets

$$Kf(G_1 \diamond G_2) = (n_1 + m_1n_2)\left[\sum_{i=2}^{n_1}\frac{2r+d_1n_2r+\mu_i^{(1)}}{2r\mu_i^{(1)}+n_2r^2\mu_i^{(1)}} + \sum_{j=2}^{n_2}\frac{m_1}{r(\mu_j^{(2)}+2)} + \frac{(m_1-n_1)}{2r} + \frac{1}{(2r+rd_1n_2)}\right]$$

The desired results thus holds.

Case 2: For the initial graph G_1 which is unicycle.

Similarly the corrected version of Theorem 5.2 presented in [3] can be stated as follows. □

Theorem 3.5. *Let G_1 be the d_1-regular graph(not unicycle graph) with order n_1 and size m_1, G_2 the any graph with order n_2 and size m_2, respectively. Assume that the Laplacian spectra of G_1 and G_2 are $l(G_1) = \{0 = \mu_1^{(1)}, \mu_2^{(1)}, \dots, \mu_{n_1}^{(1)}\}$ and $l(G_2) = \{0 = \mu_1^{(2)}, \mu_2^{(2)}, \dots, \mu_{n_2}^{(2)}\}$. Then*

(i) $\tau(G_1 \diamond G_2) = \frac{(2r+rd_1n_2)}{n_1+m_1n_2} \prod_{i=2}^{n_1}(2r\mu_i^{(1)} + n_2r^2\mu_i^{(1)}) \prod_{j=2}^{n_2}[r(\mu_j^{(2)} + 2)]^{m_1}$.

(i) $Kf(G_1 \diamond G_2) = (n_1+m_1n_2)\left[\sum_{i=2}^{n_1} \frac{2r+d_1n_2r+\mu_i^{(1)}}{2r\mu_i^{(1)}+n_2r^2\mu_i^{(1)}} + \sum_{j=2}^{n_2} \frac{m_1}{r(\mu_j^{(2)}+2)} + \frac{1}{(2r+rd_1n_2)}\right]$.

The proof is similar to the above theorem.

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DEPARTMENT OF MATHEMATICS
DIBRUGARH UNIVERSITY
DIBRUGARH,
INDIA.
Email address: igogoi1995@gmail.com

DEPARTMENT OF MATHEMATICS
DIBRUGARH UNIVERSITY
DIBRUGARH,
INDIA.
Email address: phukanbablee@gmail.com

DEPARTMENT OF MATHEMATICS
DIBRUGARH UNIVERSITY
DIBRUGARH,
INDIA.
Email address: adityapegu92@gmail.com

DEPARTMENT OF MATHEMATICS
DIBRUGARH UNIVERSITY
DIBRUGARH,
INDIA.
Email address: a.bharali@dibru.ac.in