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TOPOLOGICAL INDICES FOR LABELED GRAPHS

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ABSTRACT. Topological indices and graph labelings are independently considered as immense fields in studying graph properties. Through this work, we bridge the concepts, topological indices and graph labelings, in order to yield several new topological indices to study the labeled graphs. In this paper, we introduce new topological indices for some graphs that admit graceful, odd harmonious and cordial labelings.

1. INTRODUCTION

Topological indices (TIs) are graph invariants that help to identify the graph by means of its unique index. These indices act as peculiar features in the field of Chemical Graph Theory to study the properties of molecular graphs. Several TIs have been introduced based on parameters of graphs such as degree, eccentricity, etc [5]. In this paper, with the help of graph labels we introduce a new idea which inturn paves the route to the evolution of several TIs. Graph labeling [3] was introduced by Rosa in 1967 as an assignment of numbers to the vertices or edges or both of a graph subject to certain condition. This concept has been transformed in to a vivid branch of Graph theory through several

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labeling methods. In this paper, we focus on graceful, odd harmonious and cordial labelings. For all standard graph theoretical terminology and notation, we follow [1]. A brief summary of definitions which are useful in our work is given below:

Definition 1.1. *A function f is called graceful labeling of G with n vertices and m edges if $f : V(G) \to \{0, 1, 2, \ldots, m\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \ldots, m\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is bijective. The graph which admits graceful labeling is called graceful graph.*

Definition 1.2. *A function f is called odd harmonious labeling of G if $f : V(G) \to \{0, 1, 2, \ldots, 2m - 1\}$ is injective and the induced function $f^* : E(G) \to \{1, 3, \ldots, 2m - 1\}$ defined as $f^*(uv) = f(u) + f(v)$ is bijective. The graph which admits odd harmonious labeling is called odd harmonious graph.*

Definition 1.3. *Let f be such that $f : V(G) \to \{0, 1\}$ and for each edge uv assign the label $|f(u) - f(v)|$. The function f is called cordial labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges of G with label $i (= 0 \; or \; 1)$ respectively. The graph which admits cordial labeling is called cordial graph.*

Definition 1.4. *The square of a graph G, G^2, has the same set of vertices of G with u, v adjacent in G^2 whenever $d(u, v) \leq 2$ in G.*

Definition 1.5. *The splitting graph of a graph G, $S'(G)$, is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

Definition 1.6. *The shadow graph of a connected graph G, $D_2(G)$, is obtained by taking two copies of G say G' and G'', and joining each vertex u' in G' with the neighbors of the corresponding vertex u'' in G''.*

2. Density Based Topological Indices

A graph whose vertices and edges are assigned labels under some concrete notion is known as a *labeled graph*. In this section, we introduce the notion of *Density of a vertex* of a labeled graph and define six different types of new TIs. To introduce the new TIs, we consider only the non negative integers as labels.

Definition 2.1. Density of a vertex u, $\eta_L(u)$, *with respect to a labeling* L, *of a labeled graph* G, *is defined as* $\eta_L(u) = \sum f(uv)$ *where the sum ranges over all* $v \in N(u)$, *and* $f(uv)$ *is the label assigned to the edge* uv. *In otherwords, density of a vertex* u *is the sum of the labels of all the edges that are incident with* u.

The following are some notations used in the subsequent sections:

1. g - Graceful labeling
2. oh - Odd harmonious labeling
3. c - Cordial labeling
4. $(G)_L$ - Graph G equipped with labeling L.
5. $SQI(G)_L$ - Square Index of $(G)_L$.
6. $PI(G)_L$ - Product Index of $(G)_L$.
7. $SI(G)_L$ - Sum Index of $(G)_L$.
8. $CSQI(G)_L$ - Cluster Square Index of $(G)_L$.
9. $CPI(G)_L$ - Cluster Product Index of $(G)_L$.
10. $CSI(G)_L$ - Cluster Sum Index of $(G)_L$.
11. $\sum\limits_{v \in V(G)} \eta_L(v)$ - Sum of the densities of labeled graphs

Based on the density of the vertices of G, we introduce the following TIs:

1. Square Index of $(G)_L$, $SQI(G)_L = \sum\limits_{v \in V(G)} \eta_L(v)^2$
2. Product Index of $(G)_L$, $PI(G)_L = \sum\limits_{uv \in E(G)} \eta_L(u)\eta_L(v)$
3. Sum Index of $(G)_L$, $SI(G)_L = \sum\limits_{uv \in E(G)} (\eta_L(u) + \eta_L(v))$
4. Cluster Square Index of $(G)_L$, $CSQI(G)_L = \frac{SQI(G)_L}{\sum\limits_{v \in V(G)} \eta_L(v)}$
5. Cluster Product Index of $(G)_L$, $CPI(G)_L = \frac{PI(G)_L}{\sum\limits_{v \in V(G)} \eta_L(v)}$
6. Cluster Sum Index of $(G)_L$, $CSI(G)_L = \frac{SI(G)_L}{\sum\limits_{v \in V(G)} \eta_L(v)}$

The topological indices defined above are called the **density based topological indices of the labeled graph** G. Here on, by topological indices we mean only the density based topological indices.

In this work, we have computed the TIs of the following graphs:

1. Graceful graphs discussed in [7] and [9] such as P_n, $K_{m,n}$, W_n, $B_{n,n}$, $B^2_{n,n}$, $S'(B_{n,n})$ and $S'(K_{1,n})$.

2. Odd harmonious graphs discussed in [4] and [6] such as P_n, C_n, $K_{m,n}$, $D_2(K_{1,n})$ and $S'(K_{1,n})$.
3. Cordial graphs discussed in [2] and [8] such as $B^2_{n,n}$, $D_2(B_{n,n})$, $S'(B_{n,n})$ and C_n.

Under the standard labeling schemes given in the references, we derive the TIs for labeled graphs. It is clear that the TIs are not invariant for labeled graphs. Therefore we get different values for each graph with respect to each labeling.

Densities that are derived for graceful graphs are called *graceful densities*. Densities that are derived for odd harmonious graphs are called *odd harmonious densities*. Densities that are derived for cordial graphs are called *cordial densities*.

In this study, we consider graphs with $n \geq 3$ vertices. The following are some of the results obtained for labeled graphs and we quote them without proof:

1. The sum of the graceful densities of $(P_n)_g$,
$$\sum_{v \in V(P_n)} \eta_g(v) = n(n-1).$$
2. The sum of the graceful densities of $(K_{m,n})_g$,
$$\sum_{v \in V(K_{m,n})} \eta_g(v) = mn(mn+1).$$
3. The sum of the graceful densities of $(K_{n,n})_g$,
$$\sum_{v \in V(K_{n,n})} \eta_g(v) = n^2(n^2+1).$$
4. The sum of the graceful densities of $(W_n)_g$,
$$\sum_{v \in V(W_n)} \eta_g(v) = 2n(2n+1).$$
5. The sum of the graceful densities of $(B_{n,n})_g$,
$$\sum_{v \in V(B_{n,n})} \eta_g(v) = 2(n+1)(2n+1).$$
6. The sum of the graceful densities of $(B^2_{n,n})_g$,
$$\sum_{v \in V(B^2_{n,n})} \eta_g(v) = 2(2n+1)(4n+1).$$
7. The sum of the graceful densities of $(S'(B_{n,n}))_g$,
$$\sum_{v \in V(S'(B_{n,n}))} \eta_g(v) = 6(2n+1)(3n+2).$$
8. The sum of the graceful densities of $(S'(K_{1,n}))_g$,

$$\sum_{v \in V(S'(K_{1,n}))} \eta_g(v) = 3n(3n+1).$$

9. The sum of the odd harmonious densities of $(P_n)_{oh}$,

$$\sum_{v \in V(P_n)} \eta_{oh}(v) = 2(n-1)^2.$$

10. The sum of the odd harmonious densities of $(C_n)_{oh}$,

$$\sum_{v \in V(C_n)} \eta_{oh}(v) = 2n^2.$$

; $n \equiv 0(mod\ 4)$

11. The sum of the odd harmonious densities of $(K_{m,n})_{oh}$,

$$\sum_{v \in V(K_{m,n})} \eta_{oh}(v) = 2m^2n^2.$$

12. The sum of the odd harmonious densities of $(K_{n,n})_{oh}$,

$$\sum_{v \in V(K_{n,n})} \eta_{oh}(v) = 2n^4.$$

13. The sum of the odd harmonious densities of $(D_2(K_{1,n}))_{oh}$,

$$\sum_{v \in V(D_2(K_{1,n}))} \eta_{oh}(v) = 32n^2.$$

14. The sum of the odd harmonious densities of $(S'(K_{1,n}))_{oh}$,

$$\sum_{v \in V(S'(K_{1,n}))} \eta_{oh}(v) = 18n^2.$$

15. The sum of the cordial densities of $(B^2_{n,n})_c$,

$$\sum_{v \in V(B^2_{n,n})} \eta_c(v) = 2(2n+1).$$

16. The sum of the cordial densities of $(D_2(B_{n,n}))_c$,

$$\sum_{v \in V(D_2(B_{n,n}))} \eta_c(v) = 4(2n+1).$$

17. The sum of the cordial densities of $(S'(B_{n,n}))_c$,

$$\sum_{v \in V(S'(B_{n,n}))} \eta_c(v) = 2(3n+2).$$

18. The sum of the cordial densities of $(C_n)_c$,

$$\sum_{v \in V(C_n)} \eta_c(v) = \begin{cases} n & \text{if } n \equiv 0(mod\ 4) \\ n-1 & \text{if } n \equiv 1(mod\ 4) \\ n+1 & \text{if } n \equiv 3(mod\ 4) \end{cases}$$

Since the computation of TIs involve tedious tasks and space consuming, proofs have been avoided. The final values obtained for various labeled graphs are contained in Table 1, Table 2 and Table 3.

TABLE 1. SQI and CSQI of Labeled Graphs

	$SQI(G)_L$	$CSQI(G)_L$
$(P_n)_g$	$\frac{n(n-1)(4n-5)}{3}$	$\frac{4n-5}{3}$
$(K_{m,n})_g$	$\frac{mn}{12}\{[(4m^2-1)n^2+(3m^2+7)mn$ $+(2m^2+1)3]n+2m\}$	$\frac{1}{12(mn+1)}\{[(4m^2-1)n^2+(3m^2+7)$ $mn+(2m^2+1)3]n+2m\}$
$(K_{n,n})_g$	$\frac{n^3}{12}(n^2+1)(7n^2+5)$	$\frac{n(7n^2+5)}{12}$
$(W_n)_g$; $n \geq 4$	$\frac{1}{3}(3n^4+26n^3+66n^2-23n-78)$ where $n \equiv 0(mod\ 2)$ and $\frac{1}{3}(n-1)(3n^3+41n^2+17n+186)$ where $n \equiv 1(mod\ 2)$	$\frac{3n^4+26n^3+66n^2-23n-78}{6n(2n+1)}$ where $n \equiv 0(mod\ 2)$ and $\frac{(n-1)(3n^3+41n^2+17n+186)}{6n(2n+1)}$ where $n \equiv 1(mod\ 2)$
$(B_{n,n})_g$	$\frac{1}{6}(n+1)(15n^3+55n^2+50n+12)$	$\frac{15n^3+55n^2+50n+12}{12(2n+1)}$
$(B^2_{n,n})_g$	$2(2n+1)(10n^3+23n^2+9n+1)$	$\frac{10n^3+23n^2+9n+1}{4n+1}$
$(S'(B_{n,n}))_g$	$\frac{1}{6}(699n^4+2525n^3+2379n^2+1030n+240)$ where $n \equiv 0(mod\ 2)$ and $\frac{1}{6}(699n^4+2525n^3+2370n^2+1063n+219)$ where $n \equiv 1(mod\ 2)$	$\frac{699n^4+2525n^3+2379n^2+1030n+240}{36(2n+1)(3n+2)}$ where $n \equiv 0(mod\ 2)$ and $\frac{699n^4+2525n^3+2370n^2+1063n+219}{36(2n+1)(3n+2)}$ where $n \equiv 1(mod\ 2)$
$(S'(K_{1,n}))_g$	$\frac{n}{12}(n+1)(159n^2+107n-2)$	$\frac{(n+1)(159n^2+107n-2)}{36(3n+1)}$
$(P_n)_{oh}$	$\frac{2}{3}(8n^3-30n^2+34n-9)$	$\frac{8n^3-30n^2+34n-9}{3(n-1)^2}$
$(C_n)_{oh}$	$\frac{2}{3}(8n^3-3n^2-2n-12)$	$\frac{8n^3-3n^2-2n-12}{3n^2}$
$(K_{m,n})_{oh}$	$\frac{mn}{3}[m^2n(1+3n^2+4mn)-(m^3+n)]$	$\frac{m^2n(1+3n^2+4mn)-(m^3+n)}{6mn}$
$(K_{n,n})_{oh}$	$\frac{n^3}{3}(7n^4-1)$	$\frac{7n^4-1}{6n}$
$(D_2(K_{1,n}))_{oh}$	$\frac{8n}{3}(48n^3+64n^2+3n-4)$	$\frac{48n^3+64n^2+3n-4}{12n}$
$(S'(K_{1,n}))_{oh}$	$\frac{n}{3}(123n^3+206n^2+6n-17)$	$\frac{123n^3+206n^2+6n-17}{54n}$
$(B^2_{n,n})_c$	$2(n^2+3n+1)$	$\frac{n^2+3n+1}{2n+1}$
$(D_2(B_{n,n}))_c$	$4(n^2+3n+1)$	$\frac{n^2+3n+1}{2n+1}$
$(S'(B_{n,n}))_c$	$(n+1)(5n+4)$	$\frac{(n+1)(5n+4)}{2(3n+2)}$
$(C_n)_c$	n, where $n \equiv 0(mod\ 4)$ $n-1$, where $n \equiv 1(mod\ 4)$ $n+3$, where $n \equiv 3(mod\ 4)$	1, for $n \equiv 0,1(mod\ 4)$ $\frac{n+3}{n+1}$, where $n \equiv 3(mod\ 4)$

TABLE 2. PI and CPI of Labeled Graphs

	$PI(G)_L$	$CPI(G)_L$
$(P_n)_g$	$\frac{1}{3}(4n^3 - 12n^2 + 8n + 3)$	$\frac{4n^3-12n^2+8n+3}{3n(n-1)}$
$(K_{m,n})_g$	$\frac{m^2n^2(mn+1)^2}{4}$	$\frac{mn(mn+1)}{4}$
$(K_{n,n})_g$	$\frac{n^4(n^2+1)^2}{4}$	$\frac{n^2(n^2+1)}{4}$
$(W_n)_g$; $n \geq 5$	$\frac{1}{2}(6n^4 + 20n^3 + 27n^2 + 44n - 116)$ where $n \equiv 0(mod\ 2)$ and $\frac{1}{2}(6n^4 + 28n^3 - 9n^2 + 94n - 107)$ where $n \equiv 1(mod\ 2)$	$\frac{6n^4+20n^3+27n^2+44n-116}{4n(2n+1)}$ where $n \equiv 0(mod\ 2)$ and $\frac{6n^4+28n^3-9n^2+94n-107}{4n(2n+1)}$ where $n \equiv 1(mod\ 2)$
$(B_{n,n})_g$	$\frac{1}{4}(n+1)^2(13n^2 + 16n + 4)$	$\frac{(n+1)(13n^2+16n+4)}{8(2n+1)}$
$(B^2_{n,n})_g$	$76n^4 + 136n^3 + 69n^2 + 14n + 1$	$\frac{76n^4+136n^3+69n^2+14n+1}{2(2n+1)(4n+1)}$
$(S'(B_{n,n}))_g$	$\frac{1}{4}(929n^4 + 2311n^3 + 1790n^2 + 594n + 156)$ where $n \equiv 0(mod\ 2)$ and $\frac{1}{4}(929n^4 + 2311n^3 + 1795n^2 + 611n + 234)$ where $n \equiv 1(mod\ 2)$	$\frac{929n^4+2311n^3+1790n^2+594n+156}{24(2n+1)(3n+2)}$ where $n \equiv 0(mod\ 2)$ and $\frac{929n^4+2311n^3+1795n^2+611n+234}{24(2n+1)(3n+2)}$ where $n \equiv 1(mod\ 2)$
$(S'(K_{1,n}))_g$	$\frac{n^2}{4}(71n^2 + 52n + 9)$	$\frac{n(71n^2+52n+9)}{12(3n+1)}$
$(P_n)_{oh}$	$\frac{4}{3}(4n^3 - 18n^2 + 23n - 3)$	$\frac{2(4n^3-18n^2+23n-3)}{3(n-1)^2}$
$(C_n)_{oh}$; $n > 4$	$\frac{1}{3}(16n^3 - 15n^2 + 2n - 48)$	$\frac{16n^3-15n^2+2n-48}{6n^2}$
$(K_{m,n})_{oh}$	m^4n^4	$\frac{m^2n^2}{2}$
$(K_{n,n})_{oh}$	n^8	$\frac{n^4}{2}$
$(D_2(K_{1,n}))_{oh}$	$256n^4$	$8n^2$
$(S'(K_{1,n}))_{oh}$	$n^3(77n - 1)$	$\frac{n(77n-1)}{18}$
$(B^2_{n,n})_c$	$(n+1)(5n+1)$	$\frac{(n+1)(5n+1)}{2(2n+1)}$
$(D_2(B_{n,n}))_c$	$4(n+1)(3n+1)$	$\frac{(n+1)(3n+1)}{(2n+1)}$
$(S'(B_{n,n}))_c$	$(n+1)(7n+3)$	$\frac{(n+1)(7n+3)}{2(3n+2)}$
$(C_n)_c$	n, where $n \equiv 0(mod\ 4)$ $n-2$, where $n \equiv 1(mod\ 4)$ $n+2$, where $n \equiv 3(mod\ 4)$	1, where $n \equiv 0(mod\ 4)$ $\frac{n-2}{n-1}$, where $n \equiv 1(mod\ 4)$ $\frac{n+2}{n+1}$, where $n \equiv 3(mod\ 4)$

TABLE 3. SI and CSI of Labeled Graphs

	$SI(G)_L$	$CSI(G)_L$
$(P_n)_g$	$n(2n-3)$	$\frac{2n-3}{n-1}$
$(K_{m,n})_g$	$\frac{mn(m+n)(mn+1)}{2}$	$\frac{m+n}{2}$
$(K_{n,n})_g$	$n^3(n^2+1)$	n
$(W_n)_g$	$(n+4)(n^2+4n-3)$ where $n \equiv 0(mod\ 2)$ and n^3+10n^2+4n-3 where $n \equiv 1(mod\ 2)$	$\frac{(n+4)(n^2+4n-3)}{2n(2n+1)}$ where $n \equiv 0(mod\ 2)$ and $\frac{n^3+10n^2+4n-3}{2n(2n+1)}$ where $n \equiv 1(mod\ 2)$
$(B_{n,n})_g$	$2(n+1)(n^2+3n+1)$	$\frac{n^2+3n+1}{2n+1}$
$(B^2_{n,n})_g$	$2(4n+1)(2n^2+5n+1)$	$\frac{2n^2+5n+1}{2n+1}$
$(S'(B_{n,n}))_g$	$\frac{1}{2}(60n^3+211n^2+151n+44)$ where $n \equiv 0(mod\ 2)$ and $\frac{1}{2}(60n^3+211n^2+148n+49)$ where $n \equiv 1(mod\ 2)$	$\frac{60n^3+211n^2+151n+44}{12(3n+2)(2n+1)}$ where $n \equiv 0(mod\ 2)$ and $\frac{60n^3+211n^2+148n+49}{12(3n+2)(2n+1)}$ where $n \equiv 1(mod\ 2)$
$(S'(K_{1,n}))_g$	$\frac{n}{2}(16n^2+19n+5)$	$\frac{16n^2+19n+5}{6(3n+1)}$
$(P_n)_{oh}$	$2(n-1)(2n-3)$	$\frac{2n-3}{n-1}$
$(C_n)_{oh}$	$4n^2$	2
$(K_{m,n})_{oh}$	$m^2n^2(m+n)$	$\frac{m+n}{2}$
$(K_{n,n})_{oh}$	$2n^5$	n
$(D_2(K_{1,n}))_{oh}$	$32n^2(n+1)$	$n+1$
$(S'(K_{1,n}))_{oh}$	$2n^2(7n+8)$	$\frac{7n+8}{9}$
$(B^2_{n,n})_c$	$2(2n^2+5n+1)$	$\frac{2n^2+5n+1}{2n+1}$
$(D_2(B_{n,n}))_c$	$8(n^2+3n+1)$	$\frac{2(n^2+3n+1)}{2n+1}$
$(S'(B_{n,n}))_c$	$5n^2+16n+6$	$\frac{5n^2+16n+6}{2(3n+2)}$
$(C_n)_c$	$2n$, where $n \equiv 0(mod\ 4)$ $2(n-1)$, where $n \equiv 1(mod\ 4)$ $2(n+1)$, where $n \equiv 3(mod\ 4)$	2, for $n \equiv 0,1,3(mod\ 4)$

3. CONCLUSION

In this paper, we coupled the notion of TIs and graph labelings, to obtain several new types of TIs. Also we found the TIs SQI, PI, SI, CSQI, CPI and CSI for some graceful, odd harmonious and cordial graphs. Finding the above TIs for graphs under any such labelings can be done as future work.

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