

## EXISTENCY RESULTS OF FIRST ORDER NEUTRAL DELAY DIFFERENCE EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENT IN THE NEUTRAL TERM

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**ABSTRACT.** In this paper some criteria for oscillatory behavior of first order Neutral Delay Difference equation with the positive coefficient and the negative coefficient in the neutral term is obtained, where  $k, l > 0$ ,  $\{p_n\}$ ,  $\{q_n\}$  are positive sequences.

### 1. INTRODUCTION

First order Neutral Delay Difference Equation is gaining interest because they are the discrete analogue of differential Equations. In recent years, several papers on oscillation of solutions of Neutral Delay Difference Equations have appeared. S.S. Cheng and Y.Z. Lin [1] have provided a complete characterization of oscillation solutions of first order Neutral Delay Difference Equations with positive coefficient in the Neutral term. John R.Graef, R.Savithri, E.Thandapani [3] have analyzed the non oscillatory solutions of first order Neutral Delay Differential Equations with positive coefficient in the Neutral term.

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Elmetwally M.Elabbasy, Taher S.Hassan, Samir H.Saker [2] have provided oscillatory solutions of first order Neutral Delay Differential Equations with negative coefficient in the Neutral term.

Ozkan Ocalan [4] extensively discuss the problem of Oscillation of neutral differential equation with positive and negative coefficients.

Tanaka [5] discussed the various solutions of Oscillation First order Neutral Delay Differential Equations.

Here some oscillation results in difference equations based on the existence results of differential equations are provided. Examples are provided to illustrate the results.

## 2. SECTION I

In this section some criteria for oscillatory behavior of first order Neutral Delay Difference Equation,

$$(2.1) \quad \Delta(x_n + p_n f(x_{n-k}) + q_n f(x_{n-l})) = 0$$

is obtained where  $k, l > 0$ ,  $\{p_n\}$ ,  $\{q_n\}$  are positive sequences.

The following assumptions has been made to prove the results:

H<sub>1</sub>:  $f(u)$  is an increasing function and  $uf(u) > 0$ .

H<sub>2</sub>: There exists a function  $w$  such that  $w(u) > 0$  for  $u > 0$  and  $w(u)f(v) > f(uv)$ .

H<sub>3</sub>:  $\varphi(u)$  is an increasing function such that  $u\varphi(u) > 0$  and  $|\varphi(u+v)| < |f(u) + f(v)|$ .

## 3. EXISTENCE OF OSCILLATORY SOLUTIONS

**Theorem 3.1.** *Every solution of the equation (2.1) is oscillatory if there exists a constant  $\lambda_n$  such that  $0 < \lambda_n < 1$  for some  $n \geq n_0$  and the difference inequality*

$$(3.1) \quad \Delta z_n + Q_n \varphi(z_{n-l+k}) \leq 0$$

satisfies, where  $Q_n = \min \left\{ \lambda_n q_n, \frac{(1 - \lambda_{n-k})q_{n-l}}{wp_{n-l}} \right\}$  and  $z_n = \sum_{n_0+k}^m Q_n \phi(y_{n-l})$ .

*Proof.* Suppose to the contrary that there is a non oscillatory solution  $\{x_n\}$  of (2.1). Suppose that  $x_n > 0$  for all  $n \geq n_0$ .

Let  $y_n = x_n + p_n f(x_n - k)$ .

Then by the equation (2.1),  $\Delta y_n = -q_n f(x_n - l) < 0$ ,  $y_{n+1} < y_n$ , and  $y_n$  is a decreasing function. Further,  $y_{n+1} - y_n = -q_n f(x_n - l)$ ,

$$y_{n+1} + q_n f(x_{n-l}) = y_n,$$

and  $y_n > q_n f(x_n - l)$  for some  $n \geq n_0$ . Taking summation from  $n_0$  to  $m$ ,  $m > n_0$ :

$$\begin{aligned} \sum_{n=n_0}^m y_n &> \sum_{n=n_0}^m q_n f(x_{n-l}) \\ &> \sum_{n=n_0}^m (\lambda_n q_n f(x_{n-l}) + (1 - \lambda_n) q_n f(x_{n-l})) \\ &> \sum_{n=n_0}^m Q_n f(x_{n-l}) + \sum_{n=n_0+k}^m (1 - \lambda_{n-k}) q_{n-k} f(x_{n-k-l}) \\ &> \sum_{n=n_0}^m Q_n f(x_{n-l}) + \sum_{n=n_0+k}^m Q_n w(p_{n-l}) f(x_{n-k-l}) \\ &> \sum_{n=n_0}^m Q_n f(x_{n-l}) + \sum_{n=n_0+k}^m Q_n f(p_{n-l} x_{n-k-l}) \\ &> \sum_{n=n_0+k}^m Q_n [f(x_{n-l}) + f(p_{n-l} x_{n-k-l})] \\ &> \sum_{n=n_0+k}^m Q_n [\varphi(x_{n-l}) + p_{n-l} x_{n-k-l}] \\ &= \sum_{n=n_0+k}^m Q_n \varphi[y_{n-l}] \\ \sum_{n=n_0+k}^m Q_n \varphi[y_{n-l}] &> 0. \end{aligned}$$

Let

$$z_n = \sum_{n=n_0+k}^m Q_n \phi(y_{n-l}).$$

Also,

$$\begin{aligned}
 \Delta z_n &= z_{n+1} - z_n \\
 &= \sum_{n=n_0+k}^m (Q_{n+1}\varphi[y_{n+1-l}] - Q_n\varphi[y_{n-l}]) \\
 &= Q_{n_0+k+1}\varphi[y_{n_0+k+1-l}] + Q_{n_0+k+2}\varphi[y_{n_0+k+2-l}] + Q_{n_0+k+3}\varphi[y_{n_0+k-l}] \\
 &\quad + \cdots + Q_{m+1}\varphi[y_{m+1-l}] - Q_{n_0+k}\varphi[y_{n_0+k+1}] - Q_{n_0+k+1}\varphi[y_{n_0+k+1-l}] \\
 &\quad - Q_{n_0+k+2}\varphi[y_{n_0+k+2-l}] + \cdots - Q_m\varphi[y_{m-l}] \\
 &= Q_{m+1}\varphi[y_{m+1-l}] - Q_{n_0+k}\varphi[y_{n_0+k-l}] \\
 \Delta z_n &> -Q_{n_0+k}\varphi[y_{n_0+k-l}] \\
 &> -Q_n\varphi(y_{n-1}), \quad n \geq n_0 + k \\
 &> -Q_n\varphi(z_{n-1+k})
 \end{aligned}$$

and

$$(3.2) \quad \Delta z_n + Q_n\varphi(z_{n-1+k}) > 0.$$

Condition (3.2) holds when  $z_n$  is eventually positive solution. Contradiction to the equation (3.1). Since  $z_n = \sum_{n=n_0+k}^m Q_n\phi(y_{n-l})$  and  $y_n = x_n + p_n x_{n-k}$ , we say that  $\{x_n\}$  is an oscillatory solution of the equation (2.1).  $\square$

**Theorem 3.2.** *Every solution of the equation (2.1) is oscillatory, if it satisfies the condition  $\Delta y_n + q_n f(x_{n-1}) \leq 0$ , where  $y_n = x_n + p_n f(x_{n-k})$  and  $Q_n = \min \left\{ \lambda_n p_n, \frac{1-\lambda_n}{\omega_{n-l}} p_n \right\}$  and if there exists a function  $\omega$  such that  $\omega(u) > 0$ , for  $u > 0$  and  $f(u) = u$ ,  $|f(uv)| \leq \omega(u)|f(v)|$  and  $f(uv) \geq uv$ ,  $\lambda_n$  is positive constant.*

*Proof.* Suppose  $\{x_n\}$  is a non oscillatory solution. Let  $x_n > 0$  and let us assume  $\{x_n\}$  is eventually positive. Let

$$\begin{aligned}
 y_n &= x_n + p_n f(x_{n-k}) \\
 &> p_n f(x_{n-k}).
 \end{aligned}$$

Hence

$$\begin{aligned}
 y_n &> p_n f(x_{n-k}) \\
 &= \lambda_n p_n f(x_{n-k}) + (1 - \lambda_n) p_n f(x_{n-k}) \\
 &> Q_n f(x_{n-k}) + Q_n f(x_{n-k}) \omega_{n-1+k} \\
 &> Q_n f(x_{n-k}) + Q_n f(x_{n-k} x_{n-1+k}) \\
 &> Q_n f(x_{n-k}) + x_{n-k} x_{n-1+k} > 0,
 \end{aligned}$$

since  $y_n > 0$ ,  $\Delta y_n > 0$ , i.e.,  $\Delta y_n + q_n f(x_{n-1}) > 0$ ,  $y_n > 0$ .

This contradicts our assumption. Hence  $\{x_n\}$  is an oscillatory solution.  $\square$

**Theorem 3.3.** *If  $\lim_{n \rightarrow \infty} \sum q_n > 0$ , and  $\{x_n\}$  is an eventually positive solution of*

$$(3.3) \quad \Delta(x_n + p_n x_{n-k}) + q_n f(x_{n-1}) = 0,$$

*then  $\Delta z_n + \frac{M q_n}{p_n^2} (p_n - 1) z_{n+k-1} \leq 0$ , where*

$$(3.4) \quad z_n = x_n + p_n x_{n-k}.$$

*Proof.* Equation (3.3) becomes,

$$\begin{aligned}
 \Delta z_n + q_n f(x_{n-1}) &= 0 \\
 \Delta z_n &= -q_n f(x_{n-1}) < 0 \\
 z_{n+1} - z_n &< 0 \\
 \Rightarrow z_{n+1} &< z_n.
 \end{aligned}$$

Hence  $z_n$  is decreasing. From the equation (3.4),

$$(3.5) \quad p_n x_{n-k} = z_n - x_n.$$

From the equation (3.4),

$$z_{n+k} = x_{n+k} + p_n x_n,$$

since  $z_n$  is decreasing,  $z_n > z_{n+k} \geq p_n x_n$ .

From the equation (3.5),

$$\begin{aligned}
 (3.6) \quad p_n^2 x_{n-k} &= p_n z_n - p_n x_n \quad (\text{multiplying by } p_n) \\
 p_n x_n &= z_{n+k} - x_{n+k} \\
 p_n x_n &< z_{n+k} \\
 -p_n x_n &\geq -z_{n+k}.
 \end{aligned}$$

Substituting in the equation (3.6),

$$\begin{aligned}
 p_n^2 x_{n-k} &= p_n z_n - p_n x_n \\
 &\geq p_n z_n - z_{n+k} \\
 &\geq p_n z_{n+k} - z_{n+k} \\
 x_{n-k} &\geq \frac{p_n - 1}{p_n^2} z_{n+k} \\
 x_{n-1} &\geq \frac{p_n - 1}{p_n^2} z_{n+k-1} \\
 \Delta z_n + q_n f(x_{n-1}) &= 0 \\
 \Rightarrow \Delta z_n &= -q_n f(x_{n-1}) \\
 \Delta z_n &\leq -q_n M \frac{p_n - 1}{p_n^2} z_{n+k-1} \\
 (3.7) \quad \Delta z_n + M \frac{q_n(p_n - 1)}{p_n^2} z_{n+k-1} &\leq 0.
 \end{aligned}$$

Hence the theorem.  $\square$

#### 4. SECTION II

In this section some criteria for oscillatory behavior of first order Neutral Delay Difference Equation:

$$(4.1) \quad \Delta(x_n - q_n f(x_{n-k})) + p_n x_{n-l} = 0$$

is obtained where  $k, l > 0$  and  $\{p_n\}$  and  $\{q_n\}$  are positive sequences.

##### 4.1. Some oscillatory results.

**Theorem 4.1.** *Every solution of the equation (4.1) is oscillatory if  $k, l, p_n, q_n > 0$ .*

*Proof.*

##### Case 1

Suppose  $\{x_n\}$  is a non oscillatory solution of equation (4.1), let  $x_n > 0$  and  $\{x_n\}$  is eventually positive,  $x_{n-1} < x_n$ :

$$\begin{aligned}
 \Delta(x_n - q_n f(x_{n-k})) &= -p_n x_{n-l} \\
 &= -p_n (z_{n-l} + q_{n-l} x_{n-k-l}).
 \end{aligned}$$

Hence

$$\begin{aligned}
 \Delta z_n + p_n z_{n-l} + p_n q_{n-l} x_{n-k-l} &= 0 \\
 z_{n+1} - z_n + p_n (z_{n-l} + q_{n-l} x_{n-k-l}) &= 0 \\
 z_{n+1} + p_n (z_{n-l} + q_{n-l} x_{n-k-l}) &= z_n \\
 p_n (z_{n-l} + q_{n-l} x_{n-k-l}) &< 0 \\
 p_n (x_{n-l} - q_{n-l} x_{n-k-l} + q_{n-l} x_{n-k-l}) &< 0 \\
 p_n x_{n-l} &< 0,
 \end{aligned}$$

which is a contradiction since  $p_n$  is positive and  $x_{n-l}$  is eventually positive.

Hence equation (4.1) has an oscillatory solution.

### Case 2

Let us assume  $x_n < 0$  and  $\{x_n\}$  is eventually negative, then  $x_{n-1} > x_n$ ,

$$\begin{aligned}
 \Delta (x_n - q_n f(x_{n-k})) &= -p_n x_{n-l} \\
 &= -p_n (z_{n-l} + q_{n-l} x_{n-k-l}) \\
 \Delta z_n + p_n z_{n-l} + p_n q_{n-l} x_{n-k-l} &= 0 \\
 z_{n+1} - z_n + p_n z_{n-l} + p_n q_{n-l} x_{n-k-l} &= 0 \\
 p_n (z_{n-l} + q_{n-l} x_{n-k-l}) &> 0 \\
 p_n (x_{n-l} - q_{n-l} x_{n-k-l} + q_{n-l} x_{n-k-l}) &> 0 \\
 p_n x_{n-l} &> 0,
 \end{aligned}$$

which is a contradiction since  $p_n$  is positive and  $x_{n-l}$  is eventually negative solution.

Hence equation (4.1) has an oscillatory solution.  $\square$

### 4.2. Some Non oscillatory results.

**Theorem 4.2.** Let  $\{x_n\}$  be an eventually positive solution of  $\Delta (x_n - q_n f(x_{n-k})) + p_n x_{n-l} = 0$   $x_n > 0$  and let

$$(4.2) \quad z_n = x_n - q_n x_{n-k}.$$

Assume  $k, l > 0, p_n, q_n > 0$ . Then  $z_n$  is eventually non increasing positive function.

*Proof.* From the equation (4.1),

$$\begin{aligned}\Delta z_n &= -p_n x_{n-l} < 0 \\ z_{n+1} - z_n &< 0 \\ z_{n+1} &< z_n.\end{aligned}$$

Hence  $z_n$  is eventually non increasing positive function.  $\square$

**Theorem 4.3.** *Every non oscillatory solution of the equation (4.1) converges to zero monotonically for large  $n$  as  $n \rightarrow \infty$  if  $k, l > 0, q_n > 0, p_n > 0$ .*

*Proof.* Suppose  $\{x_n\}$  is a non oscillatory solution of (4.1). Let us assume  $\{x_n\}$  is eventually positive (if  $x_n$  is eventually negative, the proof is similar).

Hence from the theorem (3.7),  $z_n$  is eventually non increasing positive function.

From the equation (4.2),

$$x_n \geq z_n.$$

Here  $\lim_{n \rightarrow \infty} x_n \geq \lim_{n \rightarrow \infty} z_n = \alpha \geq 0$ . If  $\alpha > 0$ , then from the equation (4.1),

$$\Delta z_n = -p_n x_{n-l} < 0.$$

Hence  $\lim_{n \rightarrow \infty} z_n \rightarrow -\infty$ , which contradicts that  $z_n$  is a positive function, and further we take  $\alpha = 0$ , therefore  $\lim_{n \rightarrow \infty} x_n = 0$ .

Hence every non oscillatory solution of the equation (4.1) converges to zero monotonically as  $n \rightarrow \infty$ .  $\square$

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