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RECURRENCE RELATION UNDER EFROS THEOREM

Roshani Sharma and Bhupendra Tripathi[1]

ABSTRACT. In the present manuscript, we apply Efros theorem to establish certain recurrence relation. The established results supposed to be new and general. By giving particular values to the parameters, a number of new and known results can be established.

1. INTRODUCTION

First, we will give a brief account of the Efros theorem, Laplace transform [1] and Parseval Goldstein theorem [4], which will be used to derive our main theorem.

(a) The Efros theorem [5] states that if G(p) and q(p) are two analytic function given by:

$$F(p) = L[f(t)],$$

$$G(p)e^{-\tau q(p)} = L[g(t,\tau)],$$

then

$$G(p)F(q(p)) = L\left[\int_0^\infty f(\tau)g(t,\tau)d\tau\right].$$

[1]*corresponding author*

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(b) Laplace transform can be defined as follows

$$f(p) = L[f(t);p] = f(p) = \int_0^\infty e^{-pt} f(t)dt,$$

(c) Parseval-Goldstein theorem states that if $\phi_1(p) = L[h_1(t)]$ and $\phi_2(p) = L[h_2(t)]$, then

$$\int_0^\infty \phi_1(t)h_2(t)dt = \int_0^\infty \phi_2(t)h_1(t)dt. \tag{1.1}$$

2. Main Result

Theorem 2.1. *If* $\lambda > n-1, R(\sigma+\lambda+1) > 0$, $(p+a) > 0$,

$$F(p) = L[f(t)],$$

and

$$G(p)e^{-\tau q(p)} = L[g(t,\tau)],$$

then

$$L\left[t^n(t+a)^{-\lambda-1}G(t)e^{-\tau(q(t))};p\right] = \sum_{r=0}^{n} \frac{(-1)^{n-r}a^{n-r}}{\Gamma(\lambda-r+1)} {}^nC_r \int_0^\infty x^{\lambda-r}e^{-bx}g(x,\tau)dx,$$

where $f(t) = o(t)$ *for some* t *and* $f(t) = \mathcal{O}(e^{-at}t^{\mu})$ *for large* t.

Proof. Since from ([2], p. 127), we have

$$g(r) = \int_0^\infty e^{-pt}f(t)dt, \quad \text{taking} f(t) = t^\lambda,$$

$$g(r) = \int_0^\infty e^{-pt}t^\lambda dt.$$

Now $L\left[t^\lambda e^{-bt};p\right] = \frac{\Gamma(\lambda+1)}{(p+b)^{\lambda+1}} = \Gamma(\lambda+1)(p+b)^{-\lambda-1}$. Further, by virtue of Leibnitz theorem we have

$$\frac{d^n}{dt^n}(t^\lambda e^{-bt}) = e^{-bt}\sum_{r=0}^{n}\frac{(-1)^{n-r}\Gamma(\lambda+1)}{\Gamma(\lambda-r+1)} {}^nC_r b^{n-r}t^{\lambda-r}.$$

Therefore, if we take $f(t) = t^\lambda e^{-bt}$, $p^nL[f(t);p] = L[f^n(t);p]$, where $f(0) = f'(0) = f''(0)\cdots f^{n-1}(0)$ and $f^n(t)$ stands for $\frac{d^n}{dt^n}[f(t)]$, then

$$\Gamma(\lambda+1)p^n(p+b)^{-\lambda-1} = L\left[e^{-bt}\sum_{r=0}^{n}\frac{(-1)^{n-r}\Gamma(\lambda+1)}{\Gamma(\lambda-r+1)} {}^nC_r b^{n-r}t^{\lambda-r};p\right]. \tag{2.1}$$

Using Parseval Goldstein theorem [4] in the above equation (1.1) and (2.1), we get

$$\int_0^\infty e^{-at} t^n (t+b)^{-\lambda-1} G(t) e^{-\tau(q(t))} dt = \sum_{r=0}^{n} \frac{(-1)^{n-r} b^{n-r}}{\Gamma(\lambda - r + 1)} {}^nC_r \int_0^\infty x^{\lambda-r} e^{-bx} g(x,\tau) dx.$$

Replacing b as a and a as p then we get

$$\int_0^\infty e^{-at} t^n (t+a)^{-\lambda-1} G(t) e^{-\tau(q(t))} dt = \sum_{r=0}^{n} \frac{(-1)^{n-r} a^{n-r}}{\Gamma(\lambda - r + 1)} {}^nC_r \int_0^\infty x^{\lambda-r} e^{-bx} g(x,\tau) dx.$$

$$\begin{aligned} & L\left[t^n (t+a)^{-\lambda-1} G(t) e^{-\tau(q(t))}; p\right] \\ & = \sum_{r=0}^{n} \frac{(-1)^{n-r} a^{n-r}}{\Gamma(\lambda - r + 1)} {}^nC_r \int_0^\infty x^{\lambda-r} e^{-bx} g(x,\tau) dx, \end{aligned} \tag{2.2}$$

which is supposed to be new result. □

Taking $\tau = 0$ in the above equation (2.2), then we get

$$L[t^n (t+a)^{-\lambda-1} G(t); p] = \sum_{r=0}^{n} \frac{(-1)^{n-r} a^{n-r}}{\Gamma(\lambda - r + 1)} {}^nC_r \int_0^\infty x^{\lambda-r} e^{-bx} g(x,\tau) dx. \tag{2.3}$$

3. Example

If we take $f(t) = t$, then ([2], p. 137), we have $L[t^\nu; p] = \Gamma(\nu+1) p^{-\nu-1},\ R(\nu) > -1$ and $R(p) > 0$. Substituting this value in the above equation (2.3), we get

$$L\left[t^{n+\nu} (t+a)^{-\lambda-1}; p\right] = \Gamma(\nu+1) \sum_{r=0}^{n} \frac{(-1)^{n-r} a^{n-r}}{\Gamma(\lambda - r + 1)} {}^nC_r \int_0^\infty e^{-ax} x^{\lambda-r} p^{-\nu-1} dx.$$

Solving the right-hand side with the help of the result ([2], p. 129), we get

$$L\left[t^{\lambda-1} (t+a)^{-\nu}; p\right] = \frac{p^{-\lambda} a^{-\nu}}{\Gamma(\nu)} E[\lambda; \nu :: ap],$$

where $R(\lambda) > 0,\ R(p) > 0,$

$$\begin{aligned} & \sum_{r=0}^{n} \frac{(-1)^{n-r} (ap)^n}{\Gamma(\lambda - r + 1)}\ {}^nC_r\ E[\lambda - r + 1, \nu + 1 :: ap] \\ & = \frac{1}{\Gamma(\lambda+1)} E[n + \nu + 1, \lambda + 1 :: ap]. \end{aligned} \tag{3.1}$$

Now, with the help of the result [7]:

$$E[\mu,\lambda::x]=\Gamma(\mu)\Gamma(\lambda)e^{\frac{x}{2}}x^{-\frac{1}{2}(1-\mu-\lambda)}W^{(x)}_{\left(\frac{1-\mu-\lambda}{2},\frac{\mu-\lambda}{2}\right)},$$

(3.1) can be written as follows

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}C_r\;x^{\frac{n}{2}-\frac{r}{2}}W^{(x)}_{\left(k+\frac{r}{2},m+\frac{r}{2}\right)}=\frac{\sqrt{m+n-k+\frac{1}{2}}}{\sqrt{m-k+\frac{1}{2}}}W^{(x)}_{\left(k-\frac{n}{2},m+\frac{n}{2}\right)}. \tag{3.2}$$

On taking $n=1$, (3.2) gives rise to result ([6], p. 27)

$$\left(m-k+\frac{1}{2}\right)W^{(x)}_{\left(k-\frac{1}{2},m+\frac{1}{2}\right)}+x^{\frac{1}{2}}W^{(x)}_{k,m}=W^{(x)}_{\left(k+\frac{1}{2},m+\frac{1}{2}\right)}.$$

Now, substituting in the above result (3.2) $x=\frac{1}{2}y^2$ and multiplying both side by $e^{\frac{y^2}{4}}y^{\nu-2m-n-\frac{1}{2}}$, we get

$$\sum_{r=0}^{n}(-1)^{n-r}\,{}^{n}C_r\;\frac{e^{\frac{y^2}{4}}y^{\nu-2m-n-\frac{1}{2}}}{2^{\frac{n}{2}-\frac{r}{2}}}W^{\left(\frac{y^2}{2}\right)}_{\left(k+\frac{r}{2},m+\frac{r}{2}\right)}$$
$$=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right)}\;e^{\frac{y^2}{4}}y^{\nu-2m-n-\frac{1}{2}}W^{\left(\frac{y^2}{2}\right)}_{\left(k-\frac{n}{2},m+\frac{n}{2}\right)}. \tag{3.3}$$

Taking images in Hankel transform ([3]; p. 84) of both the side of (3.3) and substituting $\frac{k}{2}+\frac{3m}{2}-\frac{\nu}{2}-\frac{1}{4}$, $\frac{k}{2}-\frac{m}{2}+\frac{\nu}{2}+\frac{1}{4}$ and $\frac{y^2}{2}$ as k, m and x respectively, we get

$$\sum_{r=0}^{n}\frac{\Gamma\left(m-k-r+\frac{1}{2}\right)}{\Gamma\left(m-k+n+\frac{1}{2}\right)}\,(-1)^{n-r}\,{}^{n}C_r\;W^{(x)}_{(k-r,m)}=x^{\frac{n}{2}}W^{(x)}_{\left(k+\frac{n}{2},m-\frac{n}{2}\right)}. \tag{3.4}$$

On taking $n=1$, (3.4) gives rise to result ([6], p. 27).

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DEPARTMENT OF MATHEMATICS,
LAKSHMI NAIRAN COLLEGE OF TECHNOLOGY,
BHOPAL-462021, MADHYA PRADESH, INDIA.
Email address: roshnipsharma@gmail.com

DEPARTMENT OF MATHEMATICS,
LAKSHMI NAIRAN COLLEGE OF TECHNOLOGY,
BHOPAL-462021, MADHYA PRADESH, INDIA.
Email address: btripathirewa@yahoo.co.in