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FORMULA FOR THE DIMENSIONS OF THE RELATIVE INVARIANTS OF PROJECTIVE SPECIAL LINEAR GROUP $PSL_2(F_7)$

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ABSTRACT. In this paper, we have computed the dimensions of the vector space of relative symmetric polynomials with respect to all irreducible characters of the projective special linear group $PSL_2(F_7)$.

1. INTRODUCTION

Symmetric polynomials surface in many branches of mathematics such as theory of equations, algebraic combinatorics and invariant theory. In this work we focus on their generalisation under the name of relative symmetric polynomials or relative invariants given by M. Shahryari [4].

On the complex vector space $H_d(x_1, x_2, \dots, x_n)$ consisting of homogeneous polynomials of degree d in n variables, the Reynold's operator will have the space of all symmetric polynomials as the image. In the above work, Shahryari has replaced Reynold's operator by another projection operator and the image of this new operator comprise the relative invariants. Now we give the rigorous definition.

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Definition 1.1. *Let G be a subgroup of the symmetric group S_n with the latter acting on $H_d(x_1, x_2, \ldots, x_n)$ by permuting the variables. Let χ be an irreducible character of G. The projection operator $T(G, \chi)$ is defined as*

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g.$$

Babaei, Zamani and Shahryari found the dimensions of the vector space of relative invariants for various subgroups of S_n such as S_n, A_n [1] and Young subgroups [5]. Later the formulæ for the relative invariants of cyclic group [6], dicyclic group [7] and dihedral group [8] were given by Babaei and Zamani. An alternative formulæ for the dimension of the space of relative invariants for the case of dihedral groups were given by S. Radha and P. Vanchinathan [3] in terms of number-theoretic functions. They have also given generating functions for the dimensions and thereby constructed a specific supercharacter theory for the dihedral groups. Later the dimensional formulæ for the relative invariants for the Mathieu and Pre-Mathieu groups were also given by P. Vanchinathan, S. Radha and Sanjit Das [9].

The projective special linear group $PSL_2(F_7)$ is the second smallest non-abelian simple group after the alternating group A_5 and is isomorphic to $GL_3(F_2)$. It has important applications in algebra, geometry and number theory. This group has been subject of deep study by Felix Klein.

In this paper, we fix a natural embedding of this group in S_8 and with respect to that embedding we have given the dimensional formulæ for the relative invariants of $PSL_2(F_7)$ associated with all its irreducible characters. The formulæ appears as a sum of binomial coefficients.

2. Projective Special linear group $PSL_2(F_7)$

The group $PSL_2(F_p)$ is the quotient of $SL_2(F_p)$ by its centre $Z = \{\pm, I_2\}$. and has $p(p^2-1)/2$ elements, which for $p = 7$ gives 168 elements [2]. Conjugacy classes and irreducible characters are well known and can be found by routine exercise or using the software SAGE (www.sagemath.org). There are 6 conjugacy classes for $PSL_2(F_7)$ represented by

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{Z}, \; g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{Z}, \; g_3 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} \text{Z},$$
$$g_4 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \text{Z}, \; g_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{Z and } g_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{Z}$$

where $Z = \{\pm I_2\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

The character table of $PSL_2(F_7)$ is given below:

$PSL_2(F_7)$	g_1	g_2	g_3	g_4	g_5	g_6
Class Size	1	21	42	56	24	24
χ_1	1	1	1	1	1	1
χ_2	7	-1	-1	1	0	0
χ_3	8	0	0	-1	1	1
χ_4	6	2	0	0	-1	-1
χ_5	3	-1	1	0	ζ	$\overline{\zeta}$
χ_6	3	-1	1	0	ζ	$\overline{\zeta}$

$\zeta = \frac{1}{2}(-1 + \sqrt{-7})$

3. Explicit description of action of $PSL_2(F_7)$ on $P^1(F_7)$

$\mathbf{P}^1(F_7)$ is the set of all lines in a 2-dimensional vector space over F_7 and they are described by their slopes except the line with infinite slope. We denote the elements of $\mathbf{P}^1(F_7)$ by $\{0, 1, 2, 3, 4, 5, 6, \infty\}$ representing the lines $y = 0, y = x, y = 2x, y = 3x, y = 4x, y = 5x$ and $x = 0$ respectively.

We fix obvious representative points for each of these eight lines, namely, $\{(1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (0, 1)\}$ denote the lines through origin with slopes 0,1,2,3,4,5,6 and ∞ respectively. $PSL_2(F_7)$ acts on $\mathbf{P}^1(F_7)$ by permuting the lines.

For our purpose of finding relative invariants we need to write down the action explicitly which we do here for one conjugacy class.

Action of g_2 on $\mathbf{P}^1(F_7)$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \infty \text{ which is the line with slope } \infty.$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} = 6 \text{ which is the line with slope 6.}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 3 \text{ which is the line with slope 3.}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \text{ which is the line with slope 2.}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 5 \text{ which is the line with slope 5.}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 4 \text{ which is the line with slope 4.}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \text{ which is the line with slope 1.}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \text{ which is the line with slope 6.}$$

(ie) $0 \mapsto \infty$ and $\infty \mapsto 0$, $1 \mapsto 6$ and $6 \mapsto 1$, $2 \mapsto 3$ and $3 \mapsto 2$, $4 \mapsto 5$ and $5 \mapsto 4$ which can be represented by the cycle notation of the permutation $(0\infty)(16)(23)(45)$.

Similarly, when g_3, g_4, g_5, g_6 acts on $P^1(F_7)$, we get the permutation $(01\infty6)$ (2435), $(0)(\infty)(124)(365)$, $(0)(145236\infty)$ and $(0)(1\infty63254)$ respectively. g_1 fixes all the lines and we get the identity permutation. By the action of all 168 elements of $PSL_2(F_7)$ on $P^1(F_7)$, we get 168 elements of the symmetric group S_8 (as a subgroup of S_8).

Relative invariants are defined for subgroups G of symmetric group S_n with respect to the irreducible characters of G. Here we use the above constructed embedding to find the dimensional formulæ for the vector spaces of relative invariants of $PSL_2(F_7)$ with respect to all its irreducible representations. Hereafter $g_1, g_2, g_3, g_4, g_5, g_6$ respectively denote the identity permutation, permutations of the type $2^4, 4^2, 3+3+1$, $7+1$ and $7+1$.

4. Dimensional formulæ for $PSL_2(F_7)$

Before stating the main result, we give the following lemma which will be used in the main result of the paper.

Lemma 4.1. *Let $f_1(d), f_2(d), f_3(d), f_4(d), f_5(d)$ and $f_6(d)$ denote the number of monomials fixed by the action of the identity permutation, permutations of the type $2^4, 4^2, 3+3+1$, $7+1$ and $7+1$ respectively on $H_d(x_1, x_2, \ldots, x_n)$. Then we have,*

- $f_1(d) = \binom{d+7}{7}$
- $f_2(d) = \binom{\frac{d}{2}+3}{3}$
- $f_3(d) = \binom{\frac{d}{4}+1}{1}$
- $f_4(d) = \sum_{l=0}^{\lfloor d/3 \rfloor} (l+1)(d-3l+1)$
- $f_5(d) = f_6(d) = \lfloor \frac{d}{7} \rfloor + 1$

[We follow the convention that $\binom{m}{k}$ is zero if m or k is not an integer.]

Now the derivations of $f_2(d)$ and $f_4(d)$ are given here.

Derivation of the formula $f_2(d)$: For a monomial to be invariant under the action of the permutation of type 2^4 say $(12)(34)(56)(78)$, the variables x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 should assume degrees $d_1, d_1, d_2, d_2, d_3, d_3, d_4, d_4$ respectively. But sum of all the degrees is d.

Hence we have $d_1+d_1+d_2+d_2+d_3+d_3+d_4+d_4 = d$. (ie) $d_1+d_2+d_3+d_4 = d/2$. Hence the number of monomials that are invariant under the action of of the permutation of type $(12)(34)(56)(78)$ is $\binom{\frac{d}{2}+3}{3}$.

Derivation of the formula $f_4(d)$: For a monomial to be invariant under the action of the permutation of type 3^2+1+1 say $(123)(456)(7)(8)$, the variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ should assume degrees $d_1, d_1, d_1, d_2, d_2, d_2, d_3, d_4$ respectively. But sum of all the degrees is d. Hence we have $d_1+d_1+d_1+d_2+d_2+d_2+d_3+d_4 = d$ (ie) $d_1+d_2 = \frac{d-d_3-d_4}{3}$. Put $\frac{d-d_3-d_4}{3} = l$. Now $d_1+d_2 = l$ denotes the number of partitions of l into 2 parts with l ranging from 0 to $\lfloor \frac{d}{3} \rfloor$ and simultaneously dividing $d-3l$ into two parts d_3 and d_4. Hence the number of monomials fixed that are invariant under the action of 3^2+1+1 is given by $f_4(d)=\sum_{l=0}^{\lfloor d/3\rfloor}(l+1)(d-3l+1)$.

The derivations of other formulæ are easy and similar to the above derivations and hence omitted.

Here we state the main result of our paper. The dimensional formulæ for the relative invariants of the Projective Special Linear group $PSL_2(F_7)$ is given in the theorem 4.1 below.

Theorem 4.1. *Let* $\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6$ *be the irreducible characters of* $PSL_2(F_7)$ *with degrees* $1,1,1,1,2,8$ *respectively. Then the dimensions of* $H_d(PSL_2(F_7),\chi_i)$ *for* $i=1,2,3,4,5,6$*, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Projective Special Linear group* $PSL_2(F_7)$ *is given below:*

(1) $\dim(H_d(PSL_2(F_7),\chi_1))$
$= (f_1(d)+21f_2(d)+42f_3(d)+56f_4(d)+24f_5+24f_5)/168$
$= (f_1(d)+21f_2(d)+42f_3(d)+56f_4(d)+48f_5(d))/168,$

(2) $\dim(H_d(PSL_2(F_7),\chi_2))$
$= (7f_1(d)-21f_2(d)-42f_3(d)+56f_4(d)+0f_5(d)+0f_6(d))/24$
$= (7f_1(d)-21f_2(d)-42f_3(d)+56f_4(d))/24$

(3) $\dim(H_d(PSL_2(F_7),\chi_3))$
$= (8f_1(d)+0f_2(d)+0f_3(d)-56f_4(d)+24f_5(d)+24f_5(d))/21$
$= (8f_1(d)-56f_4(d)+48f_5(d))/21$

(4) $\dim(H_d(PSL_2(F_7),\chi_4))$
$= (6f_1(d)+42f_2(d)+0f_3(d)+0f_4(d)-24f_5(d)-24f_5(d))/28$
$= (6f_1(d)+42f_2(d)-48f_5(d))/28$

(5) $\dim(H_d(PSL_2(F_7),\chi_5))=\dim(H_d(PSL_2(F_7),\chi_6))$
$= (3f_1(d)-21f_2(d)+42f_3(d)+0f_4(d)+24\zeta f_5(d)+24\overline{\zeta}f_5(d))/56$
$= (3f_1(d)-21f_2(d)+42f_3(d)-24f_5(d))/56$

Using the above theorem, we have calculated the dimensions $\dim(H_d(PSL_2(F_7),\chi_i))$, $i=1,2,\ldots,6$ for degrees $d=0,1,2,\ldots,10$. In the table $\dim(\chi_i)$,$i=1,2,\ldots,6$ denote $\dim(H_d(PSL_2(F_7),\chi_i)),i=1,2,\ldots,6$ and

$\dim(H_d)$ denote the dimension of the vector space of homogeneous polynomials of degree d.

Degree d	$\dim(\chi_1)$	$\dim(\chi_2)$	$\dim(\chi_3)$	$\dim(\chi_4)$	$\dim(\chi_5)$	$\dim(\chi_6)$	$\dim(H_d)$
0	1	0	0	0	0	0	1
1	7	1	0	0	0	0	8
2	2	14	8	12	0	0	36
3	3	49	32	24	6	6	120
4	7	105	104	84	15	15	330
5	9	259	272	168	42	42	792
6	19	525	608	396	84	84	1716
7	29	1057	1248	732	183	183	3432
8	54	1911	2376	1428	333	333	6435
9	82	3430	4256	2448	612	612	11440
10	140	5740	7280	4248	1020	1020	19448

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