ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.3, 1419–1423
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.3.31

ON THE STUDY OF ASSOCIATION OF UTILITY FUNCTION WITH DIFFERENT $(h,\phi)-$ENTROPIES

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ABSTRACT. Boltzmann and Gibbs initiated the notion of entropy in the 19^{th} century. Entropy measures the randomness or uncertainity of a random phenomenon. Generalized entropies are being extensively studied now-a-days. The affluent utilization of generalizations of Shannon entropy in today's world has thought-provoked the exploration of generalized utility entropies that have been much less analysed than the classical entropies. Also (h,ϕ) entropies are gaining significance in many areas of research because of their usefulness. In this paper, much endeavour has been laid down on generalized utility entropies which can further extend the studies of Information theory in distinct fields and hence prove beneficial in solving broad-spectrum of physical problems in areas of geophysics, astronomy, medical research etc.

1. INTRODUCTION

The notion of entropy was initiated by Clausius to measure disorder in system. Further, Boltzmann, Gibbs gave atomic interpretation to the previous research works. After that C. Shannon used the concept of entropy extensively in Information theory, He modified the concept of entropy in the field of Information theory that was formally introduced by Boltzmann and Gibbs in 19^{th}

2020 *Mathematics Subject Classification.* 94A17, 94A15, 94A05.
Key words and phrases. $(h,\phi)-$entropy, utility function, generalized utility entropies.
Submitted: 22.02.2021; *Accepted:* 13.03.2021; *Published:* 22.03.2021.

century. Tsallis entropy [1] and Rényi entropy [2] are distinctive generalizations of Shannon entropy [3]. Following this, many different generalized entropies have been defined to attune to many diverse fields, like Physics, Geography, Artificial Intelligence, Statistics etc. Entropy becomes mainly noticeable in frame of references of Information theory, Mechanics, Thermodynamics etc. as a degree of disorder and uncertanity.

2. Definitions

Entropy of discrete set of probabilites is given by

$$H = -\sum p_i \log p_i,$$

where p_i represents i^{th} probability. In the same manner, entropy of a continuous distribution is given by

$$H = -\int_{-\infty}^{\infty} p(x) \log p(x) dx,$$

where $p(x)$ is its distribution function. Again, joint entropy is given by

$$H(x,y) = -\int_{-\infty}^{\infty} p(x,y) \log p(x,y) dx dy.$$

Most of the properties of discrete case are followed by the entropies of continuous distributions [3].

3. (h, ϕ)-entropies

They were defined by Salicru et al. [4] which were further motivated by research of Csiszar [5]. As in [9], we set

$$S_{h(y),\phi(x)}(v) = h\left[\sum_{i \in E} \phi\ (v_i)\right],$$

where v is any measure on a countable space E.

Here, $h : \mathbb{R} \to \mathbb{R}$ and $\phi : [0,1] \to \mathbb{R}$ are such that h may be strictly increasing with ϕ strictly concave or h may be strictly decreasing with ϕ strictly convex [6]. Some very well-known entropies can be generalized using (h, ϕ)-entropies.

(h,ϕ)-entropies are used in quantum mechanics [7] and many other uncertainity relations [8]. Utility measures are practically very useful and mathematically suitable too. Also, when worked in terms of logarithms, limiting operations may become simple.

Values of $h(y)$ and $\phi(x)$ for different (h,ϕ) entropies are given as follows:

i. For Shannon entropy, $h(y) = y$ and $\phi(x) = -x\log x$
ii. For Renyi, $h(y) = (1-s)^{-1}\log y$ and $\phi(x) = x^s$
iii. For Tsallis, $h(y) = (r-1)^{-1}(1-y)$ and $\phi(x) = x^r$
iv. For Varma, $h(y) = [t(t-r)]^{-1}\log y$ and $\phi(x) = x^{r/t}$
v. For Havrda and Charvat, $h(y) = y$ and $\phi(x) = (1-2^{1-r})^{-1}(x-x^r)$
vi. For Arimoto, $h(y) = (t-1)^{-1}(y^t-1)$ and $\phi(x) = x^{1/t}$
vii. For Taneja, $h(y) = y$ and $\phi(x) = -x^r\log x$

Utility function is associated to the above-mentioned $(h,\phi)-$entropies as depicted and a new set of (h,ϕ)-entropies are obtained in the Table 1.

TABLE 1. Generalized $(h,\phi)-$entropies using utility function.

$h(y)$	$\phi(x,u)$	$S_{h,\phi}(v)$	$(h,\phi)-$entropy with utility function u
y	$-ux\log x$	$-\sum u(i)v(i)\log v(i)$	Shannon with utility function or Guiasu
$\frac{1}{(1-s)}\log y$	$x^{(s-1)u+1}$	$\frac{1}{1-s}\log\left[\sum v(i)^{(s-1)u(i)+1}\right]$	Rényi with utility function
y	$\frac{(x-x^r)u}{1-2^{1-r}}$	$\sum\frac{(v(i)-(v(i))^r)u(i)}{1-2^{1-r}}$	Havrda and Charvat with utility function
$(r-1)^{-1}(1-y)$	$x^{(r-1)u+1}$	$h\left[\sum v(i)^{(r-1)u(i)+1}\right]$ $=\frac{1}{r-1}\left[1-\sum v(i)^{(r-1)u(i)+1}\right]$	Tsallis with utility function
$[t(t-r)]^{-1}\log y$	$x^{((r-1)u+1)/t}$	$h\left[\sum v(i)^{((r-1)u(i)+1)/t}\right]$ $=\frac{1}{t(t-r)}\log\left[\sum v(i)^{((r-1)u(i)+1)/t}\right]$	Varma with utility function
$(t-1)^{-1}(y^t-1)$	$x^{u/t}$	$h\left[\sum v(i)^{u(i)/t}\right]$ $=\frac{1}{t-1}\left[\left(\sum v(i)^{u(i)/t}\right)^t-1\right]$	Arimoto with utility function
y	$-x^{(r-1)u+1}\log x$	$-\sum v(i)^{(r-1)u(i)+1}\log u(i)$	Taneja with utility function

4. OBSERVATIONS

The first entry in the Table 1 approaches to classical (h, ϕ) entropy. For instance, when $h(y) = y$ and $\phi(x) = -ux \log x$, then

$$S_{y,-ux\log x}(v) = h\left[\sum_{i\in E}\phi(v(i))\right] = h\left[-\sum_{i\in E}u(i)v(i)\log v(i)\right]$$
$$= -\left[\sum_{i\in E}u(i)v(i)\log v(i)\right]$$

which is Guiasu entropy and it approaches to Shannon entropy as $u(i) \to 1$. The second entry in the Table 1 in which utility function is associated to Rényi entropy tends to Guiasu entropy when $s \to 1$. The third entry also tends to Guiasu entropy when $r \to 1$ [10]. In the remaining entries, utility function is associated to each of the pre-defined $(h, \phi)-$entropies but their convergence is yet to be researched.

5. CONCLUSION AND FURTHER SCOPE OF RESEARCH

This table could be of particular interest if further refinements can be thought of, which can make the function become an entropy in its usual definition. For instance, normalization could be done, maximum and minimum i.e., optimal values of entropies can be computed and some explicit formulae may be obtained that could be helpful in further research in the field of Information theory.

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