ADV MATH
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COVID-19 ANALYSIS BY SEIR MODEL WITH MULTIPLE CONTROLS

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ABSTRACT. A SEIR mathematical model with multiple controls self - prevention, treatment and vaccination is formulated. The properties of Pontryagin's maximum principle were verified and found the optimal levels of controls. Numerical simulations were shown to exhibit the flow of variables with or without control strategies.

1. INTRODUCTION

The control measures were followed to reduce the spread of the infectious diseases. As we all know 'Prevention is better than cure', it is better to be self preventive which is the primary control measure. The self prevention such as social distancing, washing hands often, using clean cloth while coughing/sneezing and self quarantine helps us to prevent the spread of the disease. The medical care helps to cure the disease and makes to feel better. So treatment plays an important role as a control measure. Vaccination is another key control measure which develops an immune in our body and control the infection of the diseases.

Mathematical modeling analysis explains the transmission process and control of the infectious diseases. Optimal control theory for infectious diseases to used

to optimize the cost and the recovery individuals. Many Mathematicians used the Optimal control theory for mathematical models for an infectious diseases. Particularly, Tunde Tajudeen Yusuf and Francis Benyah [6] derived a SIR model with vaccination and treatment as control strategies. A.Huppert [2] analysed the SIR model with vaccination as control strategy. Jakia sultana and Chandra N.Podder [3] used the optimal control theory for Nipah virus infections. Gul Zaman [1] studied the behaviour of various infectious diseases and its control strategies. Nita H.Shah [5] formulated a mathematical model for the Human - Bat population with control strategies. A. Mhlanga [4] formulated and analyzed the two patch model for Ebola virus disease.

The first COVID - 19 case was reported on 1 April 2020 in Vellore district of Tamilnadu, India. In this paper, the data of COVID - 19 infection in Vellore district has been collected from the Vellore Collectorate press release. We have formulated and analysed a SEIR model for COVID - 19 with multiple controls self - prevention, treatment and vaccination.

2. FORMULATION OF MATHEMATICAL MODEL AND CONTROL SET

The SEIR model with multiple controls for COVID-19 is represented by ordinary differential equations as follows:

(2.1)
$$\frac{dS}{dt} = \omega N(t) - \beta S(t)I(t) - \mu S(t) - (u_1 + u_3)S(t)$$
$$\frac{dE}{dt} = \beta S(t)I(t) - (\mu + \alpha)E(t) - (u_1 + u_2)E(t)$$
$$\frac{dI}{dt} = \alpha E(t) - (\gamma + \mu + d + u_2)I(t)$$
$$\frac{dR}{dt} = (\gamma + u_2)I(t) + (u_1 + u_3)S(t) + (u_1 + u_2)E(t) - \mu R(t)$$

where $S(t), E(t), I(t), R(t), N(t)$ are the Susceptible, Exposed, Infectious, Recovery state, Total population respectively, ω - Average birth rate, μ - Average death rate, β - Transistion infectious rate, γ - Recovery rate and d - disease induced death rate. Here we assume u_1 - measures for self - prevention, u_2 - treatment measures and u_3 - vaccination as the controls for the model.

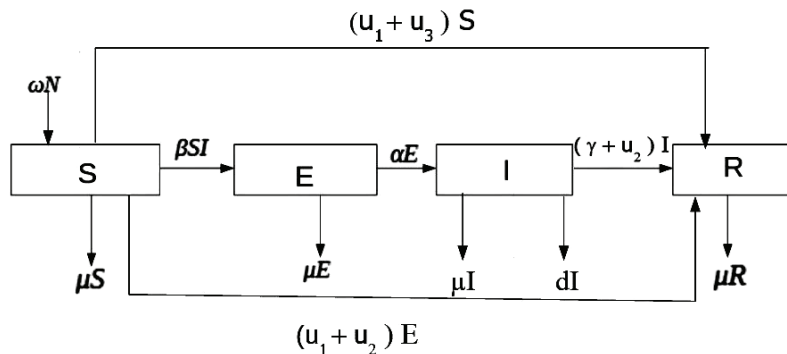The transmission of SEIR model for COVID-19 with multiple controls is diagramatically represented as follows.

FIGURE 1. SEIR model with multiple controls for COVID-19

For this system of equations, we define an objective functional similar as in [6]

$$(2.2) \qquad J = \min_{u_1, u_2, u_3 \in U} AI(t) + \frac{1}{2} \int_0^T (w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2) dt$$

which is a free problem system subject to infected.

Here $AI(t)$ is the number of individuals, w_1, w_2, w_3 are the weight parameters for the cost of awareness for self- prevention, treatment and vaccination respectively.

The control set is defined as

$$(2.3) \quad U = \{(u_1(t), u_2(t), u_3(t)) : 0 \le u_1 \le 1, 0 \le u_2 \le 1, 0 \le u_3 \le 1, t \in [0,1]\}.$$

3. PROPERTIES OF PONTRYAGIN'S MAXIMUM PRINCIPLE

Pontryagin's maximum principle is a powerful method to compute the optimal controls. To use this, we have to prove the properties [3]:

P_1: The set of controls and the corresponding state variables is non empty.

P_2: The control set U is convex and closed.

P_3: The R.H.S. of the state system is bounded by a linear function in the state and control variables.

P_4: The integrand of the objective functional is convex on U and is bounded below by $-k_2 + k_1 |U|^\eta$, with $k_1 > 0, k_2 > 0, \eta > 1$.

To prove this we need the following theorem and lemma.

Theorem 3.1. *Existence and Uniqueness Theorem Let $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ and $f(x, y)$ be continuous on a domain $\{D = (x, y)/x_0 - a < x < x_0 + a; y_0 - b < y < y_0 + b\}$ with the Lipschitz condition $||f(x, y_1) - f(x, y_2)|| \leq K||y_1 - y_2||$ where K is a positive integer, then there exist bounded solution in D. Further more, if $f(x, y)$ is Lipschitz continuous with respect to y on a rectangle $R = \{(x, y)/x_0 - c < x < x_0 + c; y_0 - b < y < y_0 + b; c < a\}$ then there is a unique solution $y(t)$ in R.*

Lemma 3.1. *If $f(x, y)$ has continuous partial derivative $\frac{\partial f_i}{\partial y_j}$ on a bounded closed domain R, is a set of real numbers. Then it satisfies a Lipschitz condition in R.*

To prove P_1: From (2.1),

(3.1) $$F_1 = \omega N(t) - \beta S(t)I(t) - \mu S(t) - (u_1 + u_3)S(t)$$

(3.2) $$F_2 = \beta S(t)I(t) - (\mu + \alpha)E(t) - (u_1 + u_2)E(t)$$

(3.3) $$F_3 = \alpha E(t) - (\gamma + \mu + d + u_2)I(t)$$

(3.4) $$F_4 = (\gamma + u_2)I(t) + (u_1 + u_3)S(t) + (u_1 + u_2)E(t) - \mu R(t)$$

From (3.1), (3.2), (3.3) and (3.4),

$$\left|\frac{\partial F_1}{\partial S}\right| < \infty, \quad \left|\frac{\partial F_1}{\partial E}\right| < \infty, \left|\frac{\partial F_1}{\partial I}\right| < \infty, \quad \left|\frac{\partial F_1}{\partial R}\right| < \infty$$

$$\left|\frac{\partial F_2}{\partial S}\right| < \infty, \quad \left|\frac{\partial F_2}{\partial E}\right| < \infty, \left|\frac{\partial F_2}{\partial I}\right| < \infty, \quad \left|\frac{\partial F_2}{\partial R}\right| < \infty$$

$$\left|\frac{\partial F_3}{\partial S}\right| < \infty, \quad \left|\frac{\partial F_3}{\partial E}\right| < \infty, \left|\frac{\partial F_3}{\partial I}\right| < \infty, \quad \left|\frac{\partial F_3}{\partial R}\right| < \infty$$

$$\left|\frac{\partial F_4}{\partial S}\right| < \infty, \quad \left|\frac{\partial F_4}{\partial E}\right| < \infty, \left|\frac{\partial F_4}{\partial I}\right| < \infty, \quad \left|\frac{\partial F_4}{\partial R}\right| < \infty$$

Here the partial derivatives are continuous and bounded. Hence there exist a unique solution of (2.1).

∴ the set of controls and the corresponding state variables is non empty, P_1 is satisfied.

To prove P_2: U is closed, by the definition. Let us take two controls $u_2, u_3 \in U$ and $\alpha \in [0, 1] \implies \alpha u_2 + (1 - \alpha)u_3 \geq 0$. As $u_2 \leq 1 \implies \alpha u_2 \leq \alpha$ and $u_3 \leq 1 \implies (1 - \alpha)u_3 \leq (1 - \alpha)$.

∴ $0 \leq \alpha u_2 + (1 - \alpha)u_3 \leq 1, \forall u_2, u_3 \in U$. Hence U is convex, P_2 is satisfied.

To prove P_3:

From (3.1), $F_1 \leq \omega N - (u_1 + u_3)S$.

From (3.2), $F_2 \leq \beta SI - (u_1 + u_2)$.

From (3.3), $F_3 \leq \alpha E - u_2 I$.

From (3.4), $F_4 \leq (\gamma + u_2)I + (u_1 + u_3)S + (u_1 + u_2)E$.

The state system can be rewrite in the matrix form,

$$
\bar{F}(t, \bar{X}, U) \leq
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \beta S & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \gamma & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
S \\ E \\ I \\ R
\end{bmatrix}
+
\begin{bmatrix}
-S \\ -E \\ -I \\ S + E + I
\end{bmatrix}
\cdot
\begin{bmatrix}
u_1(t) \\ u_2(t) \\ u_3(t)
\end{bmatrix}
$$

which can be written as the linear combination of controls,

$$
\left| \bar{G}(t, \bar{X}, U) \right| \leq \|\bar{m}\| \left| \bar{X} \right| + \left\| \bar{S} + \bar{E} + \bar{I} \right\| |u_1(t), u_2(t), u_3(t)|
$$
$$
\leq K_2 \left[\left| \bar{X} \right| + |u_1(t), u_2(t), u_3(t)| \right].
$$

As $\bar{S}, \bar{E}, \bar{I}$, are bounded and K_2 is upper bound. Hence R.H.S. is bounded by the sum of state and control variables, P_3 is satisfied.

To prove P_4: The control and state variables are non negative and the control variables $u_1, u_2, u_3 \in U$ is also convex and closed. Then the integrand of the objective functional (2.2) is convex on the set U. Now,

$$
J = AI(t) + \frac{w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2}{2}
$$
$$
\geq -AI(t) + \frac{w}{2}(u_1^2 + u_2^2 + u_3^2) = -K_2 + K_1(u_1, u_2, u_3)^2{}',
$$

where $w = w_1 + w_2 + w_3$ and $K_1 = \frac{w}{2}$

Here $K_2 > 0$ as it depends on I, $K_1 > 0$ and $\eta = 2 > 1$. Hence P_4 is satisfied.

$\therefore$ The control system satisfies the properties of Pontryagin's maximum principle then there exists an optimal control which we construct in next section.

4. EXISTENCE OF OPTIMAL CONTROL

The properties for Pontryagin's maximum principle are satisfied, which converts (2.1) into a problem of minimizing.

An Hamiltonian H is defined as

(4.1)
$$
\begin{aligned}
H(S, E, I, R) = {} & AI(t) + \frac{w_1 u_1^2}{2} + \frac{w_2 u_2^2}{2} + \frac{w_3 u_3^2}{2} \\
& + \lambda_S(\omega N - \beta SI - \mu S - (u_1 + u_3)S) \\
& + \lambda_E(\beta SI - (\mu + \alpha)E - (u_1 + u_2)E) \\
& + \lambda_I(\alpha E - (\gamma + \mu + d + u_2)I) \\
& + \lambda_R((\gamma + u_2)I + (u_1 + u_3)S + (u_1 + u_3)E - \mu R).
\end{aligned}
$$

The adjoint system is given by

$$
\begin{aligned}
\frac{d\lambda_S}{dt} &= \lambda_S(\beta I + \mu + u_1 + u_3) - \lambda_E \beta I - (u_1 + u_3)\lambda_R \\
\frac{d\lambda_E}{dt} &= \lambda_E(\mu + \alpha + u_1 + u_2) - \alpha \lambda_I - \lambda_R(u_1 + u_2) \\
\frac{d\lambda_I}{dt} &= \lambda_S \beta S - \lambda_E \beta S + \lambda_I(\gamma + \mu + d + u_2) - (\gamma + u_2)\lambda_R - A \\
\frac{d\lambda_R}{dt} &= \mu \lambda_R
\end{aligned}
$$

with the final conditions $\lambda_S(T) = \lambda_E(T) = \lambda_I(T) = \lambda_R(T) = 0$ for free problem.

The optimal controls can be obtained from the optimality conditions, $\frac{\partial H}{\partial u_1} = 0$; $\frac{\partial H}{\partial u_2} = 0$; $\frac{\partial H}{\partial u_3} = 0$ we have

$$
\begin{aligned}
u_1^* &= \frac{S(\lambda_S - \lambda_R) + E(\lambda_E - \lambda_R)}{w_1}, \\
u_2^* &= \frac{E(\lambda_E - \lambda_R) + I(\lambda_I - \lambda_R)}{w_2}, \\
u_3^* &= \frac{S(\lambda_S - \lambda_R)}{w_3}.
\end{aligned}
$$

Hence the optimal controls are

$$
\begin{aligned}
u_1^* &= min\left(1, max\left(0, \frac{S(\lambda_S - \lambda_R) + E(\lambda_E - \lambda_R)}{w_1}\right)\right), \\
u_2^* &= min\left(1, max\left(0, \frac{E(\lambda_E - \lambda_R) + I(\lambda_I - \lambda_R)}{w_2}\right)\right), \\
u_3^* &= min\left(1, max\left(0, \frac{S(\lambda_S - \lambda_R)}{w_3}\right)\right).
\end{aligned}
$$

5. Numerical Analysis

The birth rate is $\omega = 0.0477$ and death rate is $\mu = 0.0203$ for Vellore district and the transmission rate is $\beta = 0.011$, the recovery rate is $\gamma = 0.96$ and the death rate is $d = 0.017$ for COVID-19.
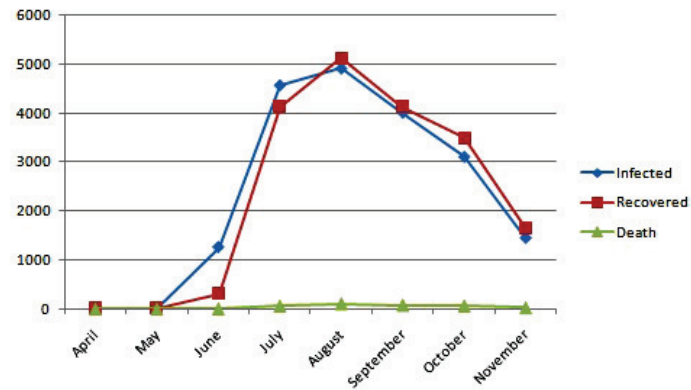

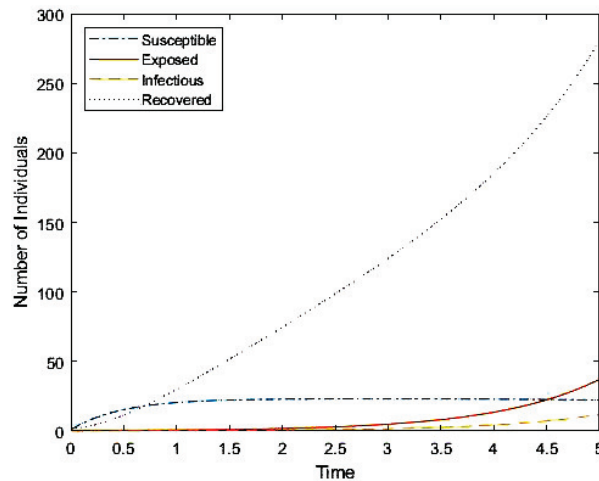FIGURE 2. Transmission of COVID-19 in Vellore district

FIGURE 3. Flow of Variables with respect to Time with multiple controls

Figure (2) represents the Confirmed cases, Recovered cases and Death cases of Covid-19 in Vellore district as on 30.11.2020. Figure (3) shows the flow of Susceptible, Exposed, Infectious and Recovery for the model.

Flow of the variables with and without controls
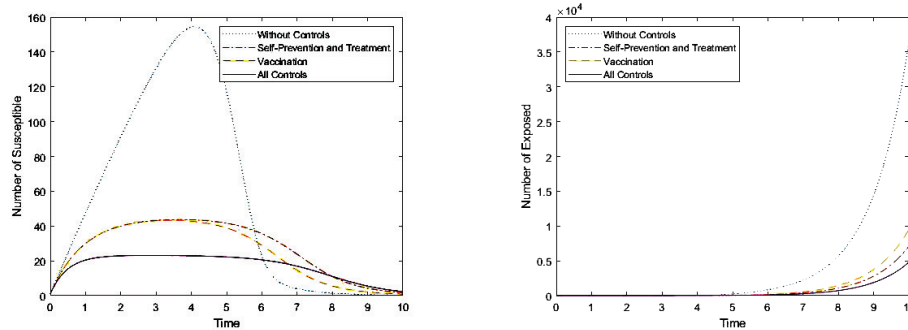

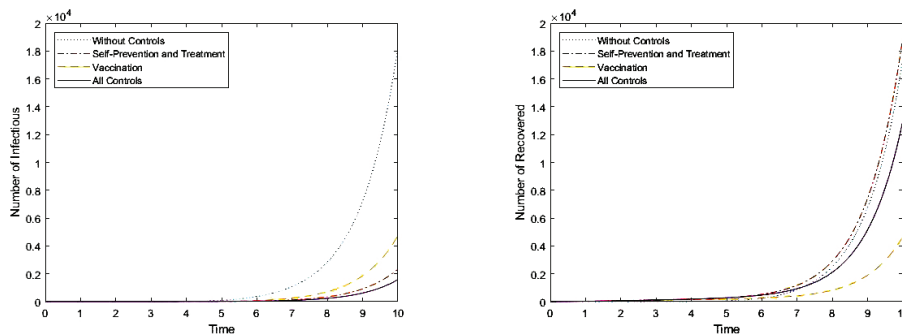FIGURE 4. Susceptible and Exposed class

FIGURE 5. Infectious and Recovered class

Figure (4) (left) shows that the Susceptible with all the controls is less than the Susceptible without controls, also Susceptible with vaccination is less than Susceptible with self prevention and treatment and figure(4) (right) shows that the Exposed with all the controls is less than the Exposed without controls, also the Exposed with vaccination is higher than the Exposed with Self-Prevention and treatment.

It is clear from figure (5) (left) that the Infectious with all the controls is less than the Infectious without controls, also the Infectious with vaccination is higher than the Infectious with Self-Prevention and treatment and from figure (5) (right) that the recovery with Self-Prevention and treatment is higher than the recovery without controls, the recovery with vaccination and the recovery with all controls.

6. Conclusion

A SEIR model with multiple control measures self - Prevention, treatment and vaccination was formulated. Properties of Pontryagin's maximum principle were verified. By Pontryagin's maximum principle, the optimal levels of controls were analysed. Transition of COVID-19 in Vellore district and the flow of variables with controls with respect to time were exhibited in numerical simulations. The graphs in the Susceptible, Exposed, Infectious and recovery class with and without controls shows Vaccination can reduce the Susceptible, Exposed and Infectious can be controlled with all the controls and Recovered can be increased by Self-Prevention and treatment.

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