Advances in Mathematics: Scientific Journal **10** (2021), no.1, 1603–1612
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.1.44

RIPPLES HAMILTONIAN GRAPH

R. Rama[1], N. jansirani, and V.R. Dare

ABSTRACT. In the topology of interconnection network designing, Hamiltonian property plays a significant control and its application involves strification of triangle meshes in computer graphics. Motivated by these studies, a new type of Ripples Graph is introduced. At each level the length of the cycle forms the famous Fibonacci Sequence and by construction, the Ripples Graph (RG) forms a Hamiltonian Circuit at each level. It is interesting to note that, the closure of RG is also a Hamiltonian. Fault-Tolerant Hamiltonian of Ripples Graph and Closure of Ripples Graph are studied. A new type of min-leaf vertex fault-tolerant Hamiltonian is introduced. Colouring and maximum (minimum) degree of each vertex have been studied for large $n, n \in \mathbb{N}$ and $n \geq 1$. Further a relation between chromatic number and maximum degree is established. The number of vertices on C_n for each $n \geq 3$ satisfies the property that atleast 3 vertices have degree less than 4. A relation between minimum degree of RG and face is obtained as $\delta(RG) = p - q + r$. Moreover, RG has no subgraph homeomorphic to $K_5, K_{3,3}$ and also satisfies kuratowski condition. Finally, the thickness and genus are also studied.

1. INTRODUCTION

The architecture of an interconnection network is always represented by a graph, where vertices represent processors and edges represent links between

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C38, 05C45.
Key words and phrases. Chromatic number, Cycle, Degree, Hamiltonian.
Submitted: 09.02.2021; *Accepted:* 13.03.2021; *Published:* 24.03.2021.

processors. It is almost impossible to design a network that is optimum from all aspects. One has to design a suitable network depending on its properties and requirements. Thus, many graphs are proposed as possible interconnection network topologies. Thus, these graphs can be referred to as good graphs. For this reason, the theory of interconnection networks is referred to as good – graph theory. The Hamiltonian properties are one of the major requirements in designing the topology of a network. For example, the Token ring approach is used in some distributed systems. An interconnection network requires the presence of Hamiltonian cycles in the structure to meet this approach [5].

In graph theory, Hamiltonian Circuit involves many applications. Some application of the Hamiltonian circuit are The Traveling Salesman Problem (TSP) is any problem where you must visit every vertex of a weighted graph once and only once, and then end up back at the starting vertex, package deliveries, fabricating circuit boards, scheduling jobs on a machine and running errands around town. And also Hamiltonian Circuit problem is among famous NP- complete problem.

In this paper, we introduce a new type of graph is called Ripples Graph (RG).

2. Basic definition

In this section, we present some basic definitions related to Hamiltonian and Fault-Tolerant Hamiltonicity. A path in a graph $G = (E, V)$ is a Hamiltonian path if its nodes are distinct and they span V. A cycle is a path with at least three nodes such that the first node is the same as the last one. A cycle is a Hamiltonian cycle if it traverses every node of G exactly once. A graph G is Hamiltonian if it has a Hamiltonian cycle, and G is Hamiltonian connected if there exists a Hamiltonian path joining any two nodes of G. A colouring of a graphs G using at most n colours is called an n – colouring. The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour G. A graph G is called n- colourable if $\chi(G) \leq n$ [4].

The closure of a graph G is the graph obtained from G by repeatedly joining pairs of nonadjacent vertices and it is denoted by $c(G)$. The minimum number of planar subgraph whose union is the given graph G is called the thickness of G and is denoted by $\theta(G)$. The genus of a graph G is defined to be the minimum number of handles to be attached to a sphere so that G can be drawn on the resulting surface without intersecting lines [1].

The graph $G = (V, E)$ is called k-vertex (or l-edge)-fault -tolerant Hamiltonian Graph if $G - V'$ (or $G - E'$) is Hamiltonian for any $V' \subseteq V$ (or $E' \subseteq E$) and $|V'| = k$ (or $|E'| = l$). The vertex fault – tolerant Hamiltonicity, $H_v(G)$ is defined as the maximum integer m such that $G - F$ remains Hamiltonian for every $F \subset V(G)$ with $|F| \leq m$ if G is Hamiltonian [5].

In literature, the following positive integers sequence called Fibonacci Sequence $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

The sequence f_n of Fibonacci numbers is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, where $f_1 = f_2 = 1$.

3. Construction of Ripples Graph

In this section, a new type of Concentric Circles pattern graph is constructed and is called Ripples Graph (RG). Each Concentric Circle(C) is considered as a single level and in the initial stage the number of the vertices on the basic inner Concentric Circle C_1 is 3(first level). In the second level the number of vertices on C_2 is 5 and the number of vertices on C_3 is 8 and so on. Hence the number of vertices on each $C_i's$ follows the famous Fibonacci sequence. Concentric Circles are linked by the edges. By a new sequence of linked edge set denoted by $E_L = \{2, 3, 5, \ldots\}$. At each level there exists a only one pair of adjacent vertices which are connected to the inner Concentric Circles which is illustrated in figure 1.

Theorem 3.1. *The total number of edges in k^{th} level of Ripples Graph (RG) is $E_T(RG) = E_C(RG) + E_L(RG) = \Sigma_{i=1}^{k} f_{i+3} + \Sigma_{j=1}^{k-1} f_{j+2}, i > j$ and $k \in \mathbb{N}$ where f_i and f_j are the i^{th} and j^{th} level of Fibonacci Sequence.*

Proof. Proof of this theorem followed by method of induction hypothesis on the level of the Ripples Graph.

Let $k = 2$. The total number of edges in RG = $\Sigma_{i=1}^{2} f_{i+3} + f_{1+2} = f_4 + f_5 + f_3 = 10$. Assume that the theorem is true for $k = n - 1$. The total number of edges in $n - 1^{th}$ level of RG = $\Sigma_{i=1}^{n-1} f_{i+3} + \Sigma_{j=1}^{n} f_{j+2}$. r Therefore the total number of

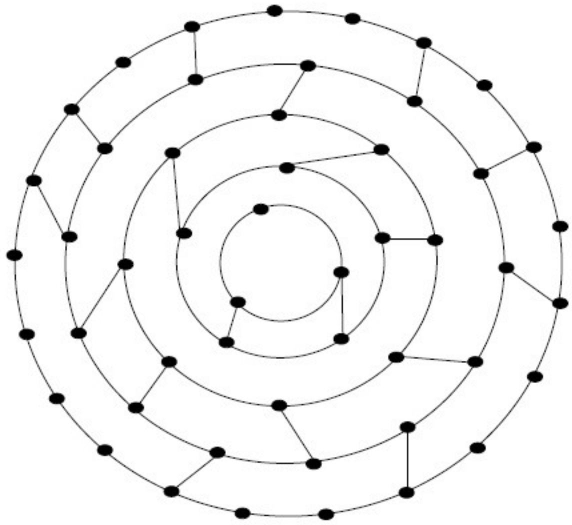

FIGURE 1. Ripples Graph (RG) at level - 5

edges in n^{th} level of RG is

$$\Sigma_{i=1}^{n} f_{i+3} + \Sigma_{j=1}^{n-1} f_{j+2}$$
$$= \{f_4 + f_5 + \ldots + f_{n+3}\} + \{f_3 + f_4 + \ldots + f_{n+1}\}$$
$$= \{f_4 + f_5 + \ldots + f_{n+2} + f_{n+3}\} + \{f_3 + f_4 + \ldots + f_{n+2} + f_{n+1}\}$$
$$= \Sigma_{i=1}^{n-1} f_{i+3} + fn + 3 + \Sigma_{j=1}^{n} f_{j+2} + f_{n+1}$$
$$= \Sigma_{i=1}^{n-1} f_{i+3} + \Sigma_{j=1}^{n} f_{j+2} + f_{n+3} + f_{n+1} \quad \text{(by assumption)}$$

Hence proved. □

Corollary 3.1. *The total number of vertices in k^{th} level of RG is V(RG) $= \Sigma_{i=1}^{k} f_{i+3}$, where f_i is the i^{th} level of Fibonacci Sequence.*

Corollary 3.2. *Let G be a Ripples Graph. Then every vertex of RG has either $\delta(v) = 2$ or $\Delta(v) = 3$.*

Theorem 3.2. *Every Ripples Graph (RG) is 3 - colourable.*

Proof. We will prove the theorem by induction on the level of vertex set p points of the Ripples Graph.

We know that, the total number ofk^{th} level of vertices in RG is $\Sigma_{i=1}^{k} f_{i+3}$, where f_i is the i^{th} term of Fibonacci Sequence.

Base case: $k = 1$.

The number of vertices in first level of RG $= f_4 = 3$. That is $p = 3$, and the first level cycle graph with 3 vertices (see figure 2). Therefore the result is obvious.

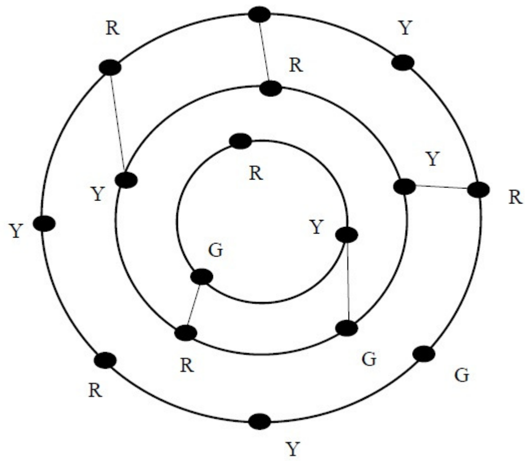

FIGURE 2. 3 colourable RG at level - 5

Here R - Red, G- Green and Y- Yellow.

Now let assume that the planar RG with p points is 3 – colourable for $p \geq 3$. Since the maximum degree of RG is 3. That is, the number of vertices in $k - 1^{th}$ level of RG is $p = \Sigma_{i=1}^{k-1} f_{i+3} \geq 3$. Then RG has a vertex v of degree 3 or less. Hence, the number of vertices in k^{th} level of RG

$$\Sigma_{i=1}^{k} f_{i+3}$$
$$= f_4 + f_5 + f_6 + \ldots + f_{k+2} + f_{k+3}$$
$$= \Sigma_{i=1}^{k-1} f_{i+3} + f_{k+3}$$
$$= \text{3- colourable } \; if \; p \geq 3 + f_{k+3} \text{ (since by assumption)}.$$

□

4. Fault Tolerant Hamiltonicity of the Ripples Graph

In a network (represented by a graph), when faults occur then that network is not reliable anymore. Thus, to solve it, we need to remove the vertices or edges. Fault-Tolerance is also desirable feature in massive parallel system that relatively high probability of failure. A number of fault-tolerant design for specific architectures have been proposed and is represented by a graph [8]. In this section, we discuss the fault tolerant Hamiltonicity of the Ripples Graph and some of the properties of Ripples Graph (RG).

Theorem 4.1. *If Ripples Graph (RG) has a spanning cycle then it is Hamiltonian.*

Proof. Given that, Ripples Graph (RG) has a cycle. This cycle contains each vertex of RG exactly once. In this condition satisfied Hamiltonian Cycle property then RG will be called Hamiltonian or Ripples Hamiltonian Graph. □

Corollary 4.1. *Every level of RG is Hamiltonian.*

Theorem 4.2. *Ripples Hamiltonian Graph is a 2- connected.*

Proof. Let G be a Ripples Hamiltonian Graph. We know that Ripples Hamiltonian Graph has a Hamiltonian cycle (by theorem 4.1). Let Z be a hamiltonian cycle in RHG. For any vertex u of RHG, $Z - u$ is connected and hence $G - u$ is also connected. Hence G has no cut points and thus RHG is 2 – connected. □

Definition 4.1. *If RG be a Ripples Hamiltonian Graph. Then RG is called mini-leaf vertex fault tolerant hamiltonian if we remove the minimum degree of vertices from RG and again we remove the leaf vertices from the RG.*

Theorem 4.3. *The total number of removed vertices of k^{th} level of mini-leaf fault tolerant Ripples Hamiltonian Graph (RHG) is $|V| = V_{mini-leaf} = C_{k-1} + C_{k-2} - 1, k \geq 3$, where C_{k-1} and C_{k-2} are number of vertices of $k-2^{th}$ concentric circle respectively.*

Proof. The Proof of the theorem is followed by method of induction hypothesis.

Base case: $k = 3$

$V_{mini-leaf} = C_{3-1} + C_{3-2} - 1 = 5 + 3 - 1 = 7.$

Assume that the theorem is true for $k = n - 1$. That is $V_{mini-leaf} = C_{n-1-1} + C_{n-1-2} - 1 = C_{n-2} + C_{n-3} - 1$. Hence the total number of n^{th} level of mini-leaf

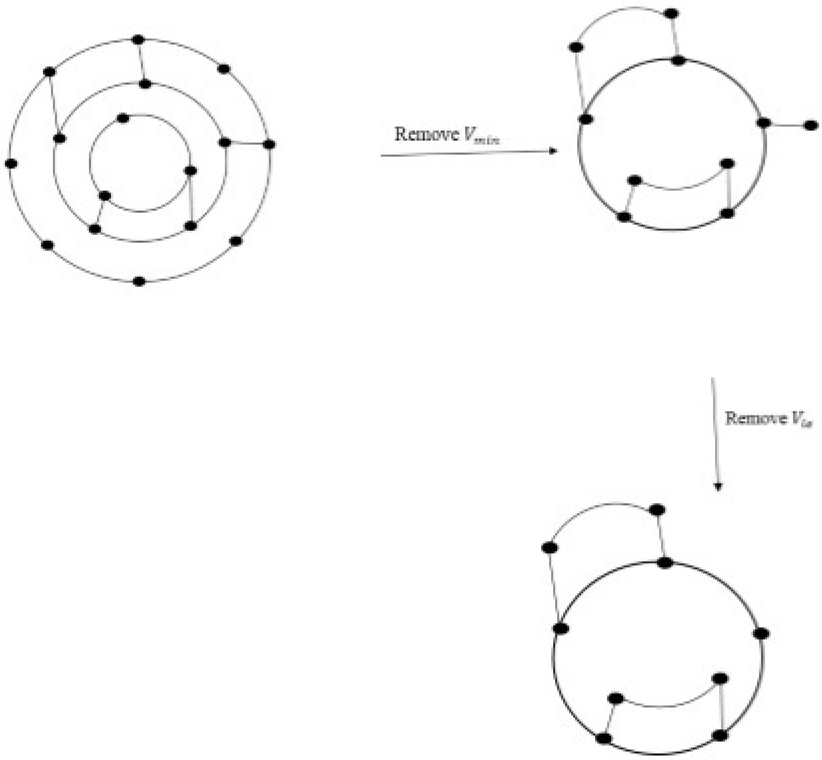

FIGURE 3. Mini-leaf vertex fault tolerant Ripples Hamiltonian Graph at level - 5

fault tolerant Ripples Hamiltonian Graph

$$V_{mini-leaf} = C_{n-1} + C_{n-2} - 1.$$

The final part of the proof follows immediately from figure 3. □

Theorem 4.4. *If G is a Ripples Graph. Then G satisfies the following properties.*

1. *G is a 2 - connected.*
2. *The closure of RG is Hamiltonian.*
3. *The genus of RG is 0.*
4. *The thickness of RG is 1. That is $\theta(RG) = 1$.*
5. *A relation between minimum degree of RG and face of RG is obtained as $\delta(RG) = p - q + r$.*

6. *Chromatic Number of RG is equal to* $\Delta(v) = 3$ *(refer Brooks Theorem).*
7. G *satisfies ORE's Theorem. That is, Suppose that RG is an simple graph with* $n \geq 3$ *vertices, and deg(x) + deg(y)* $\geq n$ *whenever* x *and* y *are nonadjacent vertices in RG, then RG has a Hamiltonian Circuit.*

Proof. The proofs are simple and follow immediately from the figures 1, 4, 5. □

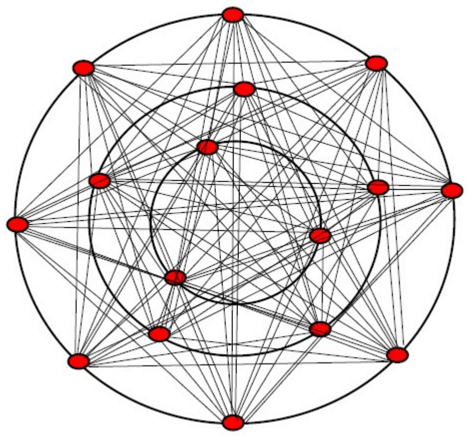

FIGURE 4. Closure of Ripples Graph (CRG) at level - 5

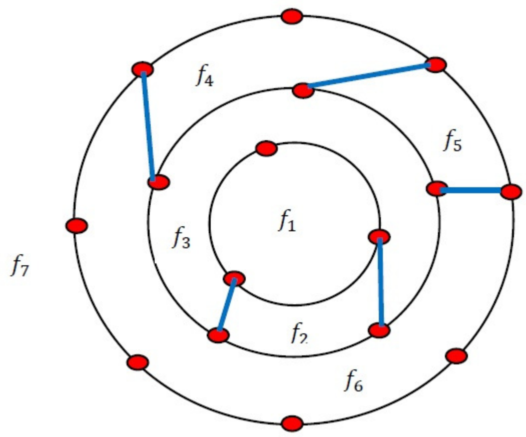

FIGURE 5. Faces of Ripples Graph (RG) at level - 5

Theorem 4.5. *The Closure of Ripples Graph (RG) is isomorphic to Complete Graph K_n, where n is the total number of vertices in CRG.*

Definition 4.2. *The Closure of Ripples Graph (CRG) is called k-fault – tolerant Hamiltonian Graph if CRG-F is Hamiltonian for any F$\subseteq$ V $\cup$ E.*

Theorem 4.6. *If Closure of Ripples Graph (CRG) is an n vertex graph with $n \geq 4$. Then CRG_n is $(n-3)$ - fault - tolerant Hamiltonian, where n is the total number of vertices in CRG.*

Proof. Suppose that F is any subset of $V(CRG) \cup E(CRG)$. Let F_V to denote $F \cap V(CRG)$. Then $CRG_n - F$ is isomorphic to $CRG_{n-f} - F'$ where $f = |F_V|$ and F' is a subset of edges in the subgraph of CRG_n induced by $< n > -F_V$. Obviously, $|F'| \leq |F| - f$. Thus, if $|F| \leq n - i$ then $\bar{E}(CRG_n - F') = |F'| \leq |F| - f \leq (n-f) - i$. Where $(n-f)$ is the numbe of vertices of $CRG_n - F'$. It follows from the Ore [7], Assume that G is an n vertex graph with $n \geq 4$. Then G is Hamiltonian if $\bar{e} \leq n-3$. Hence the proof. □

5. Conclusion

A new type of Ripples Graph is constructed based on the concentric circles pattern. The total number of vertices and edges in the n^{th} level can be determined for very large n, $n \in \mathbb{N}$. Ripples Graph (RG) forms a Hamiltonian Circuit (Cycle) is showed and Hamiltonian properties for Ripples Graph are studied. The Chromatic number of RG is obtained. And also fault-tolerant Hamiltonicity of Ripples Graph and Closure of Ripples Graph are studied. The total number of removed vertices of k^{th} level of min-leaf fault tolerant Ripples Hamiltonian Graph is obtained.

References

[1] S. Arumugam and S. Ramachandran: *Invitation to Graph Theory*, Scitech Publications (India), 2001.
[2] J.A. Bondy, U.S.R. Murty: *Graph Theory and its Application*, Springer, 2008.
[3] Y.-C. Chen, C.-H. Tsai, L.-H. Hsc, J.J.M. Tan: *On Some Super Fault-Tolerant Hamiltonian Graphs*, Journal of Applied Mathematics and Computation, **148** (2004), 729-741.

[4] G. Chartrand, S.F. Kapoor, D. R. Lick: *n-Hamiltonian Graphs*, Journal of Combinatorial Theory, **9** (1970), 308-312.

[5] L.-H.Hsc, C.-K. Lin: *Graph Theoty and Interconnection Networks*, Taylor and Francis Group, LLC, New York.

[6] N. Deo: *Graph Theory*, PHI Learning,2013.

[7] O. Ore: *Covering of Graphs*, Annali di Matematica Pura ed Applicata, **55** (1961), art.no. 315.

[8] W. Wamiliana, F. Elfaki, A. Faisol, M. Usman, I. Lestari: *Determining the Pattern for 1-fault Tolerant Hamiltonian Cycle From Generalized Petersen Graph P(n,k)*, International Journal of Computer Science and Network Security, **15**(4) (2015), 26-30.

Department of Mathematics, Queen Mary's College,
University of Madras,
Chennai,
India.
Email address: ramaraghu184@gmail.com

Department of Mathematics, Queen Mary's College,
University of Madras,
Chennai,
India.
Email address: njansirani@gmail.com

Department of Mathematics, Madras Christian College
University of Madras,
Chennai,
India.
Email address: rajkumardare@yahoo.com