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COMBINATORIAL PROPERTIES OF $K-$PERIODIC RECURRENCE WORD

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ABSTRACT. In this paper, $k-$Periodic Recurrence Word ($k-$PRW) is introduced and its Combinatorial properties are studied. $k-$PRW, its mirror image and it's factor satisfy the properrties of Rich, Balanced and Bounded are shown. Using the factor graph, $k-$PRW and the subword of PRW are in the form of Regular Trapezium and Rauzhy Graph Pattern are observed. Based on Factor Analysis, Bispecial factor is existed and the valence of $k-$PRW is examined. The existence of Upper and Lower Christoffel is also discussed for $k-$PRW and it's Mirror image. Any length of $k-$PRW is a trapezoidal shape, is shown. The regular expression for $k-$PRW over Σ are initiated and their basic properties are studied. Finite State Automaton $M(Q,\Sigma,\delta,q_0,F)$ for $k-$PRW is derived and M-ambiguity of $k-$PRW is deliberated.

1. INTRODUCTION

A finite or infinite sequence of symbols or alphabets over a finite set is a word and studied by several authors, which is used in the field of theoretical computer science, called Combinatorics on words [3,7,8]. In literature, abelian complexity, maximal pattern complexity, k-abelian complexity, periodic complexity, minimal forbidden factor complexity and palindromic complexity are the measures

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of complexity of words [4,5]. The factor complexity of a word $w = a_1a_2 \ldots \in \Sigma^\omega$, counts the number of the distinct factor of length n in w. Hence, the subword complexity for a given value of n does not count the repetitions of the subwords of length n. The study of repetitions of subwords is a classical subject of research in molecular Biology [2]. The subword complexity gives a lot of information about the structure of the words. The right special factors (left Special factors) of different valence is related to the complexity of the word and also related to the structure of the word itself. A word or string read by an automaton is called a regular expression. In formal language theory, mainly two types of mechanisms are classified which are acceptors and generators. A finite automaton is an acceptor. The regular expression and right(left) linear grammars are generators. The Finite state automata also classified into deterministic and non-deterministic finite state automata. In second section, basic definitions and preliminaries are given. In third section, $k-$PRW is introduced and it can be generated by NDFA and DFA. $k-$PRW is regular and Context Free Grammar, which are verified by pumping lemma. $k-$PRW is closed under homomorphism is shown. In fourth section, $k-$PRW is bounded by its period is shown. It has the properties of Rich, Balance is derived. Rauzy graph pattern is observed. $k-$PRW and it's mirror image are Trapezoidal and Christoffel is examined . In fifth section, $k-$PRW is unique in Parikh Matrix Mapping is studied.

2. Basic Definitions and Prelimiaries

Let Σ be a non-empty finite set of binary alphabets $\{a, b\}$. The set of all empty and non-empty words over Σ is denoted as Σ^*. $\Sigma^+ = \Sigma^* - \{\lambda\}, \lambda$ denoted as an empty word. The length of a word $w \in \Sigma^*$ is denoted by $|w|$. $|w|_a$ denotes the number of occurrences of a letter a in w. A right infinite word w is a sequence indexed by N over Σ. The set of all infinite word over Σ is denoted by Σ^ω. An infinite word w is ultimately periodic defined as $w = uv^\omega$, for some $u, v \in \Sigma^*, v \neq \lambda$, if $u = \lambda$ then w is a periodic. Let $w = w_1w_2 \ldots w_n$ then the reverse(mirror image) of w is defined as $w^R = w_nw_{n-1} \ldots w_2w_1$. If $w = w^R$ then w is a palindrome. λ is assumed to be a palindrome. A finite word w is rich if it has $|w| + 1$ distinct palindrome factors, including the empty word and an infinite word, is rich if all of its factors are rich [6]. The word $u \in \Sigma^*$ is a factor (or a subword) of w if there exist $p, q \in \Sigma^*$ such that $w = puq$. A factor u of

w is called proper if $u \neq w$. The set of all factors of w is denoted by $F(w)$. An infinite word w is said to be recurrent if any factor of w occurs infinitely often in w. The balance of a pair u and v of words of the same length as a number $\delta(u,v) = ||u|_a - |v|_a|, a \in \Sigma$. A word $w \in \Sigma^*$ is balanced if $\delta(u,v) \leq 1$ for any $u, v \in F(w)$ with $|u| = |v|$. Word s is called a right special factor of w if there exist two letters $x, y \in \Sigma, x \neq y$, such that $sx, sy \in F(w)$. A word s is called a left special factor of w if there exist two letters $x, y \in \Sigma, x \neq y$, such that $xs, ys \in F(w)$. A word of $w \in \Sigma^*$ which is a right and left special factor of w is called a Bispecial factor of w. A word $w \in \Sigma^*$ of length $|w|$ is said to be Trapezoidal if, for every integer $i \leq |w|, w$ admits one right special factor of length i, atmost [1].

Let $\Sigma = \{a_1 < a_2 < \ldots < a_n\}$. Then the Parikh mapping $\Psi : \Sigma^* \rightarrow N^n$ is given by $\Psi(x) = (|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_n})$, where N is the set of nonnegative integers. $(|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_n})$ is the Parikh vector of w. The Parikh matrix mapping is $\Psi : (\Sigma^*, ., \lambda) \rightarrow (M_{k+1}, ., I_{k+1})$ defined by $\Psi(a_q) = (m_{ij}), 1 \leq i, j \leq k+1$ such that

(i) $m_{(i,j)} = 1$

(ii) $m_{(j,j+1)} = 1$ (iii) all other elements are zero [9,10].

A word $w \in \Sigma^*$ is M-ambiguity iff it is M-equivalent to another distinct word. Otherwise, w is M-unambiguity.

3. k-Periodic Recurrence Word

In this section, $k-$Periodic Recurrence Word ($k-$PRW) is introduced and it can be generated by a NDFA and DFA. $k-$PRW is a regular and Context Free Grammar which are verified by pumping lemma. $k-$PRW is closed under homomorphism is shown.

Definition 3.1. *k-Periodic Recurrence Word (PRW) w over Σ is defined as*

$$w_i = \left\{ \begin{array}{lll} a & if & i \mod k \neq 0 \\ b & & otherwise \end{array} \right\}.$$

where $w_i = (a^{k-1}b)^n, k \geq 2, n \geq 1$ is the $k-$PRW over Σ.

For example if $k = 3$ then the $3-$PRW is $(a^2b)^n$.

Theorem 3.1. *k-PRW defined in 3.1 can be generated by*

(i) *Non Deterministic Finite State Automata(NDFA)*
(ii) *Deterministic Finite State Automata(DFA)*
(iii) *Pumping Lemma.*

Proof.

Case (i): Consider $M = (Q, \Sigma, \delta, q_0, F)$, where δ is given by
$\delta(q_0, a) = q_1, \delta(q_1, a) = q_2, \ldots,$
$\delta(q_{k-2}, a) = q_{k-1}, \delta(q_{k-1}, b) = q_k,\ \delta(q_k, a) = q_1.$
Then K-PRW constructed by the NDFA

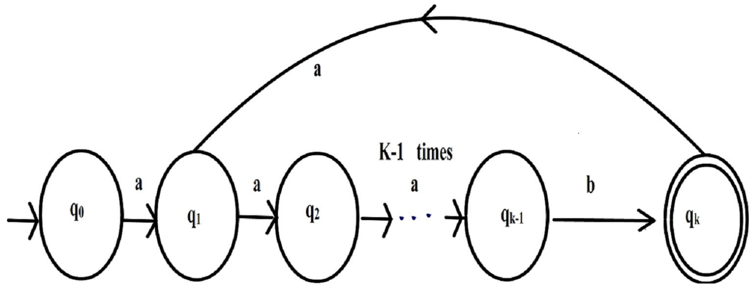

FIGURE 1. NDFA of $k-$Periodic Recurrence Word

Case (ii): Consider $M = (Q, \Sigma, \delta, q_0, F)$, where δ is given by
$\delta(q_0, a) = q_1, \delta(q_1, a) = q_2, \ldots,$
$\delta(q_{k-2}, a) = q_{k-1},$
$\delta(q_{k-1}, b) = q_k, \delta(q_k, a) = q_1.$
Then $k-$PRW constructed by the DFA

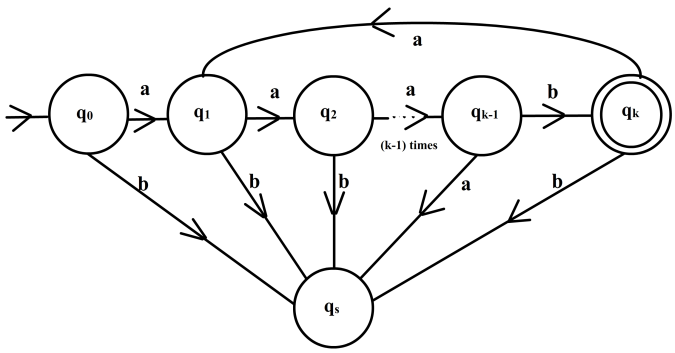

FIGURE 2. DFA of k-Periodic Recurrence Word

Case (iii): Let $L = \{(a^{k-1}b)^n : k \geq 2, n \geq 1\}$ be a $k-$PRW over Σ. L is a regular language and M its corresponding automaton with n number of states.

Let $w \in L$ and $|w| = kn$. Since k, n are positive integer, w can be decomposed as $w = xyz$, with $|xy| \leq kn$ and $|y| > 0$. $w = xyz \in L$, by pumping lemma $xy^{k-1}z$ should also in L.

If $n = 2, k = 3$ then $w = aabaab = xyz$, where $x = aa, y = baa, z = b$. Then $xy^2z = xyyz = aabaabaab \in L$. Hence, it is regular. From the three cases concluded that $k-$PRW is a regular language. This language is also accepted by a B$\ddot{u}$chi Automata. □

Theorem 3.2. *Let L be a $k-$PRW over Σ. Then L is closed under homomorphism.*

Proof. Let $L = \{(a^{k-1}b)^n : k \geq 2, n \geq 1\}$ be a $k-$PRW over Σ. The homomorphic of L is defined by $h : \Sigma \to \Gamma^*$ and $h(a) = a, h(b) = b$, where Γ is alphabets. Then the holomorphic image of L is $\{(a^{k-1}b)^n : k \geq 2, n \geq 1\}$. If $k = 3$ then $w = aabaaba \ldots ab$ and $h(aabaaba \ldots ab) = h(a)h(a)h(b) \ldots h(a)h(b) = aabaaba \ldots ab$. Hence the theorem. □

Theorem 3.3. *Let L be the collection of all $k-$PRW over Σ. Then L is a Context Free Language.*

Proof.

Case (i): Consider $L = \{(a^{k-1}b)^n : k \geq 2, n \geq 1\}$ is a $k-$PRW over Σ. L is expressed as a grammar which is defined by the production rules as $S \to a^{k-1}bA, A \to a^{k-1}bA|\lambda$. All the productions of L are in right linear. Hence, L is a Context Free Grammar. If $k = 3$ then $L = \{(a^2b)^n : n \geq 1\}$ and $S \to aabA, A \to aabA|\lambda$.

Case (ii): The Parse Tree Diagram of L exists.

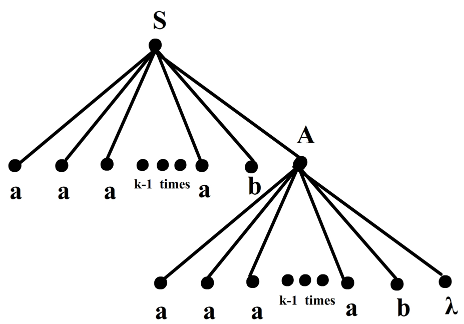

FIGURE 3. Parse Tree Diagram of k-Periodic Recurrence Word

Case (iii): Chomsky Normal Form of L has the following rules:
$S \to A_1A_2 \ldots A_{k-2}Y | A_1A_2 \ldots A_{k-2}X$,
$C \to A_1A_2 \ldots A_{k-2}Y | A_1A_2 \ldots A_{k-2}X, X \to AB, Y \to XC, A_1A_2 \ldots A_{k-2}, A \to a, B \to b$, where $S, A_1A_2 \ldots A_{k-2}, A, B, C, X, Y$ are in V and a, b are in T.

If $k = 3$, then $S \to AY|AX$, $C \to AY|AX$, $X \to AB$, $Y \to XC$, $A \to a$, $B \to b$.

Case (iv): $L = \{(a^{k-1}b)^n : k \geq 2, n \geq 1\}$, the set of string representing the language L is $\{L_1, L_2, \ldots L_{k-1}\}$, where $L_1 = \{(ab)^n\}, L_2 = \{(a^2b)^n\}, \ldots, L_{k-1} = \{(a^{k-1}b)^n\}$. $n = 3$ is the required positive integer by using pumping lemma $z = uvxyz$ with the condition as follows $u = a^{k-2}, v = aba, x = a^{k-2}, y = baa^{k-2}, z = b$ which satisfies the conditions that

(i). $|vxy| \leq kn$;

(ii). $vx \neq \varepsilon$;

(iii). For all $i \geq 0$, uv^iwx^iy is in L.

Hence, from the four cases L is a Context Free Grammar. □

4. Combinatorial Properties of $k-$ Periodic Recurrence Word

In this section, $k-$PRW is bounded by k is shown. It has the properties of Rich, Balanced are derived. The nature of $k-$PRW and its reverse are Trapezoidal and Christoffel is examined. Rauzy Graph pattern is observed.

Theorem 4.1. *Let w be a $k-$PRW over Σ. Then w is a bounded by k if $|w| \geq k-1$.*

Proof. Let $w = \{(a^{k-1}b)^n, k \geq 2, n \geq 1\}$. The proof is proceeded by Induction method. To prove this is true for $k = 3, n = 2$ for $3-$PRW w. $w = (a^2b)^2$ then the distinct subgroup of $w = \{\lambda, a, b, aa, ab, ba, aab, aba, baa, aaba, abaa, baab, aabaa, abaab, baaba, aabaab\}$. From this is we observed that every distinct length of subwords have $k = 3$ subwords. Hence it is bounded by three. Then this is true for $k = r$ for any positive integer. □

Theorem 4.2. *Let w be a $k-$PRW over Σ. Then w is Rich.*

Proof. For any length of $k-$PRW has the rich property. To prove this by the Method of Induction. Consider the distinct subwords of $k-$PRW of length 2 are $\{aa, ab, ba\}$ then the possible distinct palindrome subwords of aa are $\{a, aa, \lambda\}$.

Similarly for $ab = \{a, b, \lambda\}$ and for $ba = \{a, b, \lambda\}$. Hence $k-$PRW has property of rich. Well ordering principle this is true for the minimal value of 2. Now assume that this is true for m. Then to prove that is true for $m+1$. The $k-$PRW of length $m+1 = \{aabaaba\ldots abaabaab\}_m.a$ or $\{aabaabaa\ldots baabaa\}_m.b$ or $\{aabaaba\ldots abaabaaba\}_m.a$, in any case, the increase in the size by one then the palindrome count also increased by one. Hence this is true for all positive integers. □

Theorem 4.3. *Let w be a $k-$PRW over Σ. Then w is Balanced.*

Proof. Let $w = \{(a^{k-1}b)^n, k \geq 2, n \geq 1\}$. If $k = 4, n = 2$, then $w = aaabaaab$. Consider the distinct factors of w of length $|w| = 6$ are $\{aaabaa, aabaaa, abaaab, baaaba\}$. u and v be any factors of w Then the factors of w have the property of balance. Without Loss of generality assumed that this theorem is true for any positive integer. □

Theorem 4.4. *Let w be a $k-$PRW over Σ. Then w has a Bispecial factor if $|w| \geq k+1$.*

Proof. Let w be the $k-$PRW . $w = \{(a^{k-1}b)^n, k \geq 2, n \geq 1\}$. The right special factors of w of any length exist with the valence is $k-1$. If $k = 4, n = 1$ then $w = a^3b$. The right special factor of w is $s = aa, x = a, y = b$, then $sx = aaa \in F(w)$, $sy = aab \in F(w)$ and the other factors of right valence is zero. Similarly, The left special of w of any length exist with the valence is $k-1$ if the word of length greater than $k+1$. Then only the factor $(i.e)s = a, x = a, y = b$, then $xs = aa, ys = ba \in F(w)$. and the other factor's left valence is zero. Hence it is a Bispecial factor with the valence 2. □

Theorem 4.5. *Let w be a $k-$PRW over Σ. Then w and w^R are Trapezoidal word.*

Proof. Let $w = \{(a^{k-1}b)^n, k \geq 2, n \geq 1\}$. Then the distinct factors of w are

$$F_i(w) = i+1, \qquad 0 \leq i \leq k-2;$$
$$F_{i+1}(w) = F_i(w), \qquad k-1 \leq i \leq |w|-k-1;$$
$$F_{i+1}(w) = F_i(w) - 1, \quad |w|-k-2 \leq i \leq |w|.$$

where $F_i(w)$ represents the distinct factor of w of length i.

If $k = 3, |w| = 7$, then w=aabaaba. The distinct factors of w are $F_0(w) = 1, F_1(w) = 2, F_2(w) = F_3(w) = F_4(w) = F_5(w) = 3, F_6(w) = 2, F_1(w) = 1$.

This can be plotted in the graph, in which the sub word of length take it as in a $x-$axis and the number of distinct subwords as in a $y-$axis. The corresponding Trapezoidal graph is

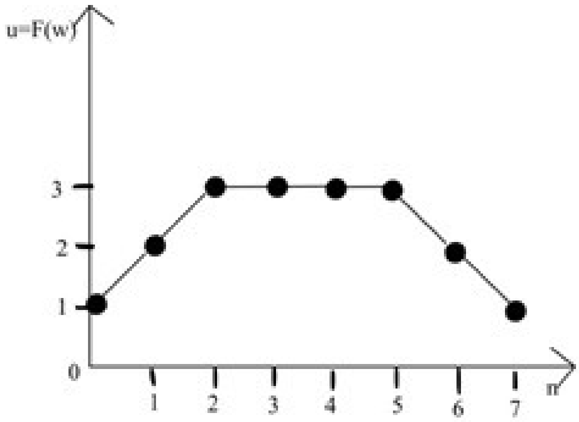

FIGURE 4. Trapezoidal Graph of $3-$Periodic Recurrence Word of length 7

After plotting the graph, observed that the graph is a Regular Trapezium. Similarly we can prove for its reverse. Rauzy Graph Pattern is existed in $k-$PRW and it is given below.

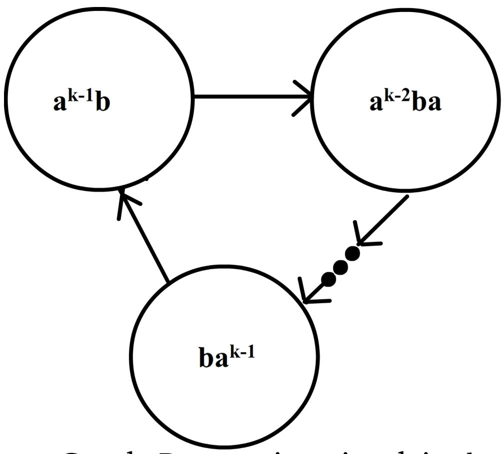

FIGURE 5. Rauzy Graph Pattern is existed in $k-$PRW and it is given below.

For example if $k = 3$ and $2 \leq |w| \leq 8$ then $3-$PRW Rauzy Graph is

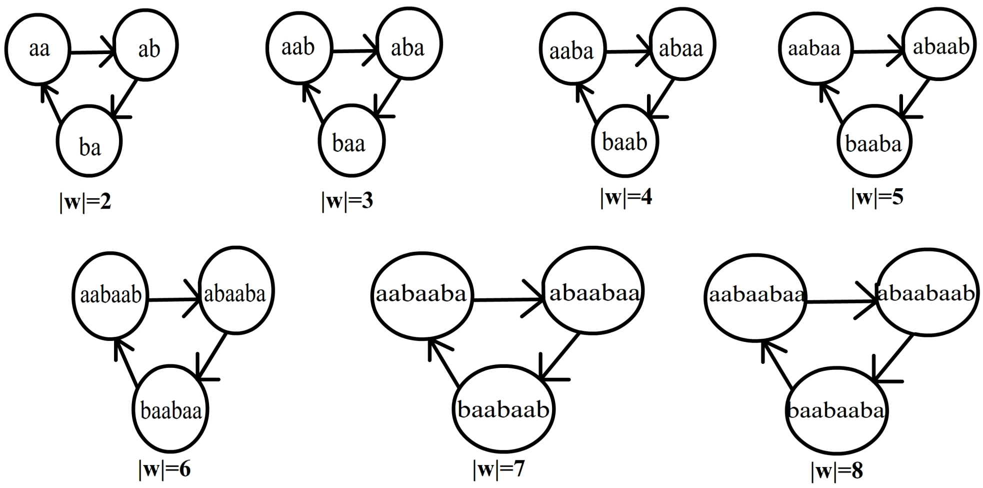

FIGURE 6. Rauzy Graph of $3-$Periodic Recurrence Word of length $2 \leq |w| \leq 8$

□

Theorem 4.6. *Let w be a $k-$PRW over Σ. Then w is a Conjugate word if the word of length is multiple of $k, k \geq 2$.*

Proof. Consider w is a $k-$PRW. w is a circular word if the length of the word is multiple of k.

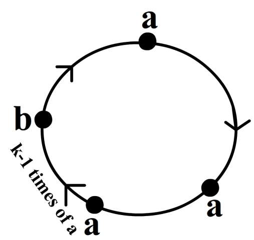

FIGURE 7. Circular pattern in $k-$Periodic Recurrence Word

Proof is by the method of contradiction. w is $3-$PRW of length 7, then $w = aabaaba$ then the first shift of w is $T_2 = abaabaa, T_3 = baabaaa \notin F(w)$. Hence it conjugates if the word of length should be in the multiple of k. □

Theorem 4.7. *Let w be a $k-$PRW over Σ. Then w and its reverse are a Christoffel word.*

Proof. The lower Christoffel word of the slope of $k-$PRW is $\dfrac{1}{k-1}$ and the upper Christoffel word of the slope of $k-$PRW is $\dfrac{k-1}{1}$. $(k-1,1)$ are relatively prime. Consider a $3-$PRW of w is $(a^2b)^2$. Then the corresponding Christoffel graph is

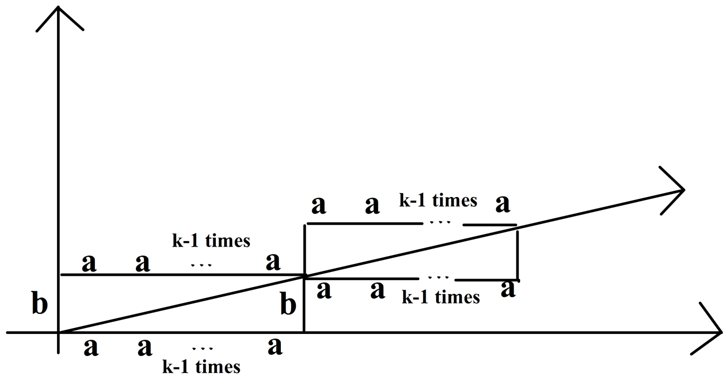

FIGURE 8. Christoffel Graph of $k-$Periodic Recurrence Word

shown in below.

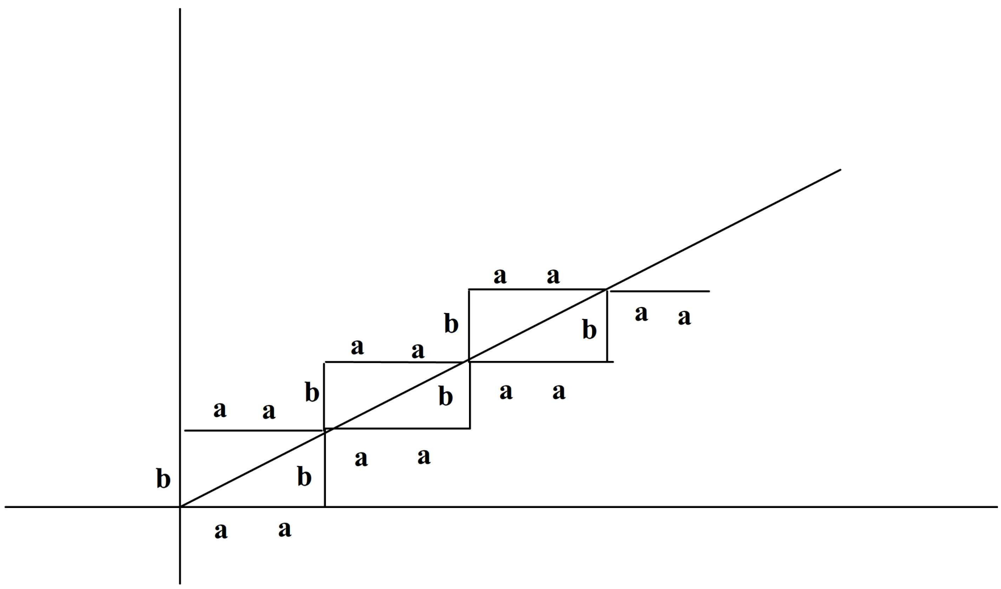

FIGURE 9. Christoffel Graph of $3-$Periodic Recurrence Word

□

5. Ambiguity of $k-$Periodic Recurrence Word

In general Parikh Matrix mapping is not injective for words of equal length. But $k-$PRW, the Parikh Matrix Mapping is injective and it is shown for $k = 3, n = 2$. Further theorems are proved for the Parikh Matrix Mapping for $k-$ PRW and it's reverse.

Theorem 5.1. *Let w be a $k-$PRW over Σ. Then Parikh Matrix Mapping of w and it's reverse are M-unambiguity.*

Proof. Let $w = (a^{k-1}b)^n$. Then

$$\Psi_\Sigma(w) = \begin{pmatrix} 1 & k-1 & k-1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n(k-1) & \dfrac{n(n+1)(k-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

and $w^R = (ba^{k-1})^n$, then the corresponding

$$\Psi_\Sigma(w^R) = \begin{pmatrix} 1 & k-1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n(k-1) & \dfrac{n(n-1)(k-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

From $\Psi_\Sigma(w)$ and $\Psi_\Sigma(w^R)$ observed that the Parikh Matrix Mapping are different. Hence the result.

For example if $k = 3, n = 2$ then

$$\Psi_\Sigma((a^2b)^2) = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\Psi_\Sigma((ba^2)^2) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$\Psi_\Sigma(w) \neq \Psi_\Sigma(w^R)$. □

Theorem 5.2. *Let w be a $k-$PRW over Σ. Then Parikh Matrix Mapping of w and w^R are commutes.*

Proof. From previous theorem,

$$\Psi_\Sigma(w) = \begin{pmatrix} 1 & n(k-1) & \dfrac{n(n+1)(k-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\Psi_\Sigma(w^R) = \begin{pmatrix} 1 & n(k-1) & \dfrac{n(n-1)(k-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Psi_\Sigma(w) \circ \Psi_\Sigma(w^R) = \begin{pmatrix} 1 & 2n(k-1) & 2n^2(k-1) \\ 0 & 1 & 2n \\ 0 & 0 & 1 \end{pmatrix} = \Psi_\Sigma(w^R) \circ \Psi_\Sigma(w).$$

Consider $k = 3$, $n = 2$. Then

$$\Psi_\Sigma((a^2b)^2) \circ \Psi_\Sigma(((a^2b)^2)^R) = \begin{pmatrix} 1 & 8 & 16 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \Psi_\Sigma(((a^2b)^2)^R) \circ \Psi_\Sigma((a^2b)^2).$$

□

Theorem 5.3. *Consider the alphabet* $\Sigma_k = \{a_1, a_2, \ldots, a_k\}$. *Let* w_1 *and* w_2 *be a* $k-$*PRW. Then the words* w_1 *with* w_2 *are M-ambiguous.*

1 *If* $u_1 = a_1a_2\ldots a_k$, $u_2 = a_2a_3\ldots a_ka_1$, $\ldots$ $u_k = a_ka_1\ldots a_{k-1}$ *and* $w_1 = u_1u_2\ldots u_k$, $w_2 = u_2u_3\ldots u_ku_1$, *for any* $k \geq 2$.

2 *If* $\mathfrak{R}u_1 = a_1a_2\ldots a_k$, $\mathfrak{R}u_2 = a_ka_1\ldots a_{k-1}$, $\ldots$, $\mathfrak{R}u_k = a_2a_3\ldots a_ka_1$ *and* $w_1 = \mathfrak{R}u_1\mathfrak{R}u_2\ldots\mathfrak{R}u_k$, $w_2 = \mathfrak{R}u_2\mathfrak{R}u_3\ldots\mathfrak{R}u_k\mathfrak{R}u_1$, *for any* $k \geq 2$

Otherwise the words w_1 *with* w_2 *are M-unambiguous.*

Proof.

Case 1: Consider $k = 4$, then $u_1 = a_1a_2a_3a_4$, $u_2 = a_2a_3a_4a_1$, $u_3 = a_3a_4a_1a_2$, $u_4 = a_4a_1a_2a_3$ and $w_1 = u_1u_2u_3u_4, w_2 = u_2u_3u_4u_1$. Then $w_1 = u_1q_1$ and $w_2 = q_1u_1$

where $q_1 = u_2u_3u_4$.

$$\Psi_4(q_1) = \begin{pmatrix} 1 & 3 & 5 & 5 & 0 \\ 0 & 1 & 3 & 5 & 5 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Psi_4(u_1) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Psi_4(w_1) = \begin{pmatrix} 1 & 4 & 9 & 14 & 14 \\ 0 & 1 & 4 & 9 & 14 \\ 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \Psi_4(w_2).$$

Hence the theorem is true for $k = 4$.

Then this is true for $k = r$ and to prove for $k = r+1$, $u_1 = a_1a_2\ldots a_ra_{r+1}$, $u_2 = a_2a_3\ldots a_ra_{r+1}a_1, \ldots, u_{r+1} = a_{r+1}a_1\ldots a_{r-1}a_r$ and $w_1 = u_1u_2\ldots u_ru_{r+1}$, $w_2 = u_2u_3\ldots u_{r+1}u_1$, and further,

$$w_1 = a_1a_2\ldots a_{r+1}a_2a_3\ldots a_{r+1}a_1\ldots a_{r+1}a_1\ldots a_r \rightarrow (5.1)$$

$$w_2 = a_2a_3\ldots a_{r+1}a_1\ldots a_{r+1}a_1\ldots a_ra_1a_2\ldots a_{r+1} \rightarrow (5.2)$$

Comparing (5.1) and (5.2) the $w_1 = u_1q_1$ and $w_2 = q_1u_1$ where $q_1 = u_2u_3u_4\ldots u_{r+1}$. Consider the Parikh Matrix of q_1,

$$\Psi_r(q_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & \ldots & p_{1,r} & p_{1,r+1} \\ 0 & 1 & p_2 & \ldots & p_{2,r} & p_{2,r+1} \\ 0 & 0 & 1 & \ldots & p_{3,r} & p_{3,r+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ldots & p_r & p_{r,r+1} \\ 0 & 0 & 0 & \ldots & 1 & p_{r+1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

and Parikh Matrix of u_1,

$$\Psi_r(u_1) = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

Parikh Matrix of w_1,

$\Psi_r(w_1) =$

$$\begin{pmatrix} 1 & p_1+1 & p_{1,2}+p_2+1 & \dots & p_{1,r}+p_{2,r}+\dots+p_r+1 & p_{1,r+1}+p_{2,r+1}+\dots+p_{r+1}+1 \\ 0 & 1 & p_2+1 & \dots & p_{2,r}+\dots+p_r+1 & p_{2,r+1}+\dots+p_{r+1}+1 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & p_r+1 & p_{r,r+1}+p_{r+1}+1 \\ 0 & 0 & 0 & \dots & 1 & p_{r+1}+1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Parikh Matrix of w_2, $\Psi_r(w_2) =$

$$\begin{pmatrix} 1 & 1+p_1 & 1+p_1+p_{1,2} & \dots & 1+p_1+p_{1,2}+\dots+p_{1,r} & 1+p_1+\dots+p_{1,r+1}+p_{r+1} \\ 0 & 1 & 1+p_2 & \dots & 1+p_2+\dots+p_{2,r} & 1+p_2+\dots+p_{2,r+1}+p_{r+1} \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & 1+p_r+p_{r,r+1} \\ 0 & 0 & 0 & \dots & 1 & 1+p_{r+1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

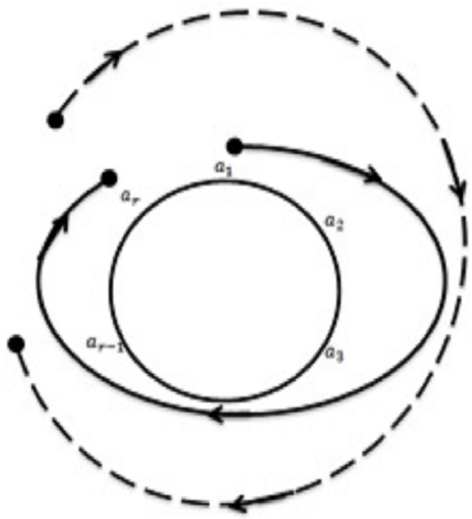

FIGURE 10. The Circulant Orientation of a Word

From the diagram q_1 is in the circulant type. Hence it is a Pascal Symmetric, then $p_{i,j} = p_{k,l}$ if $i + j = k + l$,, and further, w_1 and w_2 satisfies the Ratio Property of word, $\dfrac{p_1+1}{1+p_1} = \dfrac{p_{1,2}+p_2+1}{1+p_1+p_{1,2}} = \dots \dfrac{p_{1,r}+p_{2,r}+\dots+p_r+1}{1+p_1+p_{1,2}+\dots+p_{1,r}}$

$$= \frac{p_{1,r+1} + p_{2,r+1} + \ldots + p_{r,r+1} + p_{r+1}}{1 + p_1 + \ldots + p_{1,r+1} + p_{r+1}} = \ldots = \frac{p_{r,r+1} + p_{r+1} + 1}{1 + p_r + p_{r,r+1}} = \frac{p_{r+1} + 1}{1 + p_{r+1}} = s.$$

Hence the Statement.

Case 2: Similar proof is followed in the Case 1. Even though, in this case u_1, u_2, u_3, u_4 are in Right Circulant but the direction of the orientation (the dotted lines in the Fig.10) doesn't change. Hence the theorem. □

6. CONCLUSION

$k-$Periodic Recurrence Word ($k-$PRW) is introduced and regular expression is derived. $k-$PRW is rich and balanced is shown. Regular Trapezium, Christoffel for $k-$PRW and its reverse are existed, is verified. Rauzy Graph pattern is observed for $k-$PRW. The Parikh Matrix Mapping is unique in $k-$PRW is studied.

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