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A NOTE ON GAUSSIAN TOTAL NEIGHBOURHOOD PRIME LABELING

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ABSTRACT. Gaussian integers are the complex numbers whose real and imaginary parts are both integers. A graph G on n vertices is said to have a Gaussian neighbourhood prime labelling if there exists a labelling from the vertices of G to the first n Gaussian integers such that for each vertex in G with degree greater than one, the neighbourhood vertices have relatively prime labels. In this paper, we extend the Gaussian neighbourhood prime labelling concept to Gaussian total neighbourhood prime labelling and discuss it with some graphs.

1. INTRODUCTION

Graph labelling where the vertices are assigned values subject to certain conditions have many applications in Engineering and Science. For all terminology and notations in Graph theory, we follow [1] and for all terminology regarding graph labelling, we follow [2]. The Prime labelling concept was introduced by Roger Entriger. A graph on n vertices is said to have a prime labelling if its vertices can be labelled with the first n natural numbers in such a way that any two adjacent vertices have relatively prime labels. In [5] Steven Klee, Hunter Lehmann and Andrew Park extend the notion of prime labelling to Gaussian

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integers. They define a spiral ordering on the Gaussian integers that allow us to linearly order the Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime and Consecutive odd Gaussian integers in the spiral ordering are relatively prime. Steven Klee proved that the path graph, star graph, spider graph, n-centipede tree, double star tree and firecracker tree admits Gaussian prime labelling.

A graph G on n vertices is said to have a neighbourhood prime labelling if there exists a labelling from the vertices of G to the first n natural numbers such that for each vertex in G with degree greater than one, the neighbourhood vertices have relatively prime labels. A graph G on n vertices is said to have a Gaussian neighbourhood prime labelling if there exists a labelling from the vertices of G to the first n Gaussian integers such that for each vertex in G with degree greater than one, the neighbourhood vertices have relatively prime labels. A graph G on p vertices and q edges is said to have a total neighbourhood prime labelling if there exists a labelling from the vertices and edges of G to the first $p + q$ natural numbers such that for each vertex in G of degree at least two, the gcd of labelling on its neighbourhood vertices is one and for each vertex of degree at least two the gcd of labelling on its incident edges is one. In this Paper, the study of Gaussian neighbourhood prime labelling is extended to Gaussian total neighbourhood prime labelling and discuss the Gaussian total neighbourhood prime labelling of some graphs.

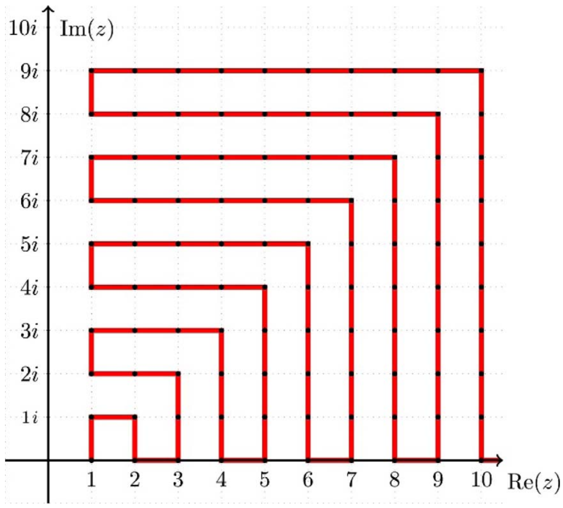
2. GAUSSIAN INTEGERS

All graphs in this paper are finite undirected graphs without loops or multiple edges. A vertex is a leaf or end vertex if it has degree one and that it is an internal node otherwise. We follow [5] for definition and information on the Gaussian integers. The Gaussian integers, denoted $Z[i]$, are the complex numbers of the form $a + bi$, where $a, b \epsilon Z$ and $i^2 = -1$. The norm of Gaussian integer $a + bi$, denoted by $N(a + bi)$, is given by $a^2 + b^2$. A Gaussian integer is even if it is divisible by $1 + i$ and odd otherwise. A unit in the Gaussian integers is one of ± 1, $\pm i$. An associate of a Gaussian integer α is $u \cdot \alpha$ where u is a Gaussian unit. A Gaussian integer ρ is prime if its only divisors are ± 1, $\pm i$, $\pm \rho$ or $\pm \rho i$. The Gaussian integers α and β are relatively prime if their only common divisors are units in $Z[i]$.

The Gaussian integers are not totally ordered. So to define the first n natural numbers we use the spiral ordering of the Gaussian integers introduced by Steven Klee in [5].

Definition 2.1. [5]. *The spiral ordering of the Gaussian integers is a recursively defined ordering of the Gaussian integers. We denote the n^{th} Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with $\gamma_1 = 1$ and continuing as:*

$$\gamma_{n+1} = \begin{cases} \gamma_n + i, & \textit{if } \mathrm{Re}(\gamma_n) \equiv 1(\mathrm{mod}2), \mathrm{Re}(\gamma_n) > \mathrm{Im}(\gamma_n) + 1 \\ \gamma_n - i, & \textit{if } \mathrm{Im}(\gamma_n) \equiv 0(\mathrm{mod}2), \mathrm{Re}(\gamma_n) \leq \mathrm{Im}(\gamma_n) + 1, \mathrm{Re}(\gamma_n) > 1 \\ \gamma_n + 1, & \textit{if } \mathrm{Im}(\gamma_n) \equiv 1(\mathrm{mod}2), \mathrm{Re}(\gamma_n) < \mathrm{Im}(\gamma_n) + 1 \\ \gamma_n + i, & \textit{if } \mathrm{Im}(\gamma_n) \equiv 0(\mathrm{mod}2), \mathrm{Re}(\gamma_n) = 1 \\ \gamma_n = i, & \textit{if } \mathrm{Re}(\gamma_n) \equiv 0(\mathrm{mod}2), \mathrm{Re}(\gamma_n) \geq \mathrm{Im}(\gamma_n) + 1, \mathrm{Im}(\gamma_n) > 0 \\ \gamma_n + 1, & \textit{if } \mathrm{Re}(\gamma_n) \equiv 0(\mathrm{mod}2), \mathrm{Im}(\gamma_n) = 0. \end{cases}$$



The first 10 Gaussian integers under this ordering are $1, 1+i, 2+i, 2, 3, 3+i, 3+2i, 2+2i, 1+2i, 1+3i, \ldots$ and $[\gamma_n]$ denote the set of the first n Gaussian integers in the spiral ordering. Here we exclude the imaginary axis to ensure that the spiral ordering excludes associates. Consecutive Gaussian integers in the spiral ordering are separated by a unit and therefore alternate parity, as in the usual ordering of N. Furthermore, odd integers with indices separated by a power of two are not guaranteed to be relatively prime to each other.

In [5] Steven Klee proved the following properties of Gaussian integers in spiral ordering.

(1) Let α be a Gaussian integer and u be a unit. Then α and $\alpha + u$ are relatively prime.
(2) Consecutive Gaussian integers in the spiral ordering are relatively prime.
(3) Consecutive odd Gaussian integers in the spiral ordering are relatively prime.
(4) Let α be a Gaussian integer and let p be a prime Gaussian integer. Then α and $\alpha + p$ are relatively prime if and only if p does not divide α.

Theorem 2.1. *Let ρ be an odd Gaussian integer, let t be a positive integer and u be a unit. Then ρ and $\rho + u.\ (1+i)^t$ are relatively prime.*

Theorem 2.2. *In the spiral ordering, consecutive even Gaussian integers are relatively prime.*

Proof. The only possible difference between two even Gaussian integers in the spiral ordering are $1 + i, 2$ or one of their associates. The differences are of the form $u \cdot (1+i)^t$, since $2 = -i \cdot (1+i)^2$. Therefore by using theorem 2.1 consecutive even Gaussian integers in the spiral ordering are relatively prime □

3. Preliminaries

Definition 3.1. *The n-centipede tree, C_n is a graph obtained by joining a single pendant edge to each vertex of a path.*

Definition 3.2. *Let n, k, m be integers with $k \leq m$. Then (n, k, m)-double star tree $\mathrm{DS}_{n,k,m}$ is the graph with $V(\mathrm{DS}_{n,k,m}) = \{v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{n+k-1}, v_{n+k,\ldots,}$ $v_{n+k+m-2}\}$ and $E(\mathrm{DS}_{n,k,m}) = \{v_j v_{j+1} : 1 \leq j \leq n-1, v_1 v_{n+j} : 1 \leq j \leq k-1, v_n v_{n+k+j} : 0 \leq j \leq m-2\}$. In the (n, k, m)-double star tree we have a path of length n whose end vertices v_1 and v_n are the central vertices for stars on k and m vertices respectively (not including the other vertices on the path).*

Definition 3.3. *A spider tree is a tree with one vertex of degree atleast 3 and all other vertices having degree 1 or 2.*

Definition 3.4. *Caterpillar tree is a tree in which all the vertices are within distance 1 of a central path.*

Definition 3.5. *For a vertex $v \epsilon V(G)$, the neighbourhood of v is the set of all vertices in G which are adjacent to v and is denoted by $N(v)$.*

Definition 3.6. [4]. *Let G be a graph on n vertices. A Gaussian neighbourhood prime labelling of G is a bijection $f : V(G) \to [\gamma_n]$ such that for each vertex $v \epsilon V(G)$ with $\deg(v) > 1$, $\{f(u_i) : u_i \epsilon N(v)\}$ are relatively prime.*

Definition 3.7. [3]. *Let $G = (V, E)$ be a graph with p vertices and q edges. A bijection $f : \mathrm{VUE} \to \{1, 2, 3, \ldots, p + q\}$ is said to be total neighbourhood prime labelling if it satisfies the following two conditions:*
(i) *For each vertex of degree at least two, the gcd of labelling on its neighbourhood vertices is one.*
(ii) *For each vertex of degree at least two, the gcd of labelling on the incident edges is one.*

A graph which admits total neighbourhood prime labelling is called total neighbourhood prime graph. Total neighbourhood prime labelling of some graphs are discussed in [3].

4. Gaussian total neighbourhood prime labelling

Using the above two definitions, introduce a new labelling called Gaussian total neighbourhood prime labelling.

Definition 4.1. *Let $G = (V, E)$ be a graph with n vertices and m edges. A bijection $f : \mathrm{VUE} \to [\gamma_{n+m}]$ is said to be Gaussian total neighbourhood prime labelling if it satisfies the following two conditions:*
(i) *For each vertex of degree at least two, the gcd of labelling on its neighbourhood vertices is one;*
(ii) *For each vertex of degree at least two, the gcd of labelling on the incident edges is one.*

A graph which admits Gaussian total neighbourhood prime labelling is called Gaussian total neighbourhood prime graph.

5. Main results

In this section, we discuss the Gaussian total neighborhood prime labelling of star graph, path graph, n-centipede tree, (n, k, m)-double star tree, spider tree and caterpillar.

Theorem 5.1. *Any star graph admits a Gaussian total neighbourhood prime labelling.*

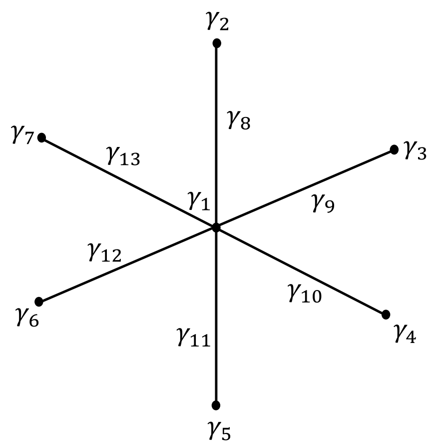


FIGURE 1.

Proof. Let S_n, $n \in N$ on n vertices be the star graph whose centre vertex is v_1 with $V(S_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(S_n) = \{v_1v_k : 2 \leq k \leq n\}$. Define the mapping $f : V \cup E \to [\gamma_{2n-1}]$ as $f(v_1) = \gamma_1$.

$$f(v_i) = \gamma_i, 2 \leq i \leq n \; f(v_1v_k) = \gamma_{n+k-1}, 2 \leq k \leq n.$$

Since consecutive Gaussian integers are relatively prime, this is a Gaussian total neighbourhood prime labelling. □

Figure 1 shows the Gaussian total neighbourhood prime labelling of S_7.

Theorem 5.2. *The path graph P_n is a Gaussian total neighbourhood prime graph for $n \in N$*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i : 1 \leq i \leq n-1\}$ where $e_i = v_iv_{i+1}$. Clearly $|V(P_n)| + |E(P_n)| = 2n-1$. Define a function $f : V \cup E \to [\gamma_{2n-1}]$ as follows.

Case 1: If n is even

$$f(v_{2i-1}) = \gamma_{\frac{n}{2}+i}, 1 \leq i \leq \tfrac{n}{2}$$
$$f(v_{2i}) = \gamma_i, 1 \leq i \leq \tfrac{n}{2}$$
$$f(e_i) = \gamma_{n+i}, 1 \leq i \leq n-1$$

Case 2: If n is odd

$$f(v_{2i-1}) = \gamma_{\frac{n-1}{2}+i}, 1 \leq i \leq \tfrac{n+1}{2}$$
$$f(v_{2i}) = \gamma_i, 1 \leq i \leq \tfrac{n-1}{2}$$
$$f(e_i) = \gamma_{n+i}, 1 \leq i \leq n-1$$

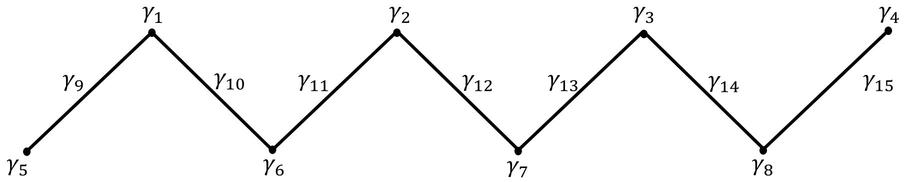


FIGURE 2.

Clearly f is a bijection. The labelling shows that adjacent vertices of all the vertices in the P_n are consecutive Gaussian integers and they are relatively prime in the spiral ordering. Also for each vertex the labelling of the incident edges are consecutive Gaussian integers and they are relatively prime. Therefore this is a Gaussian total neighbourhood prime labelling for the path P_n. Thus the path P_n is a Gaussian total neighbourhood prime graph for $n \in N$. □

Figure 2 shows the Gaussian total neighbourhood prime labelling of P_8.

Theorem 5.3. *The n-centipede tree, C_n is Gaussian total neighbourhood prime graph.*

Proof. Let $v_1, v_2, v_3 \dots, v_n$ be the vertices of the path in C_n and $u_1, u_2, u_3, \dots, u_n$ be the vertices joined with it. The edges are denoted by
$e_i = v_i v_{i+1}$, $i = 1, 2, 3, \dots, n-1$ and $e'_i = v_i u_i, i = 1, 2, 3, \dots, n$. The total number of vertices are $2n$ and total number of edges are $2n - 1$.

Define a function $f : V \cup E \to [\gamma_{4n-1}]$ as

Case 1: If n is even

$$f(v_{2i-1}) = \gamma_{4i-3}, i = 1, 2, 3, \dots, \tfrac{n}{2}$$
$$f(v_{2i}) = \gamma_{4i}, i = 1, 2, 3, \dots, \tfrac{n}{2}$$
$$f(u_{2i-1}) = \gamma_{4i-2}, i = 1, 2, 3, \dots, \tfrac{n}{2}$$
$$f(u_{2i}) = \gamma_{4i-1}, i = 1, 2, 3, \dots, \tfrac{n}{2}$$
$$f(e_i) = \gamma_{2n+2i}, i = 1, 2, 3, \dots, n-1$$
$$f(e'_i) = \gamma_{2n+2i-1}, i = 1, 2, 3, \dots, n.$$

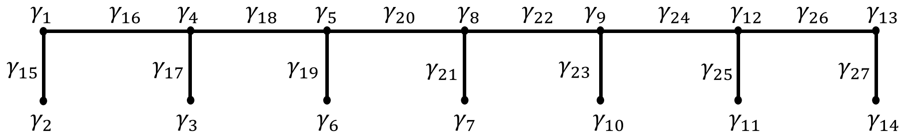


FIGURE 3.

Case 2: If n is odd

$$\begin{aligned} &f(v_{2i-1}) = \gamma_{4i-3}, i = 1, 2, 3, \dots, \tfrac{n+1}{2} \\ &f(v_{2i}) = \gamma_{4i}, i = 1, 2, 3, \dots, \tfrac{n-1}{2} \\ &f(u_{2i-1}) = \gamma_{4i-2}, i = 1, 2, 3, \dots, \tfrac{n+1}{2} \\ &f(u_{2i}) = \gamma_{4i-1}, i = 1, 2, 3, \dots, \tfrac{n-1}{2} \\ &f(e_i) = \gamma_{2n+2i}, i = 1, 2, 3, \dots, n-1 \\ &f(e_i') = \gamma_{2n+2i-1}, i = 1, 2, 3, \dots, n. \end{aligned}$$

Clearly f is a bijection. The labelling on the neighbourhood vertices of v_i, $1 \le i \le n$ are consecutive odd Gaussian integers or consecutive even Gaussian integers and they are relatively prime. Also for each vertex the labelling of the incident edges are consecutive Gaussian integers and they are relatively prime. Thus C_n is a Gaussian total neighbourhood prime graph. □

Gaussian total neighbourhood prime labelling of C_7 is shown in figure 3.

Theorem 5.4. *Any (n, k, m)-double star tree admits a Gaussian total neighbourhood prime labelling.*

Proof. Firstly label the vertices of path v_i by γ_i, $1 \le i \le n$.

Then label the $(k-1)$ vertices adjacent to v_1 by next consecutive Gaussian integers $\gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_{n+k-1}$. Also label the $(m-1)$ vertices adjacent to v_n by next consecutive Gaussian integers $\gamma_{n+k}, \gamma_{n+k+1}, \dots, \gamma_{n+k+m-2}$.

Labeling of edges are in the following way. Label the edges of path v_iv_{i+1}, $1 \le i \le (n-1)$ by $\gamma_{n+k+m-1}, \gamma_{n+k+m}, \dots, \gamma_{2n+k+m-3}$.

Then label the $(k-1)$ edges incident on v_1 by $\gamma_{2n+k+m-2}, \gamma_{2n+k+m-1}, \dots, \gamma_{2n+2k+m-4}$.

Also label the $(m-1)$ edges incident on v_n by $\gamma_{2n+2k+m-3}, \gamma_{2n+2k+m-2}, \dots, \gamma_{2n+2k+2m-5}$.

Case 1. Let $v = v_1$ or v_n

The vertices adjacent to v are labelled with consecutive Gaussian integers. Also the edges incident on v are labelled with consecutive Gaussian integers.

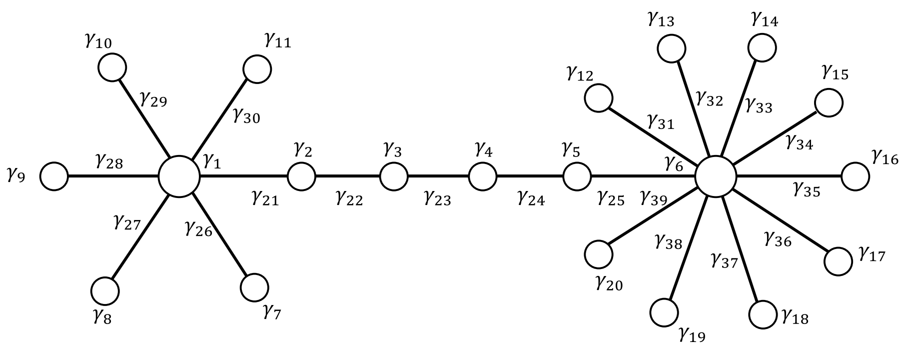


FIGURE 4.

Case 2. Let $v = v_i$, $2 \leq i \leq n-1$

The labelling on the neighbourhood vertices of v are either even or odd consecutive Gaussian integers. Also the edges incident on v are labelled with consecutive Gaussian integers.

Since consecutive Gaussian integers are relatively prime and consecutive odd or even Gaussian integers are relatively prime, any (n, k, m)-double star tree admits a Gaussian total neighbourhood prime labelling. □

Figure 4 shows the Gaussian total neighbourhood prime labelling of $(6, 6, 10)$ – double star tree.

Theorem 5.5. *Any spider tree admits a Gaussian total neighbourhood prime labelling.*

Proof. Let T be a spider tree with $|V| = n$ and $|E| = n - 1$.

Suppose the centre vertex v_1 has degree k. Then if we remove v_1 from T we are left with paths L_1, $L_2, \ldots, L_k$ with lengths a_1, $a_2, \ldots, a_k$ respectively.

Define the bijection $f : V \cup E \to [\gamma_{2n-1}]$ as follows.

Set $f(v_1) = \gamma_1$. Label the vertices of two arbitrary paths L_p and L_q with lengths a_p and a_q as follows. Label vertices of L_p with a_p consecutive Gaussian integers γ_{a_p+1}, γ_{a_p}, $\gamma_{a_p-1}, \ldots, \gamma_2$ and label vertices of L_q with next a_q consecutive Gaussian integers γ_{a_p+2}, γ_{a_p+3}, $\gamma_{a_p+4}, \ldots, \gamma_{a_p+a_q+1}$.

Label vertices of all other $(k-2)$ paths by remaining consecutive Gaussian integers in the first n Gaussian integers. The $(n-1)$ edges are labelled as follows. First label the edges of two arbitrary paths L_p and L_q with lengths a_p and a_q. Label edges of L_p with a_p consecutive Gaussian integers γ_{n+a_p},

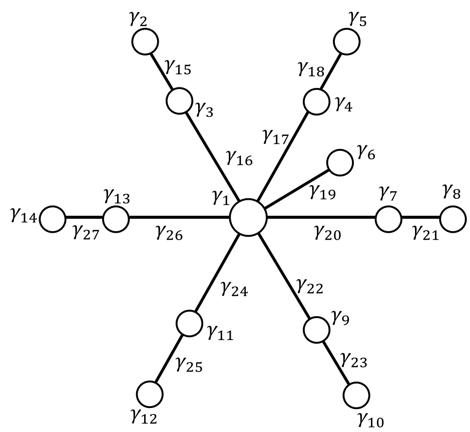


FIGURE 5.

$\gamma_{n+a_p-1}, \gamma_{n+a_p-2}, \ldots, \gamma_{n+1}$. Label edges of L_q with a_q consecutive Gaussian integers $\gamma_{n+a_p+1}, \gamma_{n+a_p+2}, \gamma_{n+a_p+3}, \ldots, \gamma_{n+a_p+a_q}$. Label edges of all other $(k-2)$ paths by remaining consecutive Gaussian integers.

For proving the Gaussian total neighbourhood prime labelling, we consider two cases. Let v be an arbitrary vertex of the graph.

Case 1. Let $v = v_1$, be the centre vertex.

Then there exists atleast two consecutive Gaussian integers in the labelling on the neighbourhood vertices of v_1. So the labelling on the neighbourhood vertices of v_1 are relatively prime. Also there exists atleast two consecutive Gaussian integers in the labelling on the edges incident on v_1. So this labelling is a Gaussian total neighbourhood prime labelling.

Case 2. Let $v = v_i$, $i = 2, 3, \ldots, n$.

Then the labelling on the neighbourhood vertices of v_i are either consecutive odd Gaussian integers or consecutive even Gaussian integers. So the labelling on the neighbourhood vertices of v_i are relatively prime. Also the labelling on the edges incident on v_i are consecutive Gaussian integers. So this labelling is a Gaussian total neighbourhood prime labelling. □

Figure 5 shows the Gaussian total neighbourhood prime labelling of spider tree with 14 vertices and 13 edges.

Lemma 5.1. *Assume G is a Gaussian total neighbourhood prime graph with n vertices and m edges. Let v be a vertex in G with $\deg(v) > 1$. The graph G' obtained by attaching a pendant edge to the vertex v is Gaussian total neighbourhood prime.*

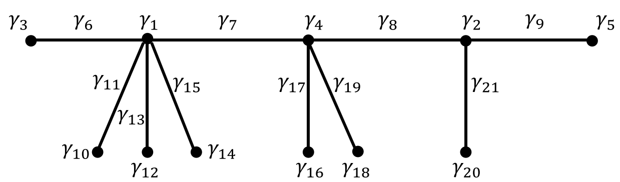
Proof. Let $f : V(G)\cup E(G) \to [\gamma_{n+m}]$ be the Gaussian total neighbourhood prime labelling for G. Let v' be the new vertex adjacent to v, then $V(G') = V(G)\cup\{v'\}$ and $E(G') = E(G) \cup \{vv'\}$, vv' is the edge connecting the vertex v to new vertex v'. We define a function $g : V(G') \cup E(G') \to [\gamma_{n+m+2}]$ by assigning $g(v') = \gamma_{n+m+1}$, $g(uv) = \gamma_{n+m+2}$, $g(u) = f(u)$ for all $u \in V(G)$ and $g(uv) = f(uv)$ for all edge $uv \in E(G)$. Let u be a vertex in G' with $\deg(u) > 1$, which is not v' since it is a pendant vertex.

For the case of $u = v$ we have $N_{G'}(v) = N_G(v) \cup \{v'\}$. Since $N_G(v) \subset N_G(v')$ and gcd of labelling on the neighbourhood vertices of v in G is one, so the gcd of labelling on the neighbourhood vertices of v in G' is also one. Also the gcd of labelling on the edges of v in G is one, the gcd of labelling on the edges of v in G' is one.

If $u \neq v$, $N_{G'}(u) = N_G(u)$. Since f is a Gaussian total neighbourhood prime labelling, the gcd of labelling on the neighbourhood vertices of v in G' one and gcd of labelling on the edges of v in G' is one. Thus g is a Gaussian total neighbourhood prime labelling. □

Theorem 5.6. *All caterpillars have Gaussian total neighbourhood prime labelling.*

Proof. Caterpillars can be constructed from a path P_n in which each of the $n-2$ interior vertices either remains as degree 2 or is adjacent to at least one pendant vertex. Consider a caterpillar with a central path P_n. First label the path using the labelling given in theorem 5.2. Then since the interior path vertices were initially degree 2, lemma 5.1 provides a way to assign labels to each pendant vertex while maintaining the Gaussian total neighbourhood prime labelling condition at each step. □



Caterpillar graph

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