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EULERIAN GRAPH OF SOME SPECIAL IDEALIZATION RINGS

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ABSTRACT. Let **R** be a ring with unity and let M be an **R**-module. Let **R**$(+)M$ be the idealization of the ring **R** by the **R**-module M. In this article, we study the Eurelian property of zero-divisor graphs. We investigate when some special idealization rings are Eulerian graphs.

1. INTRODUCTION.

In this article, all rings are a commutative ring with unity.

The notation of the zero-divisor graph of a commutative ring was introduced by I. Beck in [8], who linked some algebraic properties of G with combinatorial properties of its zero-divisor graph. Also, the context of coloring zero-divisor graph studied by D. D. Anderson and M. Naseer in [5]. The definition of zero-divisor graphs in its present form was given by Anderson and Livingston in [6, Theorem 2.3].

A zero-divisor graph of ring **R** is the graph Γ(**R**) whose vertices are the nonzero zero-divisors of **R**, with r and s adjacent if $r \neq s$ and $rs = 0$. In [6], Anderson and Livingston proved that the graph Γ(**R**) is connected with diameter at most

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3. The zero divisor graph of a commutative ring has been studied extensively by several authors $[5, 7]$.

For each **R**, let $Z(\mathbf{R})$ be the set of all zero-divisors of **R** and $Reg(\mathbf{R}) = \mathbf{R} \setminus Z(\mathbf{R})$.

Let M be an **R**- module. Consider $\mathbf{R}(+)M = \{(r_1,\ n_1) :\ r_1 \in \mathbf{R},\ n_1 \in \mathbf{M}\}$ and let $(r_1,\ n_1)$ and $(r_2,\ n_2)$ be two elements of $\mathbf{R}(+)M$. Define $(r_1,\ n_1) + (r_2, n_2) = (r_1 + r_2,\ n_1 + n_2)$ and $(r_1, n_1)(r_2, n_2) = (r_1 r_2,\ r_1 n_2 + r_2 n_1)$. Under this definition $\mathbf{R}(+)M$ becomes a commutative ring with unity. Call this ring the idealization ring of M in **R**. For more details, one can look in [9].

M. Al-Labadi in [2] studied zero-divisor graph of idealization ring $\Gamma(\mathbf{Z}_n(+)\mathbf{Z}_m)$. M. Al-Labadi in [1, 3, 4] has studied the properties of the zero-divisor graph of idealization ring when is a Planar graph and when is divisor graph.

Let **G** be a graph with the vertex set **V(G)**. The degree of a vertex u in a graph **G** is the number of edges incident with u. The degree of a vertex u is denoted by $deg(u)$. The complete graph of order m is denoted by K_m, is a graph with m vertices in which any two distinct vertices are adjacent. Recall that a graph **G** is connected if there is a path between every two distinct vertices. For every pair of distinct vertices x_1 and x_2 of **G**, let $d(x_1, x_2)$ be the length of the shortest path from x_1 to x_2 and if there is no such a path we define $d(x_1, x_2) = \infty$.

2. When the graph $\Gamma(\mathbf{R}(+)\mathbf{M})$ is Eulerian ?

In this section, we introduce when $\Gamma(\mathbf{R}(+)\mathbf{M})$ is Eulerian graph, where $\mathbf{R}(+)\mathbf{M}$ is called the idealization ring $\mathbf{R}$ by the $\mathbf{R}-$module $\mathbf{M}$.

Definition 2.1. *A graph is called **Eulerian** graph if there exists a closes trial containing every edge of the graph.*

Proposition 2.1. $[10]$ *A connected finite graph is Eulerian if and only if the degree of each vertex of the graph is even.*

Al-Labadi $[2]$ presented the following lemma when **R** is an integral domain.

Lemma 2.1. *For **R** is an integral domain and **M** is an **R**$-$module. Then we have the following:*

1. *If **R** be an integral domain such that $\mathbf{Z}_2$ is an **R**$-$module with $ann(\mathbf{Z}_2) = 0$, then $\mathbf{R} \cong \mathbf{Z}_2$.*

2. *If* $\boldsymbol{R}$ *be an integral domain such that* $\boldsymbol{Z}_3$ *is an* $\boldsymbol{R}-$*module with* $ann(\boldsymbol{Z}_2) = 0$*, then* $\boldsymbol{R} \cong \boldsymbol{Z}_3$*.*

Theorem 2.1. *Let* $\boldsymbol{R}$ *be an integral domain and* $\boldsymbol{M} \cong \boldsymbol{Z}_2$ *be an* $\boldsymbol{R}-$*module. Then we have the following:*

1. *If* $ann(\mathbf{Z}_2) = 0$*, then* $\Gamma(\boldsymbol{R}(+)\mathbf{Z}_2) = \{(0,1)\}$ *is an empty graph.*
2. *If* $ann(\mathbf{Z}_2) \neq 0$*, then* $\Gamma(\boldsymbol{R}(+)\mathbf{Z}_2)$ *is not an Eulerian graph.*

Proof. We have the following:

1: If $ann(\mathbf{Z}_2) = 0$, then $\Gamma(\mathbf{Z}_2(+)\mathbf{Z}_2) = \{(0,1)\}$ is an empty graph.
2: If $ann(\mathbf{Z}_2) \neq 0$, then $\Gamma(\mathbf{R}(+)\mathbf{Z}_2) = \{(0,1), (r_i, 0), (r_j, 1) :\ r_i,\ r_j \in \mathbf{R}\}$. So, all vertices adjacent to the vertex $(r, 0) \in \mathbf{R}(+)\mathbf{Z}_2$ is $N((r, 0)) = \{(0, 1)\}$. So, the degree of the vertex $(r, 0)$ is $deg((r, 0)) = 1$ which is an odd number.

□

Theorem 2.2. *Let* $\boldsymbol{R}$ *be an integral domain and* $\boldsymbol{M} \cong \boldsymbol{Z}_3$ *be an* $\boldsymbol{R}-$*module. Then we have the following:*

1. *If* $ann(\mathbf{Z}_3) = 0$*, then* $\Gamma(\boldsymbol{R}(+)\mathbf{Z}_3)$ *is not an Eulerian graph.*
2. *If* $ann(\mathbf{Z}_3) \neq 0$ *and* $|ann(M)| =$*odd, then* $\Gamma(\boldsymbol{R}(+)\boldsymbol{Z}_3)$ *is an Eulerian graph.*
3. *If* $ann(\mathbf{Z}_3) \neq 0$ *and* $|ann(M)| =$*even, then* $\Gamma(\boldsymbol{R}(+)\boldsymbol{Z}_3)$ *is not an Eulerian graph.*

Proof. We have the following:

1: If $ann(\mathbf{Z}_3) = 0$, then $\Gamma(\mathbf{R}(+)\mathbf{Z}_3) = \{(0, 1), (0, 2)\}$ that is not Eulerian graph.
2: If $ann(\mathbf{Z}_3) \neq 0$ and $|ann(M)| =$odd, then $\Gamma(\mathbf{R}(+)\mathbf{Z}_3) = \{(0, 1), (0, 2), (r, m) : r \in ann(\mathbf{Z}_3)\ and\ m \in \mathbf{Z}_3\}$. So, all vertices are even degree with $N((0, m)) = \{(0, n) : n \in \mathbf{Z}_3 \setminus \{0, m\}\} \bigcup \{(r, m) :\ r \in ann(\mathbf{Z}_3),\ m \in \mathbf{Z}_3\}$ i.e $deg(0, m) = 1 + 3|ann(\mathbf{Z}_3)|$ =even number. Also, $N((r, m)) = \{(0, 1), (0, 2)\} = 2$ which is an even number.
3: If $ann(\mathbf{Z}_3) \neq 0$ and $|ann(M)| =$even, then $\Gamma(\mathbf{R}(+)\mathbf{Z}_3) = \{(0, 1), (0, 2), (r, m) : r \in ann(\mathbf{Z}_3)\ and\ m \in \mathbf{Z}_3\}$. So, all the vertex adjacent to the vertex $(0, 1)$ is $N((0, 1)) = \{(0, 2), (r, m) : r \in ann(\mathbf{Z}_3)\ and\ m \in \mathbf{Z}_3\}$ i.e, $deg((0, 1)) = 1 + 3|ann(\mathbf{Z}_3)| =$ odd number.

We have the following theorem when $|\mathbf{M}| \geq 4$. □

Theorem 2.3. *Let* $\boldsymbol{R}$ *be an integral domain and* $|\boldsymbol{M}| \geq 4$ *be an* $\boldsymbol{R}-$*module. Then* $\Gamma(\boldsymbol{R}(+)\boldsymbol{M})$ *is not an Eulerian graph.*

Proof. We have the following:

1. If $|\mathbf{M}^*|$ =even, then we have all vertices adjacent to the vertex $(r, 0) \in \mathbf{R}(+)\mathbf{M}$ is $N((r, 0)) = \{(0, m) : m \in \mathbf{M}^*\}$, where $r \in ann(\mathbf{M})$. So, the degree of the vertex $(r, 0)$ is $deg((r, 0) = |\mathbf{M}^*|$ which is an odd number.

2. If $|\mathbf{M}^*|$ =odd, then we have all vertices adjacent to the vertex $(0, m) \in \mathbf{R}(+)\mathbf{M}$ is $N((0, m)) = \{(0, n) : n \in \mathrm{M}^*\} \bigcup \{(r, n) : r \in ann(\mathrm{M}), n \in \mathbf{M}\}$. So, the degree of the vertex $(0, m)$ is $deg((0, m) = |\mathbf{M}^*\{(0, m)\}| + |ann(\mathbf{M})||\mathbf{M}||\mathbf{M}^*|$ =odd+even, which is an odd number. □

Possible applications of this study can be found in problems of [11, 12].

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