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PROPER $m-$POLAR SOFT FUZZY GRAPHS

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ABSTRACT. In this research paper, $m-$polar soft fuzzy graphs, proper $m-$polar soft fuzzy graphs and totally proper $m-$polar soft fuzzy graphs have been presented. An indispensable and appropriate situation in which these graphs correspond with each other is given. There is also mention about the description of proper $3-$polar soft fuzzy graphs on cycle and peterson graph.

1. INTRODUCTION

In 1735, the concept of graph theory originated from Konigsberg bridge problem. The Eulerian graph is a resultant of this problem. The Konigsberg bridge problem was analysed by Euler and later he built a framework to rectify the problem, which was named after him as Eulerian graph. Based on Zadeh's fuzzy relations in 1971, Haufmann introduced the first definition of fuzzy graph in 1973. The concept of fuzzy graph was presented by Rosenfeld [11] in 1975. Also, he accounted the fuzzy relations amid fuzzy sets and after acquiring analogs of various graph theoretical perception he established the framework of fuzzy relations.

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1.1. **Review of Literature.** To resolve the ambiguities, a concept named soft set theory was introduced as an innovative mathematical device by Molodtsov [7]. It is thus proved that soft sets are a highly potential idea and it is applied in several domains like game theory, smoothening the functions, research operations, Probability theory, Measurement theory, Riemann integration and Perron integration [7] [8]. Soft sets based fuzzy sets and fuzzy soft sets were discussed by Ali et al. [1]. In 2014, $m-$polar fuzzy graphs were defined by Juanjuan chen at al. [5]. A few functions on soft graphs were proposed by M.Akaram, and Nawaz. S [3]. In 2008, the concept called regular fuzzy graphs were presented by A.Nagoor Gani and K.Radha [10]. Mohinta Sumit, Samanta.T.K. [9] initiated the idea of fuzzy soft graph. The concept of $m-$polar fuzzy graph was presented by J.Chen and S.Li, S.Max, X.Wang. These inventions stimulate our intention to explain $m-$polar soft fuzzy graphs and totally regular $m-$polar soft fuzzy graphs and analyse some of their features. $m = 3$ is the proven result. Right through the paper, the set of values taken by the vertex and edge membership functions is represented by m.

2. Preliminaries

Definition 2.1. *Let V be a nonempty finite set and $\sigma : V \to [0,1]$. Again,let $\mu : V \times V \to [0,1]$ such that $\mu(x,y) \leq \sigma(x)\Lambda\sigma(y)\forall(x,y) \in V \times V$.Then the pair $G : (\sigma,\mu)$ is called fuzzy graph over the set V. Here σ and μ are respectively called the fuzzy vertex and fuzzy edge of the fuzzy graph(σ,μ).*

Definition 2.2. *Let V be a non-empty set of vertices, E be the set of parameters and $A \subseteq E$.*

(i) If $\rho : A \longmapsto F(V)$ (collection of all fuzzy subsets in V), $e \to \rho(e) = \rho_e$ (say) and $\rho_e : V \to [0,1]$, $x_i \longmapsto \rho_e(x_i)$, then (A,ρ) : is a fuzzy soft vertex.

(ii) If $\mu : A \to F(V \times V)$(collection of all fuzzy subsets in E), $\rho \mapsto \mu(e)$(say) and $\mu_e : V \times V \to [0,1]$, $(x_i,x_j) \mapsto \mu_e(x_i,x_j)$, then (A,μ) : is a fuzzy soft edge.

Also, $((A,\rho),(A,\mu))$ is called fuzzy soft graph if and only if $\mu_e(x_i,x_j) \leq \rho_e(x_i) \wedge \rho_e(x_j)$ for all $e \in A$ and for all $i,j = 1,2,\ldots,n$ and this fuzzy soft graph is denoted by $G_{A,V}$.

Definition 2.3. *Let $G^* = (V,E)$ be a crisp graph and $G_{A,V}$ be a fuzzy soft graph of G^*. Then $G_{A,V}$ is said to be regular fuzzy soft graph if $H_{A,V}(e)$ is a regular fuzzy* *graph for all $e \in A$, if $H_{A,V}(e)$ is a regular fuzzy graph of degree r for all $e \in A$, then $G_{A,V}$ is a $r-$ regular fuzzy soft graph.*

Definition 2.4. *Let $G^* = (V, E)$ be a crisp graph and $G_{A,V}$ be a fuzzy soft graph of G^*. Then $G_{A,V}$ is said to be totally regular fuzzy soft graph if $H_{A,V}(e)$ is a totally regular fuzzy graph for all $e_i \in A$,if $H_{A,V}(e)$ is a totally regular fuzzy graph of degree r for all $e \in A$, then $G_{A,V}$ is a $r-$ totally regular fuzzy soft graph.*

Definition 2.5. *Let $G_{A,V} = ((A, \rho), (A, \mu))$ be a fuzzy soft graph. The degree of a vertex u is defined as $d_{G_{A,V}}(u) = \sum_{e_i \in A}(\sum_{u \neq v} \mu_{e_i}(u, v))$.*

Definition 2.6. *Let $G_{A,V} = ((A, \rho)(A, \mu))$ be a fuzzy soft graph. The total degree of a vertex u is defined as $td_{G_{A,V}} = d_{G_{A,V}}(u) + \sum_{e_i \in A} \rho_{e_i}(u)$.*

Definition 2.7. *An $m-$polar fuzzy graph with an underlying pair (V, E) (where $E \subseteq V \times V$ is symmetric) is defined to be a pair $G = (A, B)$ where $A : V \to [0, 1]^m$ and $B : E \to [0, 1]^m$ satisfying $B(xy) \leq min\{A(x), A(y)\}$ for all $x \in E$.*

Definition 2.8. *Let U be an initial universe, $\mathcal{P}$ the set of all parameters, $A \subset \mathcal{P}$ and $\mathcal{P}(U)$ the collection of all fuzzy subsets of U. Then $(\widetilde{F}, A)$ is called soft fuzzy set, where $\widetilde{F} : A \to \mathcal{P}(U)$ is a mapping called fuzzy approximate function of the fuzzy set $(\widetilde{F}, A)$.*

3. $m-$POLAR SOFT FUZZY GRAPHS

Definition 3.1. *An $m-$polar soft fuzzy graph $\widetilde{G}_{P,V} = (G^*, \widetilde{\rho}, \widetilde{\mu}, P)$ is a $4-$ tuple such that*

(i) *$G^* = (V, E)$ is a simple graph;*

(ii) *P is a nonempty set of parameters;*

(iii) *$\widetilde{\rho} : P \to F(V)$(collection of all fuzzy subsets in V), $\mathfrak{e} \to \widetilde{\rho}(\mathfrak{e}) = \widetilde{\rho}_{\mathfrak{e}}(say)$, and $\widetilde{\rho}_{\mathfrak{e}} : V \to [0, 1]^m$, $(x_1, x_2, \ldots, x_m) \mapsto \widetilde{\rho}_{\mathfrak{e}}(x_1, x_2, \ldots, x_m)$; $(\widetilde{\rho}, P)$ is a $m-$polar soft fuzzy set over V;*

(iv) *$\widetilde{\mu} : P \to F(V \times V)$ (collection of all fuzzy subsets in $V \times V$), $\mathfrak{e} \mapsto \widetilde{\mu}(\mathfrak{e}) = \widetilde{\mu}_{\mathfrak{e}}$ (say), and $\widetilde{\mu}_{\mathfrak{e}} : V \times V \to [0, 1]^m$, $(x_1, x_2, \ldots, x_m) \mapsto \widetilde{\mu}_{\mathfrak{e}}(x_1, x_2, \ldots, x_m)$; $(\widetilde{\mu}, P)$ is a $m-$polar soft fuzzy set over E;*

(v) *$(\widetilde{\rho}_e, \widetilde{\mu}_e)$ is a $m-$polar fuzzy subgraph of G^* for all $e \in P$. That is,*

$$\widetilde{\mu}_e x_1(uv) \leq (\widetilde{\rho}_e x_1(u) \wedge \widetilde{\rho}_e x_1(v))$$
$$\widetilde{\mu}_e x_2(uv) \leq (\widetilde{\rho}_e x_2(u) \wedge \widetilde{\rho}_e x_2(v))$$

$$\vdots$$
$$\widetilde{\mu}_e x_m(uv) \leq (\widetilde{\rho}_e x_m(u) \wedge \widetilde{\rho}_e x_m(v))$$
for all $e \in P$ *and* $u, v \in V$.

The $m-$*polar fuzzy graph* $(\widetilde{\rho}_e, \widetilde{\mu}_e)$ *is denoted by* $\widetilde{H}_{P,V}(e)$ *for convenience. In otherwise, a* $m-$*polar soft fuzzy graph is a parameterized family of* $m-$*polar fuzzy graphs.*

Example 1. *Consider a* $3-$*polar soft fuzzy graph* $\widetilde{G}_{P,V}$ *be a underlying on* $G^* = (V, E)$ *such that* $V = \{a_1, a_2, a_3\}$ *and* $P = \{e_1, e_2\}$ *be a parameter set and let* $E = \{a_1a_2, a_2a_3, a_3a_1\}$.

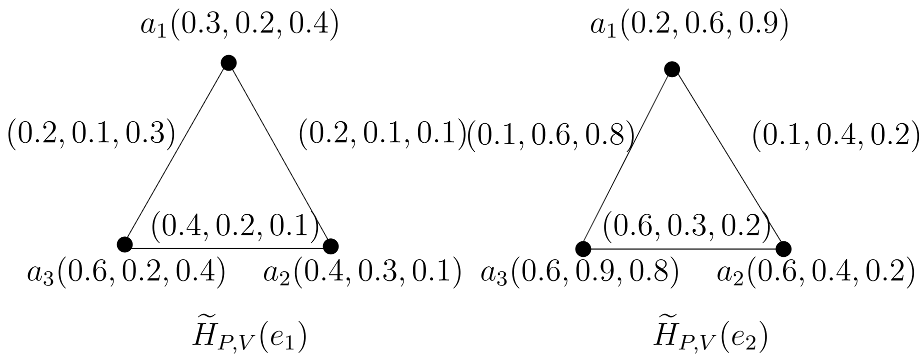

Fig.1

Thus $\widetilde{H}_{P,V}(e_1) = (\widetilde{\rho}(e_1), \widetilde{\mu}(e_1)), \widetilde{H}_{P,V}(e_2) = (\widetilde{\rho}(e_2), \widetilde{\mu}(e_2))$ are $3-$polar fuzzy graphs of G^*. It is easy to verify that $\widetilde{G}_{P,V} = (G^*, \widetilde{\rho}, \widetilde{\mu}, P)$ is a $3-$polar soft fuzzy graph.

Definition 3.2. *Let* $G^* = (V, E)$ *be a underlying graph and* $\widetilde{G}_{P,V}$ *be a* $m-$*polar soft fuzzy graph of* G^*. *Then* $\widetilde{G}_{P,V}$ *is said to be a proper* $m-$*polar soft fuzzy graph if* $\widetilde{H}_{P,V}(e)$ *is a proper* $m-$*polar fuzzy graph for all* $e \in P$. *If* $\widetilde{H}_{P,V}(e)$ *is a proper* $m-$*polar fuzzy graph of degree* $\mathcal{S}$ *for all* $e \in P$, *then* $\widetilde{G}_{P,V}$ *is a* $\mathcal{S}-$ *proper* $m-$*polar soft fuzzy graph.*

Definition 3.3. *Let* $G^* = (V, E)$ *be a underlying graph and* $\widetilde{G}_{P,V}$ *be a* $m-$*polar soft fuzzy graph of* G^*. *Then* $\widetilde{G}_{P,V}$ *is said to be a totally proper* $m-$*polar soft fuzzy graph if* $\widetilde{H}_{P,V}(e)$ *is a totally proper* $m-$*polar fuzzy graph for all* $e \in P$. *If* $\widetilde{H}_{P,V}(e)$ *is a totally proper* $m-$*polar fuzzy graph of degree* $\mathcal{S}$ *for all* $e \in P$, *then* $\widetilde{G}_{P,V}$ *is a* $\mathcal{S}-$ *totally proper* $m-$*polar soft fuzzy graph.*

4. Some Properties of Proper, Totally Proper $3-$Polar Soft Fuzzy Graph

Proposition 4.1. *A proper $3-$polar soft fuzzy graph is redundant to be a totally proper $3-$polar soft fuzzy graph.*

Example 2. *Consider a $3-$polar soft fuzzy graph $\widetilde{G}_{P,V}$ be a underlying graph on $G^*(V,E)$ such that $V=\{a_1,a_2,a_3,a_4,a_5,a_6\}$ and let $P=\{e_1,e_2\}$ be a parameter set and $E=\{a_1a_2,a_2a_3,a_3a_4,a_4a_5,a_5a_6,a_6a_1\}$.*

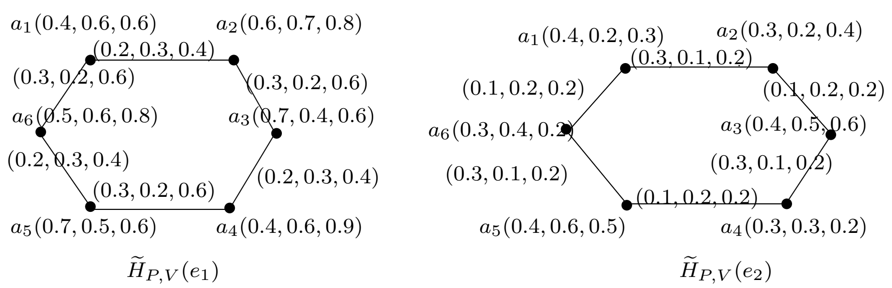

Fig.2

By performing usual calculations easy to see that $3-$polar fuzzy graph $\widetilde{H}_{P,V}(e_1)=(\widetilde{\rho}(e_1),\widetilde{\mu}(e_1)),\widetilde{H}_{P,V}(e_2)=(\widetilde{\rho}(e_2),\widetilde{\mu}(e_2))$.

In Fig.2. we have $d_{\widetilde{H}_{P,V}}(e_1)(a_j)=(0.5,0.5,1.0)\ \forall j=1,2,3,4,5,6$. Thus $\widetilde{H}_{P,V}(e_1)$ is a $(0.5,0.5,1.0)-$ proper $3-$polar fuzzy graph. and $d_{\widetilde{H}_{P,V}}(e_2)(a_j)=(0.4,0.3,0.4)$ $\forall j=1,2,3,4,5,6$. Thus $\widetilde{H}_{P,V}(e_2)$ is a $(0.4,0.3,0.4)-$ proper $3-$polar fuzzy graph. Hence, $\widetilde{G}_{P,V}$ is a $(0.9,0.8,1.4)-$ proper $3-$polar soft fuzzy graph. So, $\widetilde{G}_{P,V}=\{\widetilde{H}_{P,V}(e_1),\widetilde{H}_{P,V}(e_2)\}$ is a proper $3-$polar soft fuzzy graph but not totally proper $3-$polar soft fuzzy graph.

Proposition 4.2. *A totally proper $3-$polar soft fuzzy graph is redundant to be a proper $3-$polar soft fuzzy graph.*

Example 3. *Consider a $3-$polar soft fuzzy graph $\widetilde{G}_{P,V}$ be a underlying graph on $G^*(V,E)$ such that $V=\{a_1,a_2,a_3\}$ where $E=\{a_1a_2,a_2a_3,a_3a_1\}$and let $P=\{e_1,e_2\}$ be a parameter set.*

In Fig.3. By performing routine computations $\widetilde{H}_{P,V}(e_1)=(\widetilde{\rho}(e_1),\widetilde{\mu}(e_1))$, $\widetilde{H}_{P,V}(e_2)=(\widetilde{\rho}(e_2),\widetilde{\mu}(e_2))$ are totally proper $3-$polar fuzzy graphs. Hence $\widetilde{G}_{P,V}=\{\widetilde{H}_{P,V}(e_1),\widetilde{H}_{P,V}(e_2)\}$ is a totally proper $3-$polar soft fuzzy graph but not proper $3-$polar soft fuzzy graph.

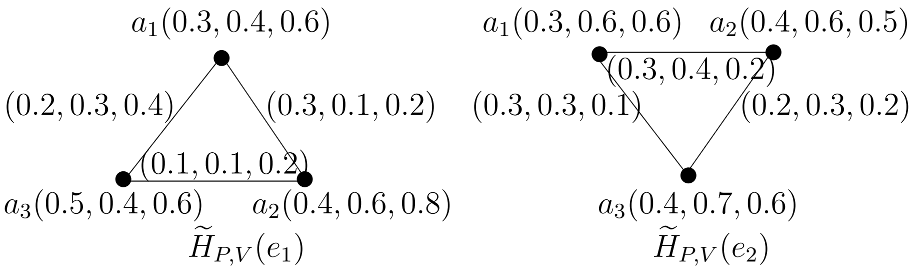

Fig.3

Proposition 4.3. *A* $3-$*polar soft fuzzy graph is both proper and totally proper* $3-$*polar soft fuzzy graph.*

Example 4. *Consider a* $3-$*polar soft fuzzy graph* $\widetilde{G}_{P,V}$ *be a underlying graph on* $G^*(V, E)$ *such that* $V = \{a_1, a_2, a_3, a_4\}$ *where* $P = \{e_1, e_2\}$ *be a parameter set and let* $E = \{a_1a_2, a_2a_3, a_3a_4, a_1a_4\}$.

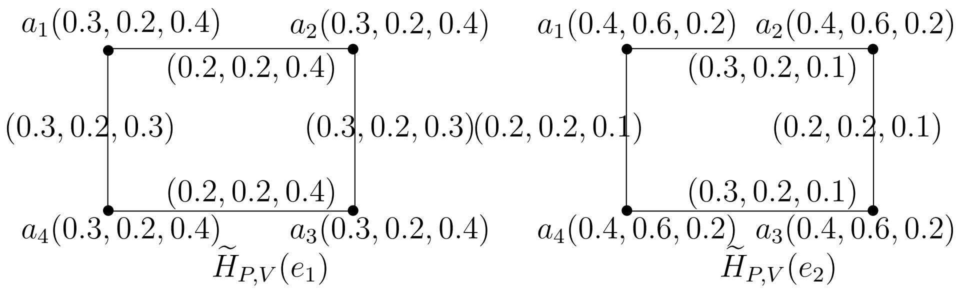

Fig.4

Hence $\widetilde{G}_{P,V} = \{\widetilde{H}_{P,V}(e_1), \widetilde{H}_{P,V}(e_2)\}$. Hence A $3-$polar soft fuzzy graph having a proper and totally proper $3-$polar soft fuzzy graph.

Remark 4.1. *It is explicit from the above examples that generally proper* $3-$*polar soft fuzzy graph and totally proper* $3-$ *polar soft fuzzy graphs are not associated with each other. Nonetheless, in the following theorem, an indispensable and appropriate situation in which both these* $3-$*polar soft fuzzy graphs are identical is given.*

5. Main Results

Theorem 5.1. *Let* $\widetilde{G}_{P,V} = ((P, \widetilde{\rho}), (P, \widetilde{\mu}))$ *be a* $3-$*polar soft fuzzy graph on* $G^* = (V, E)$. *Then* $\widetilde{\rho}_{e_i} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ *is a constant function in* $3-$*polar fuzzy graph*

$\widetilde{H}_{P,V}(e_i)$ *for all* $e_i \in P$ *for* $i = 1, 2, 3, \ldots, n$ *for all if and only if the following are equivalent:*

(i) $\widetilde{G}_{P,V}$ *is a* $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$*- proper* $3-$*polar soft fuzzy graph.*

(ii) $\widetilde{G}_{P,V}$ *is a* $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$*- a totally proper* $3-$*polar soft fuzzy graph.*

Proof. Assume that $\widetilde{\rho}_e$ is a constant function. Then $\widetilde{\rho}_{e_i}(u) = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ in which $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are constant, $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \in [0, 1]^m\ \forall\ \ e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Assume that $\widetilde{G}_{P,V}$ is a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$- proper $3-$polar soft fuzzy graph. $d_{\widetilde{G}_{P,V}}(u) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ in $3-$polar fuzzy graphs $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ $\forall u \in V$. It follows that $td_{\widetilde{G}_{P,V}}(u) = d_{\widetilde{G}_{P,V}}(u) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u)$ in $\widetilde{H}_{P,V}(e_i)$ $\forall e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$, and is equal to $d_{\widetilde{G}_{P,V}}(u) + (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

Therefore, $td_{\widetilde{G}_{P,V}}(u) = (\mathcal{P}_1 + \mathcal{C}_1, \mathcal{P}_2 + \mathcal{C}_2, \mathcal{P}_3 + \mathcal{C}_3) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $3-$polar fuzzy graphs $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ $\forall u \in V$.

$\widetilde{G}_{P,V}$ is a $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$- totally proper $3-$polar soft fuzzy graph.

Thus $(i) \Rightarrow (ii)$.

Now, suppose that $\widetilde{G}_{P,V}$ is a $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ totally proper $3-$polar soft fuzzy graph. Then $td_{\widetilde{G}_{P,V}}(u) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. It follows that $d_{\widetilde{G}_{P,V}}(u) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Further, $d_{\widetilde{G}_{P,V}}(u) + (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Therefore, $\Rightarrow d_{\widetilde{G}_{P,V}}(u) = (\mathcal{T}_1 - \mathcal{C}_1, \mathcal{T}_2 - \mathcal{C}_2, \mathcal{T}_3 - \mathcal{C}_3) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. So, $\widetilde{G}_{P,V}$ is a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$-proper $3-$polar soft fuzzy graph.

Thus $(ii) \Rightarrow (i)$ is proved.

Hence (i) and (ii) are equivalent.

On the contrarily, assume that (i) and (ii) are equivalent, i.e, $\widetilde{G}_{P,V}$ is a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ proper if and only if $\widetilde{G}_{P,V}$ is a $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ totally proper $3-$polar soft fuzzy graph.

Suppose that $\widetilde{\rho}_e$ is not a constant function. Then $\sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) \neq \sum_{e_i \in P} \widetilde{\rho}_{e_i}(v)$ for at least one pair of vertices $u, v \in V$ and for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$.

Let $\widetilde{G}_{P,V}$ be a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ proper $3-$polar soft fuzzy graph. Then $d_{\widetilde{G}_{P,V}}(u) = d_{\widetilde{G}_{P,V}}(v) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. So, $td_{\widetilde{G}_{P,V}}(u) = d_{\widetilde{G}_{P,V}}(u) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$

and for all $u, v \in V$. Therefore,

$$td_{\widetilde{G}_{P,V}}(u) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u)$$

and

$$td_{\widetilde{G}_{P,V}}(v) = d_{\widetilde{G}_{P,V}}(v) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(v)$$

in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u, v \in V$. $td_{\widetilde{G}_{P,V}}(v) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(v)$ in $\widetilde{H}_{P,V}(e_i)$for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u, v \in V$. As $\widetilde{\rho}_e(u) \neq \widetilde{\rho}_e(v)$, we have $td_{\widetilde{G}_{P,V}}(u) \neq td_{\widetilde{G}_{P,V}}(v)$. Therefore, $\widetilde{G}_{P,V}$ is not totally proper $3-$polar soft fuzzy graph, which is a contrary to our assumption. Now let $\widetilde{G}_{P,V}$ be a totally proper $3-$polar soft fuzzy graph. Then $td_{\widetilde{G}_{P,V}}(u) = td_{\widetilde{G}_{P,V}}(v) \Rightarrow d_{\widetilde{G}_{P,V}}(u) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) = d_{\widetilde{G}_{P,V}}(v) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(v)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u, v \in V$. It follows $d_{\widetilde{G}_{P,V}}(u) - d_{\widetilde{G}_{P,V}}(v) = \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) - \sum_{e_i \in P} \widetilde{\rho}_{e_i}(v) \neq 0$, and $d_{\widetilde{G}_{P,V}}(u) \neq d_{\widetilde{G}_{P,V}}(v)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and $\forall u, v \in V$. Therefore, $\widetilde{G}_{P,V}$ is not proper $3-$polar soft fuzzy graph, which is contrary to our assumption. Thus $\widetilde{\rho}$ is a constant function. □

Theorem 5.2. *If a $3-$polar soft fuzzy graph $\widetilde{G}_{P,V}$ is both $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$-proper and $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ totally proper, then $\sum_{e_i \in P} \widetilde{\rho}_{e_i}$ is a constant function in $\widetilde{H}_{P,V}(e_i)$ of G^* for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$.*

Proof. Proof. Let $\widetilde{G}_{P,V}$ is a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ proper and $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ totally proper. Then $d_{\widetilde{G}_{P,V}}(u) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$ and $td_{\widetilde{G}_{P,V}}(u) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. It follows $d_{\widetilde{G}_{P,V}}(u) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Also, $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) + \sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Therefore,$\sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) = (\mathcal{T}_1 - \mathcal{P}_1, \mathcal{T}_2 - \mathcal{P}_2, \mathcal{T}_3 - \mathcal{P}_3) = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Also follows, $\sum_{e_i \in P} \widetilde{\rho}_{e_i}(u) = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Therefore $\sum_{e_i \in P} \widetilde{\rho}_{e_i}$ is a constant function in $\widetilde{H}_{P,V}(e_i)$ of G^* for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$. □

Remark 5.1. *Generally, the contrariness of the above theorem is not factual. That is, it is redundant for $\widetilde{G}_{P,V}$ to be both proper and totally proper $3-$polar soft fuzzy graph if $\widetilde{\rho}_{e_i}(u)$ is a constant function.*

6. A Characterization of Proper 3-polar Soft Fuzzy Graph on Cycle

Next two theorems provide a characterization of a proper $3-$polar soft fuzzy graph $\widetilde{G}_{P,V}$ such that $G^*_{P,V}$ is a cycle.

Theorem 6.1. *Let on an odd cycle $G^*(V,E)$, $\widetilde{G}_{P,V}$ be a $3-$polar soft fuzzy graph. Then $\widetilde{G}_{P,V}$ is proper $3-$polar soft fuzzy graph if and only if $\widetilde{\mu}$ is a constant function in $3-$polar fuzzy subgraph $\widetilde{H}_{P,V}(e_i)$ over $H^*_{P,V}(e_i)$ is an odd cycle for all $e_i \in P$ for $i = 1,2,3,\ldots,n$.*

Proof. Suppose that $\widetilde{\mu}$ is a constant function. Then $\widetilde{\mu}(e_i)(uv) = (\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$, where $(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$ are constant, $(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3) \in [0,1]^m$ for all $e_i \in P$ and for $i = 1,2,3,\ldots,n$ in $3-$polar fuzzy graph $\widetilde{H}_{P,V}(e_i)$ and for all $uv \in E$. So $d_{\widetilde{G}_{P,V}}(u) = (2\mathcal{C}_1, 2\mathcal{C}_2, 2\mathcal{C}_3)$ in $3-$polar fuzzy graph $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1,2,3,\ldots,n$ and for all $u \in V$. Hence $\widetilde{G}_{P,V}$ is proper $3-$polar soft fuzzy graph.

On the contrarily, assume that $\widetilde{G}_{P,V}$ is a proper $3-$polar soft fuzzy graph of G^*. Let $d_1, d_2, \ldots, d_{2n+1}$ be the edges of G^* in that order. Let $\widetilde{\mu}(e_i)(d_1) = (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1,2,3,\ldots,n$. Since $\widetilde{H}_{P,V}(e_i)$ is $(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ proper $3-$polar fuzzy graph for all $e_i \in P$ for $i = 1,2,3,\ldots,n$. Then $\widetilde{\mu}(e_i)(d_2) = (\mathcal{P}_1-\mathcal{T}_1, \mathcal{P}_2-\mathcal{T}_2, \mathcal{P}_3-\mathcal{T}_3)$ for all $e_i \in P$ for $i = 1,2,3,\ldots,n$. $\widetilde{\mu}(e_i)(d_3) = (\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3) - (\mathcal{P}_1-\mathcal{T}_1, \mathcal{P}_2-\mathcal{T}_2, \mathcal{P}_3-\mathcal{T}_3) = (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$. Therefore,

$$\widetilde{\mu}(e_i)(d_j) = \begin{cases} (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) & \text{if } j \text{ is odd} \\ (\mathcal{P}_1-\mathcal{T}_1, \mathcal{P}_2-\mathcal{T}_2, \mathcal{P}_3-\mathcal{T}_3) & \text{if } j \text{ is even} \end{cases}.$$

So, $\widetilde{\mu}(e_i)(d_1) = \widetilde{\mu}(e_i)(d_{2n+1}) = (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$ for all $e_i \in P$ for $i = 1,2,3,\ldots,n$. Thus, if d_1 and d_{2n+1} incident at vertex u then $d_{\widetilde{G}_{P,V}}(u) = (\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ in $\widetilde{H}(e_i)$ for $e_i \in P$ for $i = 1,2,3,\ldots,n$. Then $\widetilde{\mu}(e_i)(d_1) + \widetilde{\mu}(e_i)(d_{2n+1}) = (\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ for all $e_i \in P$ for all $i = 1,2,3,\ldots,n$.

Further, $((\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) + (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)) = (\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ implies $(2\mathcal{T}_1, 2\mathcal{T}_2, 2\mathcal{T}_3) = (\mathcal{P}_1, \mathcal{P}_2,\mathcal{P}_3)$, and also $(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) = (\frac{\mathcal{P}_1}{2}, \frac{\mathcal{P}_2}{2}, \frac{\mathcal{P}_3}{2})$. So, $((\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3) - (\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)) = ((\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3) - (\frac{\mathcal{P}_1}{2}, \frac{\mathcal{P}_2}{2}, \frac{\mathcal{P}_3}{2})) = (\frac{\mathcal{P}_1}{2}, \frac{\mathcal{P}_2}{2}, \frac{\mathcal{P}_3}{2})$.

Therefore, $\widetilde{\mu}(e_i)(d_j) = (\frac{\mathcal{P}_1}{2}, \frac{\mathcal{P}_2}{2}, \frac{\mathcal{P}_3}{2})$ in $3-$polar fuzzy graphs $\widetilde{H}_{P,V}(e_i)$for all j and $i = 1, 2, 3, \ldots, n$.

Hence, $\widetilde{\mu}$ is a constant function. □

Theorem 6.2. *Let $\widetilde{G}_{P,V}$ be a $3-$polar soft fuzzy graph over an even cycle G^*. Then $\widetilde{G}_{P,V}$ is proper $3-$polar soft fuzzy graph if and only if $\widetilde{\mu}$ is a constant function or alternate edges have same membership degrees in $3-$polar fuzzy subgraph $\widetilde{H}_{P,V}(e_i)$ over $H^*_{P,V}(e_i)$ where $H^*_{P,V}(e_i)$ is an even cycle for all $e_i \in A$ for $i = 1, 2, 3, \ldots, n$.*

Proof. If either $\widetilde{\mu}$ is a constant function or alternate edges have same membership degree, then $\widetilde{G}_{P,V}$ is proper $3-$polar soft fuzzy graph. Conversely assume that $\widetilde{G}_{P,V}$ is a proper $3-$polar soft fuzzy graph of G^*. Let $d_1, d_2, d_3, \ldots, d_{2n}$ be the edges of G^* in that order. Let $\widetilde{\mu}(e_i)(d_1) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ in $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$. Since $\widetilde{H}_{P,V}(e_i)$ is $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$- proper $3-$polar fuzzy graphs for $i = 1, 2, 3, \ldots, n$. Then, $\widetilde{\mu}(e_i)(d_2) = (\mathcal{P}_1 - \mathcal{T}_1, \mathcal{P}_2 - \mathcal{T}_2, \mathcal{P}_3 - \mathcal{T}_3)$ for all $e_i \in P$, for $i = 1, 2, 3, \ldots, n$. $\widetilde{\mu}(e_i)(d_3) = ((\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) - (\mathcal{P}_1 - \mathcal{T}_1, \mathcal{P}_2 - \mathcal{T}_2, \mathcal{P}_3 - \mathcal{T}_3)) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ and so on. Therefore,

$$\widetilde{\mu}(e_i)(d_j) = \begin{cases} (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) & \text{if } j \text{ is odd,} \\ (\mathcal{P}_1 - \mathcal{T}_1, \mathcal{P}_2 - \mathcal{T}_2, \mathcal{P}_3 - \mathcal{T}_3) & \text{if } j \text{ is even} \end{cases}.$$

Proceeding as by a previous theorem. If $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = (\mathcal{P}_1 - \mathcal{T}_1, \mathcal{P}_2 - \mathcal{T}_2, \mathcal{P}_3 - \mathcal{T}_3)$ then $\widetilde{\mu}$ is a constant function. If $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \neq (\mathcal{P}_1 - \mathcal{T}_1, \mathcal{P}_2 - \mathcal{T}_2, \mathcal{P}_3 - \mathcal{T}_3)$, then alternate edges have same membership degrees. □

Remark 6.1. *The above theorems 3 and 4 does not hold for totally proper $3-$polar soft fuzzy graphs.*

7. A Characterization of Proper 3-polar Soft Fuzzy Graph on Peterson Graph

Theorem 7.1. *Let $\widetilde{G}_{P,V} = ((\widetilde{\rho}, P), (\widetilde{\mu}, P))$ be a $3-$polar soft fuzzy graph such that $G^* = (V, E)$ is peterson graph. If $\widetilde{\mu}$ is a constant function in $3-$polar fuzzy graphs $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$. Then $\widetilde{G}_{P,V}$ is proper $3-$polar soft fuzzy graph.*

Proof. Take peterson graph into regard on $G^* = (V, E)$. Let $\widetilde{\mu}_{e_i} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ where $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ are constant, $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3) \in [0, 1]^m$ for all $e_i \in P$ and for all $i = 1, 2, 3, \ldots, n$. Then $d_{\widetilde{G}_{P,V}}(u) = \sum_{e_i \in P} \widetilde{\mu}_{e_i}(u, v)$ in $3-$polar fuzzy graphs $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. As $d_{\widetilde{G}_{P,V}}(u) = (3\mathcal{K}_1, 3\mathcal{K}_2, 3\mathcal{K}_3)$ in $3-$polar fuzzy graphs $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Hence $\widetilde{G}_{P,V}$ is proper $3-$polar soft fuzzy graph. □

Remark 7.1. *The converse of the above theorem* 5 *need not be true.*

Theorem 7.2. *Let* $\widetilde{G}_{P,V} = ((\widetilde{\rho}, P), (\widetilde{\mu}, P))$ *be a* $3-$*polar soft fuzzy graph such that* $G^* = (V, E)$ *is peterson graph. If the edges on the cycle takes membership values* $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ *and the line joining the two cycle takes the membership values* $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$*in* $3-$*polar fuzzy graph* $\widetilde{H}_{P,V}(e_i)$ *for all* $e_i \in P$ *for* $i = 1, 2, 3, \ldots, n$. *Then* $\widetilde{G}_{P,V}$ *is proper* $3-$*polar soft fuzzy graph.*

Proof. Take peterson graph into regard on $G^* = (V, E)$. Let the edges on the cycle takes membership values $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ and the line joining the two cycle takes the membership values $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ in $3-$polar fuzzy graph $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$. Then $d_{\widetilde{G}_{P,V}}(u) = \sum_{e_i \in P} \widetilde{\mu}_{e_i}(u, v)$ in $\widetilde{H}_{P,V}(e_i)$ for all $u \in V$and for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$. $d_{\widetilde{G}_{P,V}}(u) = (2\mathcal{K}_1 + \mathcal{C}_1, 2\mathcal{K}_2 + \mathcal{C}_2, 2\mathcal{K}_3 + \mathcal{C}_3)$ in $3-$polar fuzzy graph $\widetilde{H}_{P,V}(e_i)$ for all $e_i \in P$ for $i = 1, 2, 3, \ldots, n$ and for all $u \in V$. Hence $\widetilde{G}_{P,V}$ is a proper $3-$polar soft fuzzy graph. □

8. CONCLUSION

In this paper, we have defined proper and totally proper m-polar soft fuzzy graphs and proved some results. We have provided characterization of proper 3-polar soft fuzzy graphs for cycle and peterson graph. The result can be characterized for some other standard graphs also.

REFERENCES

[1] M.I. ALI: *A note on soft sets, rough sets and fuzzy soft sets*, Applied soft computing, 11(2011),3329-3332

[2] M. AKRAM, S. NAWAZ: *On Fuzzy Soft Graphs*, Italian Journal of Pure and Applied Mathematics, **N34** (2015), 497-514.

[3] M. AKRAM, S. NAWAZ: *Operations on soft graphs*, Fuzzy information and Engineering, **7**(4) (2015), 423-449.

[4] A. AL-MASARWAH, M. ABUQAMAR: *Certain Types of Fuzzy Soft Graphs*, New Mathematics and Natural Computation, (2017).

[5] J. CHEN, S. LI, S. MA, X. WANG: *$m-$polar fuzzy sets: an extension of bipolar fuzzy sets*, Hindwai Publishing Corporation Sci. Word J., (2014), p8.

[6] P.K. MAJI, A.R. ROY, R. BISWAS: *Fuzzy soft set*, Journal of Fuzzy Mathematics, **9**(3) (2001), 589-602.

[7] D.A> MOLODTSOV: *Soft set theory -first results*, Computers and Mathematics with Applications, **37** (1997), 19-31.

[8] D.A. MOLODTSOV: *The theory of soft sets* (in Russian), URSS Publishers, Moscow, 2004.

[9] S. MOHINTO, T.K. SAMANTA: *An Introduction to Fuzzy Soft Graph*, Mathematica Moravica, **19**(3) (2015), 35-48.

[10] A. NAGOOR GANI, K. RADHA: *On Regular fuzzy graphs*, Journal of Physical Sciences, **12** (2008), 33-40.

[11] A. ROSENFELD, L.A. ZADEH, K.S. FU, M. SHIMURA, [EDS] : *Fuzzy graphs in Fuzzy sets and their Applications*, Academic Press, New York 1975, 77-95.

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