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ON τ^*-B-CLOSED SETS AND τ^*-B-CONTINUOUS

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ABSTRACT. The purpose of this paper is to define and study a new class of set called τ^*-b-closed set, and investigating the characteristics of τ^*-b-closed set. Furthermore, the new type of continuous function is introduce and find some of its basic properties.

1. INTRODUCTION

In 1996, Andrijevie [1] introduced one of the most well known notion called b-open sets. In 2009, Omari and Noorani [2] introduced and studied the concept of generalized b-closed set and generalized b-continuous function in topological spaces. In 2015, Ibrahim [3] introduced and discussed a new class of sets called τ^*-open sets.

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in X, respectively. A subset A of X is called b-open if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$ [1].

Definition 1.1. *[3] Let (X, τ) and (Y, δ) be two topological spaces and let f be a function from X into Y, then a subset A in τ is called τ^*-open if there exists an open*

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set G in δ such that $A = f^{-1}(G)$, that is $\tau^* = \{f^{-1}(G) : G \in \delta$ and $f^{-1}(G) \in \tau\}$. The family of all τ^*-open sets in X is denoted by τ^*. The complement of τ^*-open is τ^*-closed.

Definition 1.2. *[3] Let $f : X \to Y$ be any function and A be any subset of a topological space (X, τ).*

(1) *The union of all τ^*-open sets contained in A is called the τ^*-interior of A and denoted by τ^*-Int(A).*
(2) *The intersection of all τ^*-closed sets containing A is called the τ^*-closure of A and denoted by τ^*-Cl(A).*

Definition 1.3. *[3] Let $f : X \to Y$ be any function. A subset A of a space X is said to be:*

(1) *τ^*-α-open if $A \subseteq \tau^*$-Int(τ^*-Cl(τ^*-Int(A))).*
(2) *τ^*-semiopen set if $A \subseteq \tau^*$-Cl(τ^*-Int(A)).*
(3) *τ^*-preopen is $A \subseteq \tau^*$-Int(τ^*-Cl(A)).*

The complement of a τ^*-α-open (resp. τ^*-preopen and τ^*-semiopen) set is τ^*-α-closed (resp. τ^*-preclosed and τ^*-semiclosed).

2. τ^*-b-closed Sets

Definition 2.1. *[3] Let $f : X \to Y$ be any function. A subset A of a space X is called τ^*-b-open if $A \subseteq \tau^*\text{-}Int(\tau^*\text{-}Cl(A)) \cup \tau^*\text{-}Cl(\tau^*\text{-}Int(A))$.*

The complement of a τ^*-b-open set is τ^*-b-closed.

Remark 2.1. *It can be shown that a subset B of X is τ^*-b-closed if and only if $\tau^*\text{-}Cl(\tau^*\text{-}Int(B)) \cap \tau^*\text{-}Int(\tau^*\text{-}Cl(B)) \subseteq B$.*

Remark 2.2. *The concepts of τ^*-b-closed and b-closed are independent.*

Example 1. *Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\delta = \{\phi, Y, \{3\}\}$. Let $f : X \to Y$ be a function such that $f(a) = 2, f(b) = 1$ and $f(c) = 3$, then $\tau^* = \{\phi, X, \{c\}\}$. If $A = \{b, c\}$, then A is b-closed but A is not τ^*-b-closed.*

Example 2. *Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\tau = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$ and $\delta = \{\phi, Y, \{1, 2\}\}$. Let $f : X \to Y$ be a function* *such that* $f(a) = 3, f(b) = 2$ *and* $f(c) = 1$*, then* $\tau^* = \{\phi, X, \{b, c\}\}$*. If* $A = \{c\}$*, then* A *is* τ^**-b-closed but* A *is not b-closed.*

Theorem 2.1. *An arbitrary union of* τ^**-b-open sets is* τ^**-b-open.*

Proof. Let $\{A_i : i \in I\}$ be a family of τ^*-b-open sets. Then for each i, $A_i \subseteq \tau^*$-$Int(\tau^*$-$Cl(A_i)) \cup \tau^*$-$Cl(\tau^*$-$Int(A_i))$ and so

$$\begin{aligned}
\cup_{i\in I} A_i &\subseteq \cup_{i\in I}[\tau^* - Int(\tau^* - Cl(A_i)) \cup \tau^* - Cl(\tau^* - Int(A_i))] \\
&\subseteq [\cup_{i\in I}(\tau^* - Int(\tau^* - Cl(A_i)))] \cup [\cup_{i\in I}(\tau^* - Cl(\tau^* - Int(A_i)))] \\
&\subseteq [\cup_{i\in I}\tau^* - Int(\tau^* - Cl(A_i))] \cup [\cup_{i\in I}\tau^* - Cl(\tau^* - Int(A_i))] \\
&\subseteq [\tau^* - Int(\cup_{i\in I}\tau^* - Cl(A_i))] \cup [\tau^* - Cl(\cup_{i\in I}\tau^* - Int(A_i))] \\
&\subseteq [\tau^* - Int(\tau^* - Cl(\cup_{i\in I}A_i))] \cup [\tau^* - Cl(\tau^* - Int(\cup_{i\in I}A_i))].
\end{aligned}$$

Thus, $\cup_{i\in I} A_i$ is a τ^*-b-open set. □

Remark 2.3.

(1) *An arbitrary intersection of* τ^**-b-closed sets is* τ^**-b-closed.*
(2) *The intersection of two* τ^**-b-open sets may not be* τ^**-b-open.*
(3) *The union of two* τ^**-b-closed sets may not be* τ^**-b-closed.*

Example 3. *Consider* $X = \{a, b, c\}$ *and* $Y = \{1, 2, 3\}$ *with* $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ *and* $\delta = \{\phi, Y, \{1\}, \{1, 2\}\}$*. Let* $f : X \to Y$ *be a function such that* $f(a) = f(b) = 1$ *and* $f(c) = 2$*, then* $\tau^* = \{\phi, X, \{a, b\}\}$*. Let* $A = \{a, c\}$ *and* $B = \{b, c\}$*, then* A *and* B *are* τ^**-b-open but* $A \cap B$ *is not* τ^**-b-open.*

Definition 2.2. *Let* $f : X \to Y$ *be any function. Then:*

(1) *The union of all* τ^**-b-open sets contained in* A *is called the* τ^**-b-interior of* A *and denoted by* τ^**-*$bInt(A)$*.*
(2) *The intersection of all* τ^**-b-closed sets containing* A *is called the* τ^**-b-closure of* A *and denoted by* τ^**-*$bCl(A)$*.*

Now, we state the following theorem without proof.

Theorem 2.2. *Let* $f : X \to Y$ *be any function. For any subsets* A, B *of* X*, we have the following:*

(1) $A \subseteq \tau^*$*-*$bCl(A)$ *and* τ^**-*$bInt(A) \subseteq A$*.*
(2) A *is* τ^**-b-open if and only if* $A = \tau^*$*-*$bInt(A)$*.*
(3) A *is* τ^**-b-closed if and only if* $A = \tau^*$*-*$bCl(A)$*.*

(4) *If* $A \subseteq B$*, then* τ^**-*$bInt(A) \subseteq \tau^*$*-*$bInt(B)$ *and* τ^**-*$bCl(A) \subseteq \tau^*$*-*$bCl(B)$.
(5) τ^*-$bInt(A) \cup \tau^*$-$bInt(B) \subseteq \tau^*$-$bInt(A \cup B)$.
(6) τ^*-$bInt(A \cap B) \subseteq \tau^*$-$bInt(A) \cap \tau^*$-$bInt(B)$.
(7) τ^*-$bCl(A) \cup \tau^*$-$bCl(B) \subseteq \tau^*$-$bCl(A \cup B)$.
(8) τ^*-$bCl(A \cap B) \subseteq \tau^*$-$bCl(A) \cap \tau^*$-$bCl(B)$.
(9) τ^*-$bInt(X \setminus A) = X \setminus \tau^*$-$bCl(A)$.
(10) τ^*-$bCl(X \setminus A) = X \setminus \tau^*$-$bInt(A)$.
(11) $X \setminus \tau^*$-$bCl(X \setminus A) = \tau^*$-$bInt(A)$.
(12) $X \setminus \tau^*$-$bInt(X \setminus A) = \tau^*$-$bCl(A)$.
(13) $x \in \tau^*$-$bInt(A)$ *if and only if there exists a* τ^**-b-open set* L *such that* $x \in L \subseteq A$.

Theorem 2.3. *Let* $f : X \to Y$ *be any function and* A *a subset of* X*. Then,* $x \in \tau^*$*-*$bCl(A)$ *if and only if for every* τ^**-b-open subset* L *of* X *containing* $x \in X$*,* $A \cap L \neq \phi$*.*

Proof. Let $x \in \tau^*$-$bCl(A)$ and suppose that $L \cap A = \phi$ for some τ^*-b-open set L which contains x. Then, $(X \setminus L)$ is τ^*-b-closed and $A \subseteq (X \setminus L)$, thus τ^*-$bCl(A) \subseteq (X \setminus L)$. But this implies that $x \in (X \setminus L)$, a contradiction. Thus, $L \cap A \neq \phi$.

Conversely, let $A \subseteq X$ and $x \in X$ such that for each τ^*-b-open set L_1 which contains x, $L_1 \cap A \neq \phi$. If $x \notin \tau^*$-$bCl(A)$, there is a τ^*-b-closed set F such that $A \subseteq F$ and $x \notin F$. Then, $(X \setminus F)$ is a τ^*-b-open set with $x \in (X \setminus F)$, and thus $(X \setminus F) \cap A \neq \phi$, which is a contradiction. □

Theorem 2.4. *Let* $f : X \to Y$ *be any function and* $V \subseteq X$*. Then,* V *is* τ^**-b-open if and only if for each* $s \in V$*, there exists a* τ^**-b-open set* K *such that* $s \in K \subseteq V$*.*

Proof. It is obvious. □

Theorem 2.5. *Let* $f : X \to Y$ *be any function.*

(1) *Then, either* $\{x\}$ *is* τ^**-preopen or* τ^**-Int(*τ^**-Cl({x}))=* ϕ *for any* $x \in X$*.*
(2) *For each point* $x \in X$*,* $\{x\}$ *is* τ^**-preopen or* τ^**-preclosed.*

Proof.

(1) If τ^*-Int(τ^*-$Cl(\{x\})) \neq \phi$, then $x \in \tau^*$-Int(τ^*-Cl($\{x\}$)) and so $\{x\}$ is τ^*-preopen. If $\{x\}$ is not τ^*-preopen, then $x \notin \tau^*$-Int(τ^*-$Cl(\{x\}))$ and hence τ^*-Int(τ^*-Cl({x}))= ϕ.

(2) Obvious.

$\square$

Theorem 2.6. *Let $f : X \to Y$ be any function. A subset A of X is τ^*-b-closed if and only if τ^*-$bCl(A) \subseteq U$, whenever $A \subseteq U$ and U is τ^*-b-open.*

Proof. Let A be τ^*-b-closed, then $A = \tau^*$-$bCl(A)$ and for all τ^*-b-open set U with $A \subseteq U$, we have τ^*-$bCl(A) \subseteq U$.

Conversely, by Theorem 2.2 (1), $A \subseteq \tau^*$-$bCl(A)$, it is enough to prove that τ^*-$bCl(A) \subseteq A$. Let $x \in \tau^*$-$bCl(A)$. By Thorem 2.5, either $\{x\}$ is τ^*-preopen or τ^*-Int(τ^*-$Cl(\{x\})) = \phi$. Then, we have the two cases:

(1) If $\{x\}$ is τ^*-preopen, then it is τ^*-b-open and by Theorem 2.3, $\{x\} \cap A \neq \phi$ and hence $x \in A$.
(2) If τ^*-Int(τ^*-$Cl(\{x\})) = \phi$, then by Theorem 2.5 (2), $X \setminus \{x\}$ is τ^*-preopen and so it τ^*-b-open. If $x \notin A$, then $A \subseteq X \setminus \{x\}$. By assumption, τ^*-$bCl(A) \subseteq X \setminus \{x\}$ and so $x \notin \tau^*$-$bCl(A)$ it is contradiction and hence $x \in A$.

From (1) and (2) we get τ^*-$bCl(A) \subseteq A$. Therefore, τ^*-$bCl(A) = A$ and thus A is τ^*-b-closed. $\square$

Theorem 2.7. *Every τ^*-closed set is τ^*-b-closed.*

Proof. Let A be a τ^*-closed set in X such that $A \subseteq U$, where U is τ^*-b-open. Since A is τ^*-closed, and τ^*-$bCl(A) \subseteq \tau^*$-$Cl(A) = A$. Therefore τ^*-$bCl(A) \subseteq U$ and hence A is τ^*-b-closed. $\square$

Remark 2.4. *The converse of the above theorem need not be true in general as it is shown below.*

Example 4. *Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\delta = \{\phi, Y, \{2\}, \{2, 3\}\}$. Let $f : X \to Y$ be a function such that $f(a) = f(c) = 2$, and $f(b) = 3$, then $\tau^* = \{\phi, X, \{a, c\}\}$. If $A = \{a\}$, then A is τ^*-b-closed but not τ^*-closed.*

Remark 2.5. *Let $f : X \to Y$ be any function. Then:*

(1) *Every τ^*-α-closed set is τ^*-b-closed.*
(2) *Every τ^*-preclosed set is τ^*-b-closed.*
(3) *Every τ^*-semiclosed set is τ^*-b-closed.*

Remark 2.6. *The converse of the above remark need not be true in general as it is shown below.*

Example 5. *Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $\delta = \{\phi, Y, \{1\}, \{1, 2\}\}$. Let $f : X \to Y$ be a function such that $f(a) = f(b) = 1$ and $f(c) = 2$, then $\tau^* = \{\phi, X, \{a, b\}\}$. If $A = \{a\}$, then A is τ^*-b-closed but neither τ^*-α-closed nor τ^*-semiclosed.*

Example 6. *Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ with $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$ and $\delta = \{\phi, Y, \{2\}, \{3\}\{2, 3\}\}$. Let $f : X \to Y$ be a function such that $f(a) = 1$, $f(b) = 2$, and $f(3) = 3$, then $\tau^* = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$. If $A = \{b\}$, then A is τ^*-b-closed but not τ^*-preclosed.*

Theorem 2.8. *A subset $A \subseteq X$ is τ^*-b-open if and only if $F \subseteq \tau^*$-bInt(A) whenever F is τ^*-b-closed and $F \subseteq A$.*

Proof. Let A be τ^*-b-open and suppose $F \subseteq A$ where F is τ^*-b-closed. Then $X \setminus A$ is τ^*-b-closed and by Theorem 2.6, τ^*-$bCl(X \setminus A) \subseteq X \setminus F$ and $X \setminus \tau^*$-$bInt(A) \subseteq X \setminus F$. Thus $F \subseteq \tau^*$-$bInt(A)$.

Conversely, let $X \setminus A \subseteq G$ where G is τ^*-b-open, then $X \setminus G$ is τ^*-b-closed and $X \setminus G \subseteq A$ and hence $X \setminus G \subseteq \tau^*$-$bInt(A)$. Thus τ^*-$bCl(X \setminus A) \subseteq G$. Therefore, $X \setminus A$ is τ^*-b-closed and so A is a τ^*-b-open. □

3. τ^*-B-CONTINUOUS

Definition 3.1. *A map $f : X \to Y$ is called τ^*-continuous (resp. τ^*-b-continuous) if the inverse image of every closed set in Y is τ^*-closed (resp. τ^*-b-closed) in X.*

Theorem 3.1. *If a map $f : X \to Y$ is τ^*-continuous, then it is τ^*-b-continuous.*

Proof. Let f be τ^*-continuous and M be any closed set in Y. Then the inverse image $f^{-1}(M)$ is τ^*-closed in X. Since every τ^*-closed is τ^*-b-closed, so $f^{-1}(M)$ is τ^*-b-closed in X. Therefore, f is τ^*-b-continuous. □

Remark 3.1. *The converse of the above theorem need not be true in general as it is shown below.*

Example 7. *Consider $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{a, b\}\}$ and $\delta = \{\phi, Y, \{c\}, \{a, b\}\}$. Let $f : X \to Y$ be a function such that $f(a) = c$, $f(b) = b$ and $f(c) = a$,* *then $\tau^* = \{\phi, X\}$, and f is τ^*-b-continuous but f is not τ^*- continuous, becuse $f^{-1}(\{c\}) = \{a\}$ is not τ^*-closed.*

Theorem 3.2. *For a map $f : X \to Y$ the following are equivalent:*

(1) *f is τ^*-b-continuous.*
(2) *The inverse image of each open set in Y is τ^*-b-open in X.*

Proof. Let f be τ^*-b-continuous and G be any open set in Y. Then $Y \setminus G$ is closed in Y. Since f is τ^*-b-continuous, then $f^{-1}(Y \setminus G)$ is τ^*-b-closed in X. But $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$, thus $X \setminus f^{-1}(G)$ is τ^*-b-closed in X, and so $f^{-1}(G)$ is τ^*-b-open in X.

Conversely, assume that the inverse image of each open set in Y is τ^*-b-open in X. Let F be any closed set in Y, then $Y \setminus F$ is open in Y. By assumption, $f^{-1}(Y \setminus F)$ is τ^*-b-open in X. But $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$, then $X \setminus f^{-1}(F)$ is τ^*-b-open in X and so $f^{-1}(F)$ is τ^*-closed in X. Thus, f is τ^*-b-continuous. □

Remark 3.2. *A map $f : X \to Y$ is τ^*-continuous if and only if the inverse image of each open set in Y is τ^*-open in X.*

Theorem 3.3. *If a map $f : X \to Y$ is τ^*-continuous, then it is continuous.*

Proof. Let f be τ^*-continuous and K be any closed set in Y. Then, $f^{-1}(K)$ is τ^*-closed in X. Since every τ^*-closed is closed, so $f^{-1}(K)$ is closed in X. Thus, f is continuous. □

Remark 3.3. *The converse of the above theorem is true because if H is any open set in Y and f is continuous, then $f^{-1}(H)$ is open in X and by Definition 1.1, $f^{-1}(H)$ must be in τ^*.*

Theorem 3.4. *For a map $f : X \to Y$ the following are equivalent:*

(1) *f is τ^*-b-continuous.*
(2) *$f(\tau^*\text{-}bCl(B)) \subseteq Cl(f(B))$, for every subset B of X.*
(3) *$\tau^*\text{-}bCl(f^{-1}(A)) \subseteq f^{-1}(Cl(A))$, for each subset A of Y.*

Proof.

$(1) \Rightarrow (2)$. Let B be any subset of X. Then $f(B) \subseteq Cl(f(B))$ and $Clf(B)$ is closed in Y. Hence $B \subseteq f^{-1}(Clf(B))$ and since f is τ^*-b-continuous, then $f^{-1}(Clf(B))$ is τ^*-b-closed set in X. Therefore, $\tau^*\text{-}bCl(B) \subseteq f^{-1}(Cl(f(B)))$. Hence $f(\tau^*\text{-}bCl(B)) \subseteq Cl(f(B))$.

$(2) \Rightarrow (3)$. Let A be any subset of Y, then $f^{-1}(A)$ is a subset of X. By (2), we have $f(\tau^*\text{-}bCl(f^{-1}(A))) \subseteq Cl(f(f^{-1}(A))) \subseteq Cl(A)$. It follows that τ^*-$bCl(f^{-1}(A)) \subseteq f^{-1}(Cl(A))$.

$(3) \Rightarrow (1)$. Let A be any closed set in Y. By (3), we have $\tau^*\text{-}bCl(f^{-1}(A)) \subseteq f^{-1}(Cl(A)) = f^{-1}(A)$, but $f^{-1}(A) \subseteq \tau^*\text{-}bCl(f^{-1}(A))$. Therefore, $f^{-1}(A)$ is τ^*-b-closed in X. Hence, f is τ^*-b-continuous. □

Theorem 3.5. *If $f : X \to Y$ and $g : Y \to Z$ be any two functions, then $g \circ f : X \to Z$ is τ^*-b-continuous if g is continuous and f is τ^*-b-continuous.*

Proof. Let A be any closed set in Z. Since g is continuous function, then $g^{-1}(A)$ is closed in Y, and also since f is τ^*-b-continuous, then $f^{-1}(g^{-1}(A))$ is τ^*-b-closed in X. Hence $g \circ f$ is τ^*-b-continuous. □

Remark 3.4. *If $f : X \to Y$ and $g : Y \to Z$ be any two functions, then $g \circ f : X \to Z$ is τ^*-b-continuous if g is τ^*-continuous and f is τ^*-b-continuous.*

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