Advances in Mathematics: Scientific Journal **10** (2021), no.4, 1879–1898
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.4

NUMERICAL QUENCHING FOR A NON-NEWTONIAN FILTRATION EQUATION WITH SINGULAR BOUNDARY FLUX

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ABSTRACT. This paper concerns the study of the numerical approximation for the following initial-boundary value problem.

$$\begin{cases} u_t = \left(|u_x|^{p-2}u_x\right)_x + (1-u)^{-h}, & 0<x<1,\ t>0, \\ u_x(0,t)=0, \quad u_x(1,t) = -u^{-q}(1,t), & t>0, \\ u(x,0)=u_0(x)>0, \quad 0\le x\le 1, \end{cases}$$

where $p \geq 2$, $h > 0$, $q > 0$, $u_0 : [0,1] \to (0,1)$ and satisfies compatibility conditions. We find some conditions under which the solution of a semidiscrete form of above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

1. INTRODUCTION

In this paper, we consider the following boundary value problem

$$u_t = (|u_x|^{p-2}u_x)_x + (1-u)^{-h}, \quad 0<x<1, t>0, \tag{1.1}$$

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2020 *Mathematics Subject Classification.* 35B50, 35K55, 35K20, 65M06.

Key words and phrases. Non-Newtonian filtration equations, semidiscretization, discretization, semidiscrete quenching time, convergence.

Submitted: 01.03.2021; *Accepted:* 22.03.2021; *Published:* 03.04.2021.

$$u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{1.3}$$

where $p \ge 2$, $h > 0$, $q > 0$, $u_0 : [0,1] \to (0,1)$ and satisfies some compatibility conditions such that $u_0'(0) = 0$, $u_0'(1) = -u_0^{-q}(1)$, $u_0'(x) \le 0$ and $(|u_0'(x)|^{p-2}u_0'(x))' + (1-u_0(x))^{-h} \ge 0$, $0 \le x \le 1$.

The quenching behavior describes the phenomenon that there exists a finite time T_q such that the solution of the problem (1.1)–(1.3) satisfied the following definition

Definition 1.1. *We say that the classical solution u of the problem* (1.1)–(1.3) *quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0,T_q)$ but*

$$\lim_{t \to T_q} \|u(.,t)\|_\infty = 1,$$

where $\|u(.,t)\|_\infty = \max_{0 \le x \le 1} |u(x,t)|$. The time T_q is called the quenching time of the solution u.

The problem (1.1)–(1.3) may be rewritten in the following form

$$u_t = (p-1)|u_x|^{p-2}u_{xx} + (1-u)^{-h}, \quad 0 < x < 1, t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{1.6}$$

where $p \ge 2$, $h > 0$, $q > 0$, $u_0 : [0,1] \to (0,1)$ and satisfies some compatibility conditions such that $u_0'(0) = 0$, $u_0'(1) = -u_0^{-q}(1)$, $u_0'(x) \le 0$ and $(p-1)|u_0'(x)|^{p-2}u_0''(x) + (1-u_0(x))^{-h} \ge 0$, $0 \le x \le 1$.

Equation (1.1) is known as the classical non-Newtonian filtration equation that incorporates the effects of nonlinear reaction source and nonlinear boundary outflux. Kawarada [8] first studied the quenching phenomenon for semilinear heart equation $u_t = u_{xx} + (1-u)^{-1}$. He obtained the results that, when the solution reaches level $u = 1$, the reaction term and the time derivative blow up. Since then, the theoretical study of quenching phenomena for semilinear parabolic equations have been the subject of investigations of many researchers(see for examples [4–6,8,9,15–21] and the references therein). Concerning problem

(1.1), Ying Yang proves under certain conditions that quenching occurs in finite time and he shows that the only quenching point is $x = 0$. He has also established the bounds for quenching rate and the lower bound for the quenching time.

In this paper, we are interested in the numerical study of the phenomenon of quenching using a semidiscrete form of (1.4)–(1.6). We give some conditions under which the solution of the semidiscrete form of (1.4)–(1.6) quenches in finite time and estimate its semidiscrete quinching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. This paper is organised as follows. In the next section, we give some properties concerning our semidiscrete sheme. In section 3, under some conditions, we prove that the solution of a semidiscrete form of (1.4)–(1.6) quenches in a finite time and estimate its semidiscrete quenching time. In section 4, we show that the quenching time converges to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the semidiscrete scheme. Let $I \geq 3$ be a positive integer and let $s = 1/I$. Define the grid $x_i = is$, $0 \leq i \leq I$. Approximate the solution u of (1.4)–(1.6) by the solution $U_s = (U_0, U_1, \dots, U_I)^T$ and approximate the initial condition u_0 of (1.4)–(1.6) by the initial condition $\varphi_s = (\varphi_0, \varphi_1, \dots, \varphi_I)^T$ of the following semidiscrete equations

$$\begin{aligned} \frac{dU_i(t)}{dt} &= (p-1)|\delta^0 U_i(t)|^{p-2}\delta^2 U_i(t) \\ &+ (1-U_i(t))^{-h},\ 0 \leq i \leq I-1,\ t \in [0, T_q^s), \end{aligned} \tag{2.1}$$

$$\begin{aligned} \frac{dU_I(t)}{dt} &= (p-1)|U_I^{-q}(t)|^{p-2}\delta_*^2 U_I(t) \\ &+ (1-U_I(t))^{-h},\ t \in [0, T_q^s), \end{aligned} \tag{2.2}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \leq i \leq I, \tag{2.3}$$

where

$$\begin{aligned}
\delta^2 U_i(t) &= \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{s^2}, \quad 1 \le i \le I-1, \\
\delta^2 U_0(t) &= \frac{2U_1(t) - 2U_0(t)}{s^2}, \quad \delta_*^2 U_I(t) = \delta^2 U_I(t) - \frac{2}{s} U_I^{-q}(t), \\
\delta^2 U_I(t) &= \frac{2U_{I-1}(t) - 2U_I(t)}{s^2}, \quad \delta^0 U_0(t) = 0,\ \delta^0 U_i(t) = \frac{U_{i+1}(t) - U_{i-1}(t)}{2s},
\end{aligned}$$

$1 \le i \le I-1$, $0 < \varphi_s < 1$, $\varphi_{i+1} < \varphi_i$, $0 \le i \le I-1$.. Here, $[0, T_q^s)$ is the maximal time interval on which $\|U_s(t)\|_\infty < 1$, where $\|U_s(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|$.

When the time T_q^s is finite, then we say that the solution $U_s(t)$ of (2.1)–(2.3) quenches in a finite time, and the time T_q^s is called the quenching time of the solution $U_s(t)$.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let* $\alpha_s(t), a_s(t) \in C^0([0, T_q^s), \mathbb{R}^{I+1})$ *and let* $V_s(t) \in C^1([0, T_q^s), \mathbb{R}^{I+1})$ *with* $\alpha_s(t) \ge 0$ *such that*

$$\frac{d}{dt} V_i(t) - \alpha_i(t) \delta^2 V_i(t) + a_i(t) V_i(t) \ge 0,\ 0 \le i \le I,\ t \in [0, T_q^s), \tag{2.4}$$

$$V_i(0) \ge 0, \quad 0 \le i \le I. \tag{2.5}$$

Then we have

$$V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in [0, T_q^s). \tag{2.6}$$

Proof. Let T_0 be any quantity satisfying the inequality $T_0 < T_q^s$ and define the vector $Z_s(t) = e^{\lambda t} V_s(t)$ where λ is such that

$$a_i(t) - \lambda > 0 \quad \text{for} \quad 0 \le i \le I, \quad t \in [0, T_0].$$

Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. Since, for $i \in \{0, \dots, I\}$, $Z_i(t)$ is a continuous function on the compact $[0, T_0]$, there exists $i_0 \in \{0, \dots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I, \tag{2.7}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{s^2} \ge 0,\ 1 \le i_0 \le I-1, \tag{2.8}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{s^2} \geq 0 \quad if \quad i_0 = 0, \tag{2.9}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{s^2} \geq 0 \quad if \quad i_0 = I. \tag{2.10}$$

From (2.4), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \alpha_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{2.11}$$

It follows from (2.7)–(2.11) that

$$(a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0, \tag{2.12}$$

which implies that $Z_{i_0}(t_0) \geq 0$ because $a_{i_0}(t_0) - \lambda > 0$. We deduce that $V_s(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete. □

Another form of the maximum principle for semidiscrete equations is the comparison lemma below

Lemma 2.2. *Let* $V_s(t)$, $W_s(t) \in C^1([0, T_q^s), \mathbb{R}^{I+1})$ *and* $f, b_s(t) \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ *with* $b_s(t) \geq 0$ *such that for* $t \in [0, T_q^s)$

$$\frac{dV_i(t)}{dt} - b_i(t)\delta^2 V_i(t) + f(V_i(t)) < \frac{dW_i(t)}{dt} - b_i(t)\delta^2 W_i(t) + f(W_i(t)), \tag{2.13}$$

$0 \leq i \leq I$,

$$V_i(0) < W_i(0), \quad 0 \leq i \leq I. \tag{2.14}$$

Then we have

$$V_i(t) < W_i(t), \quad 0 \leq i \leq I, t \in [0, T_q^s).$$

Proof. Define the vector $Z_s(t) = W_s(t) - V_s(t)$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \leq i \leq I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. We remark that

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \ 0 \leq i_0 \leq I, \\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{s^2} \geq 0, \ 1 \leq i_0 \leq I - 1, \\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_1(t_0) - 2Z_0(t_0)}{s^2} \geq 0 \quad if \quad i_0 = 0, \\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{s^2} \geq 0 \quad if \quad i_0 = I.
\end{aligned}$$

Therefore, we have

$$\frac{dZ_{i_0}(t_0)}{dt} - b_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + f(Wi_0(t_0)) - f(V_{i_0}(t_0)) \leq 0,$$

which contradicts the first strict inequality of the lemma and this ends the proof ☐

Lemma 2.3. *Let* $V_s(t)$*,* $W_s(t) \in C^1([0, T_q^s), \mathbb{R}^{I+1})$ *and* $f, b_s(t) \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ *with* $b_s(t) \geq 0$ *such that* $t \in [0, T_q^s)$

$$\frac{dV_i(t)}{dt} - b_i(t)\delta^2 V_i(t) + f(V_i(t)) \leq \frac{dW_i(t)}{dt} - b_i(t)\delta^2 W_i(t) + f(W_i(t)), \tag{2.15}$$

$0 \leq i \leq I$,

$$V_i(0) \leq W_i(0), \quad 0 \leq i \leq I. \tag{2.16}$$

Then we have

$$V_i(t) \leq W_i(t), \quad 0 \leq i \leq I, t \in [0, T_q^s).$$

The next lemma shows that when i is between 0 and I, then $U_i(t)$ is positive where $U_s(t)$ is the solution of the semidiscrete problem.

Lemma 2.4. *Let* U_s *be the solution of the problem* (2.1)–(2.3)*. Then we have*

$$U_i(t) > 0 \quad for \quad 0 \leq i \leq I, \quad t \in [0, T_q^s). \tag{2.17}$$

Proof. Assume that there exists a time $t_0 \in [0, T_q^s)$ such that $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. We observe that

$$\begin{aligned}
\frac{dU_{i_0}(t_0)}{dt} &= \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \leq 0, \ 0 \leq i_0 \leq I, \\
\delta^2 U_{i_0}(t_0) &= \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{s^2} > 0, \ 1 \leq i_0 \leq I-1, \\
\delta^2 U_{i_0}(t_0) &= \frac{2U_1(t_0) - 2U_0(t_0)}{s^2} > 0 \quad if \quad i_0 = 0, \\
\delta^2 U_{i_0}(t_0) &= \frac{2U_{I-1}(t_0) - 2U_I(t_0)}{s^2} > 0 \quad if \quad i_0 = I,
\end{aligned}$$

which implies that

$$\frac{dU_{i_0}(t_0)}{dt} - (p-1)|\delta^0 U_{i_0}(t_0)|^{p-2}\delta^2 U_{i_0}(t_0) - (1 - U_{i_0}(t_0))^{-h} < 0, \ 0 \leq i_0 \leq I-1,$$

$$\frac{dU_I(t_0)}{dt} - (p-1)|U_I^{-q}(t_0)|^{p-2}\delta^2 U_I(t_0) + \frac{2(p-1)}{s}|U_I^{-q}(t_0)|^{p-2}U_I^{-q}(t_0) \\ -(1-U_I(t_0))^{-h} < 0,$$

But these inequalities contradict (2.1)–(2.2) and we are so proved that $U_i(t) > 0$, $0 \leq i \leq I$, $t \in [0, T_q^s)$. $\square$

Lemma 2.5. *Let* U_s *be the solution of the problem* (2.1)–(2.3). *Then we have*

$$U_{i+1}(t) < U_i(t) \quad for \quad 0 \leq i \leq I-1, \quad t \in [0, T_q^s). \tag{2.18}$$

Proof. Introduce the vector $Z_s(t)$ defined as follows $Z_i(t) = U_{i+1}(t) - U_i(t)$ for $0 \leq i \leq I-1$. Let t_0 be the first t>0 such that $Z_i(t) < 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I-1\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the above equality. It follows that

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k\to 0}\frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \geq 0, \ 0 \leq i_0 \leq I-1,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{s^2} < 0, \ 1 \leq i_0 \leq I-2,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_1(t_0) - 3Z_0(t_0)}{s^2} < 0 \quad if \quad i_0 = 0,\\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{s^2} < 0 \quad if \quad i_0 = I-1,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= (p-1)|\delta^0 U_{i_0}(t_0)|^{p-2}\delta^2 Z_{i_0}(t_0)\\
&\quad - (p-1)(p-2)|\delta^0 U_{i_0}(t_0)|^{p-3}\delta^2 U_{i_0+1}(t_0)\delta^0 Z_{i_0}(t_0)\\
&\quad + h(1-\theta_{i_0}(t_0))^{-h-1}Z_{i_0}(t_0) < 0 \quad 0 \leq i_0 \leq I-2\\
\frac{dZ_{I-1}(t_0)}{dt} &= (p-1)|U_I^{-q}(t_0)|^{p-2}\delta_*^2 Z_{I-1}(t_0)\\
&\quad - q(p-1)(p-2)U_I^{-q(p-2)-1}(t_0)\delta^0 U_I^{-q(p-2)}(t_0)\\
&\quad + h(1-\xi_I(t_0))^{-h-1}Z_{I-1}(t_0) < 0
\end{aligned}$$

where $\theta_{i_0}(t_0) \in (U_{i_0+1}(t_0), U_{i_0}(t_0))$ and $\xi_I(t_0) \in (U_I(t_0), U_{I-1}(t_0))$.

Therefore, we have a contradiction because of (2.1)–(2.2). This ends the proof. $\square$

Lemma 2.6. *Let U_s be a solution of the problem* (2.1)–(2.3) *and the initial data at* (2.3) *verifies some compatibility conditions. Then,* $\frac{dU_i(t)}{dt} > 0$ *for* $0 \le i \le I$*,* $t \in (0, T_q^s)$.

Proof. Consider the vector $Z_s(t)$ such that $Z_i(t) = \frac{dU_i(t)}{dt}$ for $0 \le i \le I$ and $t \in (0, T_q^s)$. Let t_0 be the first $t \in (0, T_q^s)$ such that $Z_i(t) > 0$ for $t \in (0, t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that:

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, 0 \le i_0 \le I \\
\delta^2 Z_{i_0}(t_0) &= \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{s^2} > 0, 1 \le i_0 \le I - 1 \\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_1(t_0) - 2Z_0(t_0)}{s^2} > 0, i_0 = 0 \\
\delta^2 Z_{i_0}(t_0) &= \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{s^2} > 0, i_0 = I.
\end{aligned}$$

Moreover, by a straightforward computation, we get

$$\begin{aligned}
\frac{dZ_{i_0}(t_0)}{dt} &= (p-1)(p-2)\delta^0 U_{i_0}(t_0)|\delta^0 U_{i_0}(t_0)|^{p-2}\delta^0 Z_{i_0}(t_0)\delta^2 U_{i_0}(t_0) \\
&\quad + (p-1)|\delta^0 U_{i_0}(t_0)|^{p-2}\delta^2 Z_{i_0}(t_0) \\
&\quad + h(1 - U_{i_0}(t_0))^{-h-1} Z_{i_0}(t_0) < 0, 0 \le i_0 \le I - 1 \\
\frac{dZ_I(t_0)}{dt} &= -q(p-1)(p-2)U_I^{-q(p-2)-1} Z_I(t_0)\delta^2 U_I(t_0) + (p-1)U_I^{-q(p-2)}\delta^2 Z_I(t_0) \\
&\quad + \frac{2q(p-1)^2}{s} U_I^{-q(p-1)-1}(t_0) Z_I(t_0) + h(1 - U_I(t_0))^{-h-1} Z_I(T_0) < 0.
\end{aligned}$$

But these inequalities contradict (2.1)–(2.2) and this proof is complete. □

Lemma 2.7. *Let $U_s \in \mathcal{R}^{I+1}$ be such that $\|U_s\|_\infty < 1$ and let h be a positive constant. Then we have*

$$\delta^2(1 - U_i)^{-h} \ge h(1 - U_i)^{-h-1}\delta^2 U_i, \ 0 \le i \le I.$$

Proof. Let us introduce function $f(x) = (1-x)^{-h}$. We observe that f is a convex function for $0 \leq x < 1$. Using Taylor's expansion, we get

$$\begin{aligned}
f(U_1) &= f(U_0) + (U_1 - U_0)f'(U_0) + \frac{(U_1 - U_0)^2}{2} f''(\eta_0) \\
f(U_{I-1}) &= f(U_I) + (U_{I-1} - U_I)f'(U_I) + \frac{(U_{I-1} - U_I)^2}{2} f''(\eta_I) \\
f(U_{i+1}) &= f(U_i) + (U_{i+1} - U_i)f'(U_i) + \frac{(U_{i+1} - U_i)^2}{2} f''(\theta_i),\ 1 \leq i \leq I-1 \\
f(U_{i-1}) &= f(U_i) + (U_{i-1} - U_i)f'(U_i) + \frac{(U_{i-1} - U_i)^2}{2} f''(\eta_i),\ 1 \leq i \leq I-1,
\end{aligned}$$

where θ_i is an intermediate between U_{i+1} and U_i and η_i the one between U_i and U_{i-1}. The first and the second equalities imply that

$$\begin{aligned}
\delta^2 f(U_0) &= f'(U_0)\delta^2 U_0 + \frac{(U_1 - U_0)^2}{s^2} f''(\eta_0) \\
\delta^2 f(U_I) &= f'(U_I)\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{s^2} f''(\eta_I)
\end{aligned}$$

Combining the third and the last equalities, we see that

$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2s^2} f''(\theta_i) + \frac{(U_{i-1} - U_i)^2}{2s^2} f''(\eta_i) \quad 0 \leq i \leq I-1,$$

Use the fact that $f'(x) = h(1-x)^{-h-1}$, $f''(x) = h(h+1)(1-x)^{-h-2}$ and $\|U_s\|_\infty < 1$ to complete the proof. $\square$

3. Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_s of (2.1)–(2.3) quenches in a finite time and estimate its semidiscrete quenching time.

Theorem 3.1. *Let U_s be the solution of* (2.1)–(2.3) *and assume that there exists a constant $A > 0$ such that the initial data at* (2.3) *satisfies*

$$(p-1)|\delta^0\varphi_i|^{p-2}\delta^2\varphi_i + (1-\varphi_i)^{-h} \geq A(1-\varphi_i)^{-h}, \quad 0 \leq i \leq I-1 \tag{3.1}$$

$$(p-1)|\varphi_I^{-q}|^{p-2}\delta^2\varphi_I - \frac{2(p-1)}{s}\varphi_I^{-q(p-1)} + (1-\varphi_I)^{-h} \geq \mathcal{A}(1-\varphi_I)^{-h}. \tag{3.2}$$

Then, there exists a finite time T_q^s such that U_s quenches in this time and we have the following estimate

$$T_q^s \leq \frac{(1-\|\varphi_s\|_\infty)^{h+1}}{A(h+1)}. \tag{3.3}$$

Proof. Let $[0,T_q^s)$ be the maximal time interval on which $\|U_s\|_\infty < 1$. Our objectif is to show that T_q^s is finite and satisfies the inequality (3.3). Introduce the function $J_s(t)$ such that

$$J_i(t) = \frac{dU_i(t)}{dt} - A(1-U_i(t))^{-h}, \quad 0 \leq i \leq I.$$

A straightforward computation gives

$$\frac{dJ_i(t)}{dt} - (p-1)|\delta^0 U_i(t)|^{p-2}\delta^2 J_i(t) = \frac{d^2U_i(t)}{dt^2} - hA(1-U_i(t))^{-h-1}\frac{dU_i(t)}{dt}$$
$$-(p-1)|\delta^0 U_i(t)|^{p-2}\delta^2(\frac{dU_i(t)}{dt}) + A(p-1)|\delta^0 U_i(t)|^{p-2}\delta^2(1-U_i(t))^{-h}.$$

From Lemma 2.7, we have $\delta^2(1-U_i(t))^{-h} \geq h(1-U_i(t))^{-h-1}\delta^2 U_i(t)$, which implies that

$$\begin{aligned}
&\frac{dJ_i(t)}{dt} - (p-1)|\delta^0 U_i(t)|^{p-2}\delta^2 J_i(t)\\
\geq\ & \frac{d^2U_i(t)}{dt^2} - hA(1-U_i(t))^{-h-1}\frac{dU_i(t)}{dt}\\
-\ & (p-1)|\delta^0 U_i(t)|^{p-2}\delta^2(\frac{dU_i(t)}{dt})\\
+\ & hA(p-1)|\delta^0 U_i(t)|^{p-2}(1-U_i(t))^{-h-1}\delta^2 U_i(t).
\end{aligned}$$

Using (2.1), we arrive at

$$\frac{dJ_i(t)}{dt} - (p-1)|\delta^0 U_i(t)|^{p-2}\delta^2 J_i(t) \geq h(1-U_i(t))^{-h-1}J_i(t),$$

$0 \leq i \leq I-1,\ t \in [0,T_q^s)$,

$$\begin{aligned}
&\frac{dJ_I(t)}{dt} - (p-1)|U_I^{-q}|^{p-2}\delta^2 J_I(t) - h(1-U_I)^{-h-1}J_I(t)\\
\geq\ & \frac{2q(p-1)^2}{s}U_I^{-q(p-1)-1}g(U_I(t)),
\end{aligned}$$

where $g(U_I(t)) = \dfrac{dU_I(t)}{dt} + \dfrac{Ah}{q(p-1)s}U_I(t)(1-U_I(t))^{-h-1}$. It is not hard to see that

$$\frac{dJ_i(t)}{dt} - (p-1)|\delta^0 U_i(t)|^{p-2}\delta^2 J_i(t) - h(1-U_i(t))^{-h-1}J_i(t) \geq 0,$$

$0 \leq i \leq I-1,\ t \in [0, T_q^s)$,

$$\frac{dJ_I(t)}{dt} - (p-1)|U_I^{-q}|^{p-2}\delta^2 J_I(t) - h(1-U_I)^{-h-1}J_I(t) \geq 0, \quad t \in [0, T_q^s).$$

From (3.1) and (3.2), we see that $J_s(0) \geq 0$. We deduce from Lemma 2.1 that $J_s(t) \geq 0$, for $t \in [0, T_q^s)$, which implies that

$$\frac{dU_i(t)}{dt} \geq A(1-U_i(t))^{-h}, \quad 0 \leq i \leq I, \quad t \in [0, T_q^s).$$

These estimate may be rewritten in the following form

$$(1-U_i(t))^h dU_i(t) \geq Adt, \quad 0 \leq i \leq I, \quad t \in [0, T_q^s).$$

Integrating the above inequalities over the interval (t, T_q^s), we get

$$T_q^s - t \leq \frac{(1-U_i(t))^{h+1}}{A(h+1)}, \quad 0 \leq i \leq I, \quad t \in [0, T_q^s). \tag{3.4}$$

From Lemma 2.5, we have $\|\varphi_s\|_\infty = U_0(0)$, taking $t = 0$ and $i = 0$ in (3.4), we obtain the desired result

$$T_q^s \leq \frac{(1-\|\varphi_s\|_\infty)^{h+1}}{A(h+1)}.$$

□

Remark 3.1. *The inequalities* (3.4) *imply that*

$$T_q^s - t_0 \leq \frac{(1-U_s(t_0))^{h+1}}{A(h+1)}, \quad t_0 \in [0, T_q^s).$$

This remark is crucial to prove the convergence of the semidiscrete quenching time.

4. Convergence of the semidiscrete quenching time

In this section, under some assumptions, we show that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. We denote by

$$u_s(t) = (u(x_0,t), u(x_1,t), \dots, u(x_I,t))^T \text{ and } \|U_s(t)\|_\infty = \max_{0\le i\le I} |U_i(t)|.$$

We first prove the following theorem on the convergence of the semidiscrete scheme which will then allow us to prove the main theorem of this section, namely the convergence of the semidiscrete quenching time.

Theorem 4.1. *Assume that the problem* (1.4)–(1.6) *has a solution* $u \in C^{4,1}([0,1]\times[0,T])$ *such that* $\sup_{t\in[0,T]} \|u(.,t)\|_\infty = \lambda < 1$. *Suppose that the initial data at* (2.3) *satisfies*

$$\|\varphi_s - u_s(0)\|_\infty = o(1) \quad as \quad s \to 0, \tag{4.1}$$

Then, for s sufficiently small, the problem (2.1)–(2.3) *has a unique solution* $U_s \in C^1([0,T], \mathbb{R}^{I+1})$ *such that*

$$\max_{0\le t\le T} \|U_s(t) - u_s(t)\|_\infty = O(\|\varphi_s - u_s(0)\|_\infty + s) \quad as \quad s \to 0, \tag{4.2}$$

$where\ T < \min\{T_q, T_q^s\}$.

Proof. Let $\rho > 0$ be such that $\rho + \lambda < 1$. Let $M > 0$ such that

$$h(1-\rho-\lambda)^{-h-1} < M. \tag{4.3}$$

The problem (2.1)–(2.3) has for each s, a unique solution $U_s \in C^1([0,T], \mathbb{R}^{I+1})$. Let $t(s)$ the greatest value of $t > 0$ such that

$$\|U_s(t) - u_s(t)\|_\infty < \rho \quad for \quad t \in (0, t(s)). \tag{4.4}$$

The relation (4.1) implies that $t(s) > 0$ for s small enough. Let $t^*(s) = \min\{t(s), T\}$. By the triangular inequality, we obtain

$$\|U_s(t)\|_\infty \le \|u(.,t)\|_\infty + \|U_s(t) - u_s(t)\|_\infty \quad for \quad t \in (0, t^*(s)),$$

which implies that

$$\|U_s(t)\|_\infty \le \lambda + \rho, \quad for \quad t \in (0, t^*(s)). \tag{4.5}$$

Let $e_s(t) = U_s(t) - u_s(t)$ be the error of discretization.

Using Taylor's expansion, we have for $t \in (0, t^*(s))$, $\alpha_i(t) = (p-1)|\delta^0 U_i(t)|^{p-2}$, $\beta_I(t) = (p-1)|U_I^{-q}(t)|^{p-2}$

$$\frac{d}{dt}e_0(t) - \alpha_0(t)\delta^2 e_0(t) = h(1-\theta_0(t))^{-h-1}e_0(t) + \frac{s}{3}\alpha_0(t)u_{xxx}(\widetilde{x}_0, t),$$

$$\frac{d}{dt}e_i(t) - \alpha_i(t)\delta^2 e_i(t) = h(1-\theta_i(t))^{-h-1}e_i(t) + \frac{s^2}{12}\alpha_i(t)u_{xxxx}(\widetilde{x}_i, t),$$

$1 \le i \le I-1$,

$$\begin{aligned}\frac{d}{dt}e_I(t) - \beta_I(t)\delta^2 e_I(t) &= h(1-\xi_I(t))^{-h-1}e_I(t) + \frac{2}{s}\beta_I(t)f'(\xi_I(t))e_I(t)\\ &- \frac{s}{3}\beta_I(t)u_{xxx}(\widetilde{x}_I, t),\end{aligned}$$

where $\theta_i(t)$ is an intermediate value between $u(x_i, t)$ and $U_i(t)$ for $i \in \{0, \ldots, I-1\}$ and $\xi_I(t)$ is an intermediate value between $u(x_I, t)$ and $U_I(t)$. Since $u \in C^{4,1}$, using (4.5), there existes a constant $K > 0$ such that

$$\frac{d}{dt}e_0(t) - \alpha_0(t)\delta^2 e_0(t) \le \frac{M}{s}|e_0(t)| + Ks \tag{4.6}$$

$$\frac{d}{dt}e_i(t) - \alpha_i(t)\delta^2 e_i(t) \le M|e_i(t)| + Ks^2,\ 1 \le i \le I-1, \tag{4.7}$$

$$\frac{d}{dt}e_I(t) - \beta_I(t)\delta^2 e_I(t) \le \frac{M}{s}|e_I(t)| + Ks. \tag{4.8}$$

Consider the vector $W_s(t)$ such that

$$W_i(t) = e^{(L+1)t}(\|\varphi_s - u_s(0)\|_\infty + Ks), \quad 0 \le i \le I.$$

A direct calculation yields

$$\frac{d}{dt}W_0(t) - \alpha_0(t)\delta^2 W_0(t) > \frac{M}{s}|W_0(t)| + Ks, \tag{4.9}$$

$$\frac{d}{dt}W_i(t) - \alpha_i(t)\delta^2 W_i(t) > M|W_i(t)| + Ks^2, 1 \le i \le I-1, \tag{4.10}$$

$$\frac{d}{dt}W_I(t) - \beta_I(t)\delta^2 W_I(t) > \frac{M}{s}|W_I(t)| + Ks, \tag{4.11}$$

$$W_0(t) > e_0(t), \quad W_I(t) > e_I(t), \quad t \in (0, t^*(s)) \tag{4.12}$$

$$W_i(0) > e_i(0), \quad 0 \le i \le I. \tag{4.13}$$

Applying comparison Lemma 2.2, we arrive at

$$W_i(t) > e_i(t) \quad for \quad t \in (0, t^*(s)), \quad 0 \leq i \leq I.$$

In the same way, we also prove that

$$W_i(t) > -e_i(t) \quad for \quad t \in (0, t^*(s)), \quad 0 \leq i \leq I,$$

which implies that

$$W_i(t) > |e_i(t)| \quad for \quad t \in (0, t^*(s)), \quad 0 \leq i \leq I.$$

We deduce that

$$\|U_s(t) - u_s(t)\|_\infty \leq e^{(M+1)T}(\|\varphi_s - u_s(0)\|_\infty + Ks), \quad t \in (0, t^*(s)).$$

To complete the proof of this theorem, we have to show that for s sufficiently small $t^*(s) = T$. But if it is not true, for some s, as small as we like, $t^*(s) < T$ and by (4.2) we obtain

$$\begin{aligned} \frac{\varrho}{2} &= \|U_s(t^*(s)) - u_s(t^*(s))\|_\infty \\ &\leq e^{(M+1)T}(\|\varphi_s - u_s(0)\|_\infty + Ks). \end{aligned} \tag{4.14}$$

Since the term on the right hand side of the above inequality goes to zero as s tends to zero, we deduce that $\frac{\varrho}{2} \leq 0$, which is impossible. □

Theorem 4.2. *Suppose that the solution u of* (1.4)–(1.6) *quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$ and the initial condition at* (2.3) *satisfies*

$$\|\varphi_s - u_s(0)\|_\infty = \circ(1) \quad s \to 0.$$

Under the assumptions of Theorem 3.1, the solution U_s of the problem (2.1)–(2.3) *quenches in a finite time T_q^s and*

$$\lim_{s \to 0} T_q^s = T_q.$$

Proof. Let $\gamma > 0$. There exists a constant $R > 0$ such that

$$\frac{(1-z)^{h+1}}{A(h+1)} \leq \frac{\gamma}{2}, \quad z \in [0, R]. \tag{4.15}$$

Since $u(.,t)$ quenches in a finite time T_q, there exists a time $T_1 < T_q$ such that $|T_1 - T_q| < \frac{\gamma}{2}$ and $0 \leq \|u(.,t)\|_\infty \leq \frac{R}{2}$ for $t \in [T_1, T_q]$. Setting $T_2 = \frac{T_1 + T_q}{2}$, it is not hard to see that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_2]$. From Theorem 4.1, we have $\|U_s(T_2) - u_s(T_2)\|_\infty \leq \frac{R}{2}$. Applaying the triangle inequality, we get

$$\|U_s(T_2)\|_\infty \leq \|U_s(T_2) - u_s(T_2\|_\infty + \|u_s(T_2)\|_\infty \leq R.$$

From Theorem 3.1, U_s quenches in a finite time T_q^s. We deduce from Remark 3.1 and (4.15) that

$$|T_q^s - T_q| \leq |T_q^s - T_2| + |T_2 - T_q| \leq \frac{(1 - \|U_s(T_2)\|_\infty)^{h+1}}{A(h+1)} + \frac{\gamma}{2} \leq \gamma.$$

This inequality gives the desired result. □

5. Numerical experiments

In this section, we study the quenching phenomenon using full discrete schemes (explicit and implicit) of (1.4)–(1.6). At first, we approximate the solution u of (1.4)–(1.6) by the solution $U_s^{(n)} = \left(U_0^{(n)}, U_1^{(n)}, \cdots, U_I^{(n)}\right)^T$ of the following explicit scheme

$$\begin{aligned}
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} &= (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 U_i^{(n)} \\
&+ (1 - U_i^{(n)})^{-h}, \quad 0 \leq i \leq I-1, \\
\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n^e} &= (p-1)|(U^{-q})_I^{(n)}|^{p-2}\delta^2 U_I^{(n)} + (p-1)|(U^{-q})_I^{(n)}|^{p-2}(\frac{-(U^{-q})_I^{(n)}}{s}) \\
&+ (1 - U_I^{(n)})^{-h}, \\
U_i^{(0)} &= \varphi_i > 0, \quad 0 \leq i \leq I,
\end{aligned}$$

where $n \geq 0$, $\delta^0 U_0^{(n)} = 0$, $\delta^0 U_i^{(n)} = \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2s}$, $\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{s^2}$, for $1 \leq i \leq I-1$,

$$\delta^2 U_I^{(n)} = \frac{2}{s^2}\left(U_{I-1}^{(n)} - U_I^{(n)}\right),$$

$$\Delta t_n^e = \min\left(\frac{s^2}{2(p-1)\max\{a(j-1,1)\}}, \tau(1 - \|U_s^{(n)}\|_\infty^{h+1})\right),$$

with $\tau = const \in (0,1)$ and $a(j-1,1) = \left(\frac{|U_{j+1}^{(n)} - U_{j-1}^{(n)}|}{2s}\right)^{p-2}$ for $2 \leq j \leq I$.

Now, approximate the solution u of (1.4)–(1.6) by the solution $U_s^{(n)} = (U_0^{(n)}, U_1^{(n)}, \cdots, U_I^{(n)})^T$ of the following implicit scheme

$$\begin{aligned}
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= (p-1)|\delta^0 U_i^{(n+1)}|^{p-2}\delta^2 U_i^{(n+1)} \\
&+ (1 - U_i^{(n)})^{-h}, \quad 0 \le i \le I-1, \\
\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} &= (p-1)|(U^{-q})_I^{(n)}|^{p-2}\delta^2 U_I^{(n+1)} \\
&+ (p-1)|(U^{-q})_I^{(n)}|^{p-2}\left(\frac{-(U^{-q})_I^{(n)}}{s}\right) \\
&+ (1 - U_I^{(n)})^{-h}, \\
U_i^{(0)} &= \varphi_i > 0, \quad 0 \le i \le I,
\end{aligned}$$

where $n \ge 0$, $\delta^0 U_i^{(n+1)} = 0$,

$$\delta^0 U_i^{(n+1)} = \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2s},$$

$$\delta^2 U_i^{(n+1)} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{s^2},$$

$$\delta^2 U_I^{(n+1)} = \frac{2}{s^2}\left(U_{I-1}^{(n+1)} - U_I^{(n+1)}\right),$$

$$\Delta t_n = \tau(1 - \|U_s^{(n)}\|_\infty^{h+1})$$

with $\tau = const \in (0,1)$. In the following tables, in rows, we present the numerical quenching times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. The numerical time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s_0 = \frac{\log((T_{4s} - T_{2s})/(T_{2s} - T_s))}{\log(2)}.$$

For the numerical value, we take:

$$\varphi_i = 0.5 + \frac{1}{6\pi} cos\left(\frac{\pi(is)}{2}\right) - \frac{1}{3}(is)^{4.5},$$

for $i = 0, \cdots, I$, $\tau = \dfrac{s^2}{2}$.

TABLE 1. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method for $q = -\left(\log(\frac{13}{8})/\log(\frac{1}{6})\right)$ $p = 2$ $h = 7$

I	T^n	n	$CPU\ time$	s_0
16	0.000200417708905	1545	0.156	-
32	0.000199393432328	5829	0.296	-
64	0.000199138064753	21900	0.578	2.003
128	0.000199074266382	81926	4.528	2.000
256	0.000199058318375	304993	29.687	2.000
512	0.000199054326936	1129121	219.891	2.000
1024	0.000199053310654	4153077	1842.141	2.000

TABLE 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method for $q = -\left(\log(\frac{13}{8})/\log(\frac{1}{6})\right)$ $p = 2$ $h = 7$

I	T^n	n	$CPU\ time$	s_0
16	0.000200417708905	1545	0.141	-
32	0.000199393432328	5829	0.703	-
64	0.000199138064753	21900	30.375	2.003
128	0.000199074266382	81926	293.984	2.000
256	0.000199058318375	304993	4457.843	2.000
512	0.000199054326936	1129121	71839.984	2.000
1024	0.000199053310654	4153077	927158.782	2.000

Remark 5.1. *The two tables show that the solution of the problem quenches in a finite time. We estimate this time at* 2.10^{-4}.

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 16$ and $(q, p, h) = \left(-\left(\log(\frac{13}{8})/\log(\frac{1}{6})\right), 2, 7\right)$.

In the figures 1 and 2, we can appreciate the quenching of the discrete solution, the figures 3 and 4 show that the decrease of the discrete solution and in the figures 5 and 6, we observe that the discrete solution quenches at the finite time $T_q^s = 2.10^{-4}$.

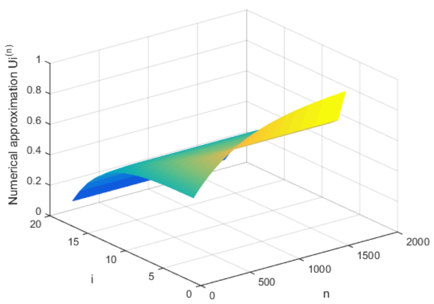

FIGURE 1. Evolution of the discrete solution (explicit scheme).

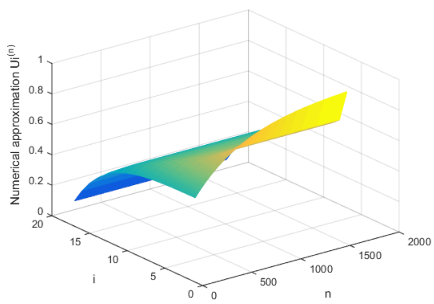

FIGURE 2. Evolution of the discrete solution (implicit scheme).

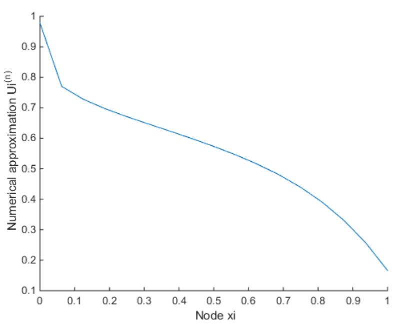

FIGURE 3. Evolution of the discrete solution according to the node (explicit scheme).

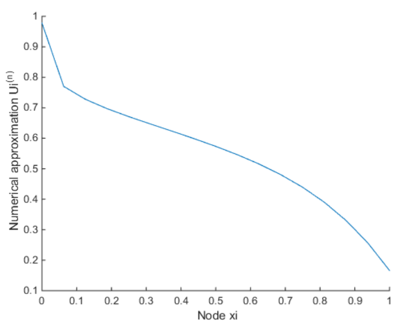

FIGURE 4. Evolution of the discrete solution according to the node (implicit scheme).

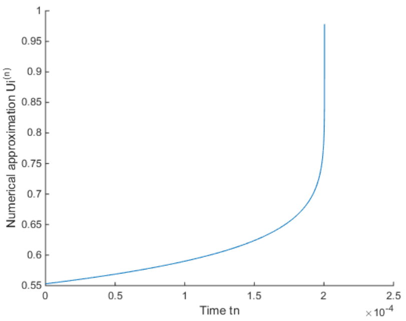

FIGURE 5. Evolution of the norm of the discrete solution according to the time (explicit scheme).

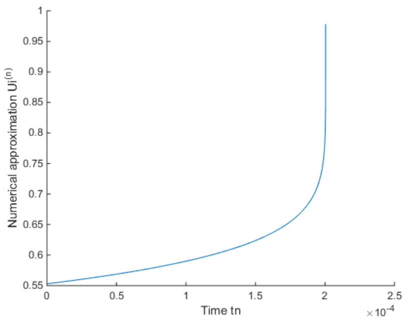

FIGURE 6. Evolution of the norm of the discrete solution according to the time (implicit scheme).

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