ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.4, 1899–1914
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.5

NUMERICAL QUENCHING OF A SEMILINEAR HEAT EQUATION WITH A SINGULAR BOUNDARY OUTFLUX

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ABSTRACT. In this paper, we study the semidiscrete approximation for the following semilinear heat equation with a singular boundary outflux

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t)^{-q}, \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1. \end{cases}.$$

We find some conditions under which the solution of a semidiscrete form of above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

1. INTRODUCTION

In this paper, we consider the semilinear heat equation with a singular boundary outflux

$$u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

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2020 *Mathematics Subject Classification.* 35B50, 35B51, 35K05, 35K55, 65M06.
Key words and phrases. Numerical quenching, heat equation, singular boundary, semidiscrete quenching time, convergence.
Submitted: 02.03.2021; *Accepted:* 24.03.2021; *Published:* 04.04.2021.

(1.2) $$u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t)^{-q}, \quad t > 0,$$

(1.3) $$u(x,0) = u_0(x), \quad 0 \le x \le 1,$$

where p, q are a positive constants and $u_0 : [0,1] \longrightarrow (0,1)$ is nonincreasing and satisfies the compatibility conditions:

$$u_0'(0) = 0,\ u_0'(1) = -u_0(1)^{-q}.$$

Definition 1.1. *We say that the solution u of* (1.1)–(1.3) *quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$ but*

$$\lim_{t\to T_q} \|u(.,t)\|_\infty = 1,$$

where $\|u(.,t)\|_\infty = \max\{|u(x,t)| : 0 \le x \le 1\}$. The time T_q is called quenching time of the solution u.

The theoretical study of quenching problems with various boundary has been the subject of investigations of many authors (see [2], [3], [4], [5], [7], [8], [9], [12] and the references cited therein). In [12], B. Selcuk and N. Ozalp prove a finite-time quenching for the solution of (1.1)–(1.3) Under certain conditions, they show that $x = 0$ is the only quenching point and they get a quenching rate and a lower bound for the quenching time. In this paper, we are interested in the numerical study using a semidiscrete scheme of (1.1)–(1.3). For previous study on numerical approximations of parabolic system with non-linear boundary conditions we refer to ([6], [10], [11])

We organise this paper as follows: In the next section, we give some lemmas which will be used throughout the paper. In the fourth section, under some hypotheses, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. In the fifth section, we give a result about the convergence of the semidiscrete quenching time to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. The Semidiscrete Problem

Let I be a nonnegative integer, we set $h = \frac{1}{I}$, and we define the grid, $x_i = ih$, $i = 0, \dots, I$. We approximate the solution u of the problem (1.1)–(1.3) by the solution $U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T$. For $t \in (0, T_q^h)$, we have

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) + (1 - U_i(t))^{-p}, \quad 0 \le i \le I - 1, \tag{2.1}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + (1 - U_I(t))^{-p} - \frac{2}{h} U_I(t)^{-q}, \tag{2.2}$$

$$U_i(0) = \varphi_i, \quad 0 \le i \le I, \tag{2.3}$$

where

$$\begin{aligned}
\delta^+\varphi_i &= \frac{\varphi_{i+1} - \varphi_i}{h}, 0 \le i \le I-1, \delta^+\varphi_i \le 0, 0 \le i \le I-1, \\
\delta^2 U_i(t) &= \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1, \\
\delta^2 U_0(t) &= \frac{2U_1(t) - 2U_0(t)}{h^2}, \\
\delta^2 U_I(t) &= \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.
\end{aligned}$$

3. Properties of the Semidiscrete Problem

In this section, we give some important results which will be used later.

Lemma 3.1. *Let $b_h(t) \in C^0([0,T], \mathbb{R}^{I+1})$ and $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + b_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in (0, T], \tag{3.1}$$

$$V_i(0) \ge 0, \quad 0 \le i \le I. \tag{3.2}$$

Then we have $V_i(t) \ge 0$, $0 \le i \le I$, $t \in [0, T]$.

Proof. Let $T_0 < T$. Define the vector $Z_h(t) = e^{\lambda} V_h(t)$ where λ is such that $b_i(t) - \lambda > 0 \ \forall t \in [0, T_0]$

Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. $Z_i(t)$ is continous on the compact $[0, T_0]$, there exists $i_0 \in \{0, \dots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$.

We observe that:

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I, \tag{3.3}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, 1 \le i_0 \le I-1, \tag{3.4}$$

(3.5) $$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad i_0 = 0,$$

(3.6) $$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad i_0 = I.$$

Moreover, by a straightforward computation, we get

(3.7) $$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (b_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0.$$

Using (3.3)–(3.6), we deduce from (3.7) that $(b_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. We deduce that $V_h(t) \geq 0$, $\forall t \in [0, T_0]$ and the proof is complete. □

Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

Lemma 3.2. *Let* $g \in C^0(\mathbb{R}, \mathbb{R})$ *and* $V_h(t)$, $W_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$*, such that for* $0 \leq i \leq I$

(3.8) $$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t)) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t)), t \in (0, T]$$

(3.9) $$V_i(0) < W_i(0).$$

Then $V_i(t) < W_i(t)$*,* $0 \leq i \leq I$*,* $t \in [0, T]$.

Proof. Let $Z_h(t)$ a vector such that $Z_i(t) = W_i(t) - V_i(t)$ and let t_0, be the first $t > 0$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. We observe that:

(3.10) $$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

(3.11) $$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, 1 \leq i_0 \leq I - 1,$$

(3.12) $$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad i_0 = 0,$$

(3.13) $$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad i_0 = I.$$

Which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + g(W_{i_0}(t_0)) - g(V_{i_0}(t_0)) \leq 0.$$

This inequality contradicts (3.8) which ends the proof. □

Lemma 3.3. *Let U_h be the solution of* (2.1)–(2.3). *We assume that the initial data at* (2.3) *satisfies $\varphi_i > 0$, $0 \le i \le I$. Then for $t \in (0, T_q^h)$ and $0 \le i \le I$, we have*

$$U_i(t) > 0.$$

Proof. Let t_0, be the first $t > 0$ such that $U_{i_0}(t) > 0$, $\forall t \in (0, t_0)$ but $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} < 0, \quad 0 \le i_0 \le I, \tag{3.14}$$

$$\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} > 0, 1 \le i_0 \le I - 1, \tag{3.15}$$

$$\delta^2 U_{i_0}(t_0) = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} > 0, \quad i_0 = 0, \tag{3.16}$$

$$\delta^2 U_{i_0}(t_0) = \frac{2U_{I-2}(t_0) - 2U_{I-1}(t_0)}{h^2} > 0, \quad i_0 = I. \tag{3.17}$$

By a straightforward computation, we get

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) - (1 - U_{i_0}(t_0))^{-p} < 0,$$

for $0 \le i_0 \le I - 1$,

$$\frac{dU_I(t)}{dt} - \delta^2 U_I(t) - (1 - U_I(t))^{-p} + \frac{2}{h} U_I(t)^{-q} < 0.$$

But these inequalities contradict (2.1)–(2.2) and this proof is complete. □

Lemma 3.4. *Let U_h be the solution of* (2.1)–(2.3). *Then we have for $t \in [0, T_q^h)$*

$$U_i(t) > U_{i+1}(t), 0 \le i \le I - 1.$$

Proof. Introduce the vector $Z_h(t)$ such that $Z_i(t) = U_i(t) - U_{i+1}(t)$ for $t \in [0, T_h^q)$, $i = 0, \dots, I - 1$. Let t_0, be the first $t > 0$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I - 1, \tag{3.18}$$

(3.19) $$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, 1 \leq i_0 \leq I-2,$$

(3.20) $$\delta^2 Z_{i_0}(t_0) = \frac{Z_1(t_0) - 3Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,$$

(3.21) $$\delta^2 Z_{i_0}(t_0) = \frac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{h^2} > 0, \quad i_0 = I-1.$$

Moreover, by a straightforward computation, we get for $0 \leq i_0 \leq I-2$

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - p(1-\zeta_{i_0}(t_0))^{-p-1} Z_{i_0}(t_0) < 0.$$

Where ζ_{i_0} is an intermediate value between $U_i(t)$ and U_{i+1}. And

$$\frac{dZ_{I-1}(t_0)}{dt} - \delta^2 Z_{I-1}(t_0) - \frac{2}{h} U_I(t)^{-q} - p(1-\theta_I(t_0))^{-p-1} Z_{I-1}(t_0) < 0.$$

Where θ_I is an intermediate value between $U_{I-1}(t)$ and U_I.
But these inequalities contradict (2.1)–(2.2) and this proof is complete. □

Lemma 3.5. *Let U_h be the solution of* (2.1)–(2.3). *Then we have*

$$\frac{dU_i(t)}{dt} > 0, 0 \leq i \leq I, t \in (0, T_q^h).$$

Proof. Consider the vector $Z_h(t)$ such that $Z_i(t) = \dfrac{dU_i(t)}{dt}, t \in (0, T_h^q), i = 0, \ldots, I$. Let t_0, be the first $t \in (0, T_h^q)$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that:

(3.22) $$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

(3.23) $$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, 1 \leq i_0 \leq I-1,$$

(3.24) $$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,$$

(3.25) $$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} > 0, \quad i_0 = I.$$

Moreover, by a straightforward computation, we get

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - p(1-U_{i_0}(t_0))^{-p-1} Z_{i_0}(t_0) < 0, \quad 0 \leq i_0 \leq I-1,$$

$$\frac{dZ_I(t_0)}{dt} - \delta^2 Z_I(t_0) - p(1-U_I(t_0))^{-p-1}Z_I(t_0) - \frac{2}{h}qU_I(t)^{-q-1}Z_I(t_0) < 0.$$

But these inequalities contradict (2.1)–(2.2) and this proof is complete. □

4. QUENCHING SOLUTIONS

In this section, we show that under some assumptions, the solution U_h of (2.1)–(2.3) quenches in a finite time and estimate its semidiscrete quenching time.

Lemma 4.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $\|U_h\|_\infty < 1$ and let p be a positive constant. Then, we have*

$$\delta^2(1-U_i)^{-p} \geq p(1-U_i)^{-p-1}\delta^2 U_i,\ 0 \leq i \leq I.$$

Proof. Let us introduce $f(s) = (1-s)^{-p}$. We observe that f is a convex function for nonnegative values of s. Apply Taylor's expansion to obtain

$$\delta^2 f(U_0) = f'(U_0)\delta^2 U_0 + \frac{(U_1-U_0)^2}{h^2}f''(\theta_0).$$

$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + \frac{(U_{i+1}-U_i)^2}{2h^2}f''(\theta_i) + \frac{(U_{i-1}-U_i)^2}{2h^2}f''(\eta_i), 1 \leq i \leq I-1.$$

$$\delta^2 f(U_I) = f'(U_I)\delta^2 U_I + \frac{(U_{I-1}-U_I)^2}{h^2}f''(\eta_I).$$

where θ_i is an intermediate between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. Use the fact that $\|U_h\|_\infty < 1$ to complete the proof. □

Theorem 4.1. *Let U_h be the solution of* (2.1)–(2.3)*, and assume that there exist a nonnegative constant A such that the initial data at* (2.3) *satisfies*

$$\delta^2\varphi_i + (1-\varphi_i)^{-p} \geq A(1-\varphi_i)^{-p}, \quad 0 \leq i \leq I-1. \tag{4.1}$$

$$\delta^2\varphi_I + (1-\varphi_I)^{-p} - \frac{2}{h}\varphi_I^{-q} \geq A(1-\varphi_I)^{-p}. \tag{4.2}$$

Then, the solution U_h quenches in a finite time T_q^h and we have the following estimate

$$T_q^h \leq \frac{(1-\|\varphi_h\|_\infty)^{p+1}}{A(p+1)}.$$

Proof. Let $[0, T_q^h)$ be the maximal time interval on which $||U_h||_\infty < 1$. We consider the function $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} - A(1 - U_i(t))^{-p}, \quad 0 \leq i \leq I. \tag{4.3}$$

By a straightforward computation we get

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) = \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) - pA(1 - U_i(t))^{-p-1}\frac{dU_i(t)}{dt} + A\delta^2(1 - U_i(t))^{-p}, 0 \leq i \leq I.$$

From Lemma 4.1, we have $A\delta^2(1-U_i(t))^{-p} \geq pA(1-U_i(t))^{-p-1}\delta^2 U_i(t), 0 \leq i \leq I.$ Which implies that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \geq p(1 - U_i(t))^{-p-1}\frac{dU_i(t)}{dt} - pA(1 - U_i(t))^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)),$$

$0 \leq i \leq I - 1,$

$$\frac{dJ_I(t)}{dt} - \delta^2 J_I(t) \geq p(1 - U_I(t))^{-p-1}\frac{dU_I(t)}{dt} + \frac{2q}{h}U_I^{-q-1}(t)\frac{dU_I(t)}{dt} - pA(1 - U_I(t))^{-p-1}(\frac{dU_I(t)}{dt} - \delta^2 U_I(t)).$$

We deduce that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \geq p(1 - U_i(t))^{-p-1}J_i(t), \quad 0 \leq i \leq I - 1,$$

$$\frac{dJ_I(t)}{dt} - \delta^2 J_I(t) \geq p(1 - U_I(t))^{-p-1}J_I(t) + \frac{2}{h}U_I(t)^{-q-1}(q\frac{dU_I(t)}{dt} + pA(1 - U_I(t))^{-p-1}U_I(t)).$$

From (4.1)–(4.2), we observe that $J_i(0) \geq 0$ for $0 \leq i \leq I$. We deduce from Lemma 3.1 that $J_i(t) \geq 0$, $0 \leq i \leq I$, which implies that

$$dU_i(t) \geq A(1 - U_i(t))^{-p}dt, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h).$$

Integrating the above inequalities over the interval $[t, T_q^h)$, we get

$$T_q^h - t \leq \frac{(1 - U_i(t))^{p+1}}{A(p + 1)}, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h). \tag{4.4}$$

Taking $t = 0$, we obtain:

$$T_q^h \leq \frac{(1 - \varphi_i)^{p+1}}{A(p + 1)}, \quad 0 \leq i \leq I.$$

Using the fact that $\|\varphi_h\|_\infty = \varphi_0$, we get:

$$T_q^h \leq \frac{(1-\|\varphi_h\|_\infty)^{p+1}}{A(p+1)}.$$

We have the desired result. □

Remark 4.1. *Integrating the inequality* (4.4) *over interval* $[t_0, T_q^h)$*, we have*

$$T_q^h - t_0 \leq \frac{(1-U_i(t_0))^{p+1}}{A(p+1)}, \quad t_0 \in [0, T_q^h), \quad 0 \leq i \leq I$$

and

$$\|U_h\|_\infty \leq 1 - C_1(T_q^h - t_0)^{\frac{1}{p+1}},$$

where $C_1 = (A(p+1))^{\frac{1}{p+1}}$.

The Remark 4.1 is crucial to prove the convergence of the semidiscrete quenching time.

5. Convergence of Semidiscrete Quenching Times

Theorem 5.1. *Assume that the problem* (1.1)–(1.3) *has a solution* $u \in C^{4,1}([0,1]\times[0,T])$ *such that* $\sup_{t\in[0,T]}\|u\| = \lambda < 1$ *and and the initial data at* (2.3) *verifies*

(5.1) $$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \longrightarrow 0,$$

where $u_h(t) = (u(x_0,t),\ldots,u(x_I,t))^T$, $t \in [0,T]$. *Then, for* h *small enough, the semidiscrete problem* (2.1)–(2.3) *has a unique solution* $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ *such that*

$$\max_{t\in[0,T]}(\|U_h(t) - u_h(t)\|_\infty) = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad as \quad h \to 0.$$

Proof. Let $\rho > 0$ be such that $\rho + \lambda < 1$. The problem (2.1)–(2.3) has for each h, a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

(5.2) $$\|U_h(t) - u_h(t)\|_\infty < \rho \quad for \quad t \in (0, t(h)).$$

The relation (5.1) implies that $t(h) > 0$ for h small enough. Let $t^*(h) = \min\{t(h), T\}$. By the triangular inequality, we obtain

$$\|U_h(t)\|_\infty \leq \|u(.,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad for \quad t \in (0, t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leq \lambda + \rho, \quad for \quad t \in (0, t^*(h)). \tag{5.3}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0, t^*(h))$,

$$\frac{de_0(t)}{dt} - \delta^2 e_0(t) = p(1-\beta_0)^{-p-1}e_0(t) + h\left(\frac{h}{12}u_{xxxx}(\tilde{x}_0, t) + \frac{2}{3}u_{xxx}(x_0, t)\right),$$

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = p(1-\beta_i(t))^{-p-1}e_i(t) + \frac{h^2}{12}u_{xxxx}(\tilde{x}_i, t), 1 \leq i \leq I-1$$

$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) = \left(p(1-\beta_I)^{-p-1} + \frac{2q}{h}\mu_I^{-q-1}(t)\right)e_I(t) + h\left(\frac{h}{12}u_{xxxx}(\tilde{x}_I, t) - \frac{2}{3}u_{xxx}(x_I, t)\right).$$

where $\beta_i(t)$ is an intermediate value between $U_i(t)$ and $u_i(t)$, $0 \leq i \leq I$ and $\mu_I(t)$ the one between $U_I(t)$ and $u_I(t)$.

Using (5.3), there exist nonnegative constants K, M such that

$$\frac{de_0(t)}{dt} - \delta^2 e_0(t) \leq M|e_0(t)| + Kh. \tag{5.4}$$

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq M|e_i(t)| + Kh^2, \quad 1 \leq i \leq I-1, \tag{5.5}$$

$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) \leq \frac{M}{h}|e_I(t)| + Kh. \tag{5.6}$$

Let $Z_h(t)$ the vector defined by

$$Z_i(t) = e^{(M+1)t}(||\varphi_h - u_h(0)||_\infty + Kh), 0 \leq i \leq I.$$

A simple calculation give

$$\frac{dZ_0(t)}{dt} - \delta^2 Z_0(t) > M|Z_0(t)| + Kh, \tag{5.7}$$

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) > M|Z_i(t)| + Kh^2, \quad 1 \leq i \leq I-1, \tag{5.8}$$

$$\frac{dZ_I(t)}{dt} - \delta^2 Z_I(t) > \frac{M}{h}|Z_I(t)| + Kh, \tag{5.9}$$

$$Z_i(0) > e_i(0), \quad 0 \leq i \leq I. \tag{5.10}$$

From Lemma 3.2, we obtain

$$Z_i(t) > e_i(t),\, t \in (0, t^*(h)),\, 0 \le i \le I.$$

By analogy, we also prove that

$$Z_i(t) > -e_i(t),\, t \in (0, t^*(h)),\, 0 \le i \le I.$$

Hence we have

$$Z_i(t) > |e_i(t)|,\, t \in (0, t^*(h)),\, 0 \le i \le I.$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + Kh)e^{(M+1)t},\, t \in (0, t^*(h)).$$

Next we prove that $t^*(h) = T$. Suppose that $t(h) < T$ From (5.2), we obtain

$$\rho \le \|U_h(t(h)) - u_h(t(h))\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + Kh)e^{(M+1)T}. \tag{5.11}$$

Since $(\|\varphi_h - u_h(0)\|_\infty + Kh)e^{(M+1)T} \longrightarrow 0$ as $h \longrightarrow 0$ we deduce from (5.11) that $\rho \le 0$ which is impossible and we conclude the proof. □

Theorem 5.2. *Suppose that the solution u of problem* (1.1)–(1.3) *quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$ and the iniatial data at* (2.3) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \text{ as } h \longrightarrow 0.$$

Under the assumptions of Theorem 4.1, the solution U_h of problem (2.1)–(2.3) *quenches in finite time T_q^h and we have*

$$\lim_{h\to 0} T_q^h = T_q.$$

Proof. Set $\varepsilon > 0$. There exists $\eta > 0$ such that

$$\frac{(1-\varrho)^{p+1}}{A(p+1)} < \frac{\varepsilon}{2},\quad 0 \le \varrho \le \eta. \tag{5.12}$$

Since u quenches in a finite time T_q, there exists a time $T_0 < T_q$ such that $|T_0 - T_q| < \frac{\varepsilon}{2}$ and $0 \le \|u(.,t)\|_\infty \le \frac{\eta}{2}$ for $t \in [T_0, T_q)$. Setting $T_1 = \frac{T_0 + T_q}{2}$, it is not hard to see that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_1]$.

From Theorem 5.1, we have $\|U_h(T_1) - u_h(T_1)\|_\infty \le \frac{\eta}{2}$. Applying the triangle inequality, we get

$$\|U_h(T_1)\|_\infty \le \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_\infty \le \eta.$$

From Theorem 4.1, U_h quenches in a finite time T_q^h. We deduce from Remark 4.1 and (5.12) that

$$|T_q^h - T_q| \leq |T_q^h - T_1| + |T_1 - T_q| \leq \frac{(1 - U_h(T_1))^{p+1}}{A(p+1)} + \frac{\varepsilon}{2} \leq \varepsilon.$$

□

6. Numerical experiments

In this section, we present some numerical approximations of the quenching time of the problem (1.1)–(1.3) in the case where $u_0(x) = 0.7 - \frac{1}{2}x^4$, $p = 8.03$ and $q = -\log(2)/\log(0.2)$. Firstly, we consider the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + (1 - U_i^{(n)})^{-p}, \quad 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n^e} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n^e} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (1 - U_I^{(n)})^{-p} - \frac{2}{h}(U_I^{(n)})^{-q},$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n^e = \min\left\{\frac{h^2}{2}, h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}\right\}$. We also consider the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + (1 - U_i^{(n)})^{-p}, 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + (1 - U_I^{(n)})^{-p} - \frac{2}{h}(U_I^{(n)})^{-q},$$

$$U_i^{(0)} = \varphi_i, 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}$. In the following tables, in rows, we present the numerical quenching times, the numbers of iterations and the orders of the approximations corresponding to meshes $16, 32, 64, 128, 256, 512$. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order s of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

For the discret initial data we take $\varphi_i = 0.7 - \frac{1}{2}(ih)^4$

TABLE 1. Numerical quenching times obtained with the explicit Euler method $p = 8.03$ and $q = -\log(2)/\log(0.2)$

I	T^n	n	s
16	0.000002136	578	-
32	0.000002111	2158	-
64	0.000002104	8006	2.01
128	0.000002103	29512	2.00
256	0.000002103	107986	2.00
512	0.000002102	391701	1.97

TABLE 2. Numerical quenching times obtained with the implicit Euler method $p = 8.03$ and $q = -\log(2)/\log(0.2)$

I	T^n	n	s
16	0.000002136	578	-
32	0.000002111	2158	-
64	0.000002104	8006	2.03
128	0.000002103	29512	2.01
256	0.000002103	107986	1.99
512	0.000002102	391701	1.88

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 16$, $p = 8.03$ and $q = -\log(2)/\log(0.2)$. In figures 1, 2 and figures 3, 4, we can appreciate that the discrete solution is nonincreasing and reaches the value one at the first node. In figures 5 and 6, we see that the approximation of $\|U_h^{(n)}\|_\infty$ is nondecreasing and tends to the value one when t tends to 2.5×10^{-6}.

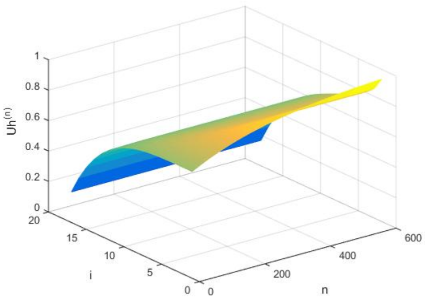

FIGURE 1. Evolution of the numerical solution (explicit scheme).

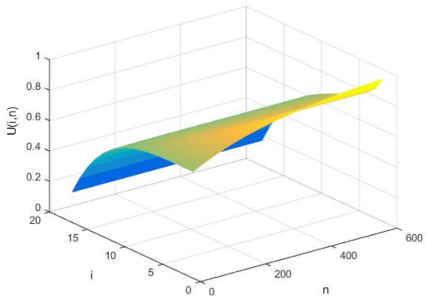

FIGURE 2. Evolution of the numerical solution (implicit scheme).

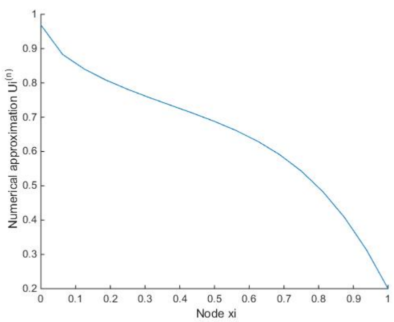

FIGURE 3. The profil of the approximation of u(x,T) where, T is the quenching time (explicit scheme).

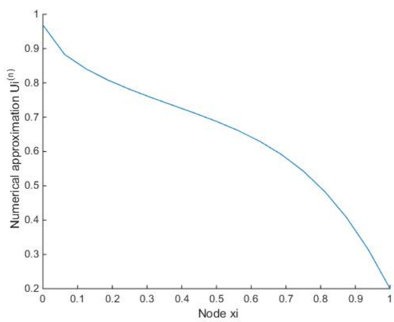

FIGURE 4. The profil of the approximation of u(x,T) where, T is the quenching time (implicit scheme).

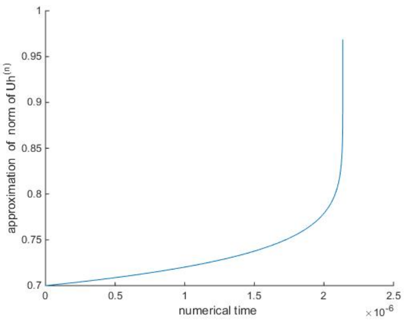

FIGURE 5. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (explicit scheme).

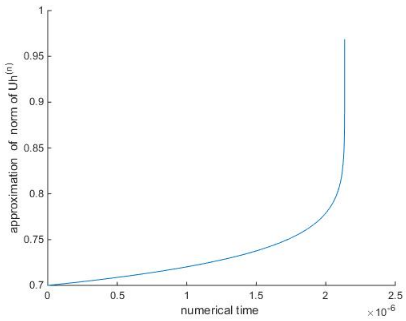

FIGURE 6. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (implicit scheme).

ACKNOWLEDGMENT

The authors want to thank the anonymous referees for the throughout reading of the manuscript.

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