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ON $N(k)$-CONTACT METRIC MANIFOLD ENDOWED WITH PSEUDO-QUASI-CONFORMAL CURVATURE TENSOR

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Abstract. In this paper, a few properties of $N(k)$-contact metric manifolds equipped with pseudo qausi-conformal curvature tensor were deeply studied. Firstly, it has been shown that a globally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold is turns into ϕ-symmetric. Further, we describe 3-dimensional $N(k)$-contact metric manifold, characterizing the locally ϕ-pseudo-quasi-conformally symmetric and pseudo-quasi-conformally ϕ-recurrent structures. Finally, we pay a special attention to the existence of 3-dimensional case by giving suitable examples.

1. Introduction

In 1968, Yano and Sawaki [7] have originate and deeply studied a type of tensor field, called quasi-coformal curvature tensor C on a Riemannian manifold. As a generalization, recently Shaikh and Jana [5] have formulated the framework of pseudo-quasi-conformal curvature tensor which comprises the structures of concircular, projective, quasi-conformal and Weyl conformal curvature

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tensors as a special cases given by:

$$
\begin{aligned}
(1.1)\qquad \tilde{C}(U,V)W &= (p+d)R(U,V)W + q[g(V,W)QU - g(U,W)QV] \\
&\quad + \left[q - \frac{d}{2n}\right][S(V,W)U - S(U,W)V] \\
&\quad - \frac{r}{2n(2n+1)}\{p+4nq\}[g(V,W)U - g(U,W)V],
\end{aligned}
$$

where $U, V, W \in \chi(M)$, S is the Ricci tensor, r is the scalar curvature, Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S [1] called Ricci operator i.e., $g(QV,W) = S(V,W)$ and p, q, d are real constants such that $p^2 + q^2 + d^2 > 0$.

In particular, if (1) $p = 1, q = d = 0$; (2) $p = q = 0, d = 1$; (3) $p \neq 0, q \neq 0, d = 0$; (4) $p = 1, q = \frac{-1}{2n-1}, d = 0$; then pseudo-quasi-conformal curvature tensor $\tilde{C}$ reduces to the concircular curvature tensor; projective curvature tensor; quasi-conformal curvature tensor and conformal curvature tensor respectively. Also from (1.1), we can easily found that:

$$
\begin{aligned}
(1.2)\quad (\nabla_X\tilde{C})(U,V)W &= (p+d)(\nabla_X R)(U,V)W \\
&\quad + q[g(V,W)(\nabla_X Q)U - g(U,W)(\nabla_X Q)V] \\
&\quad + \left[q - \frac{d}{2n}\right][(\nabla_X S)(V,W)U - (\nabla_X S)(U,W)V] \\
&\quad - \frac{dr(X)\{p+4nq\}}{2n(2n+1)}[g(V,W)U - g(U,W)V].
\end{aligned}
$$

Based on the above, the present paper is organized in the following way: In Section 2, we formulated the definitions and preliminary results that will be needed thereafter. In Section 3, we have proved that a globally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ [2] or the manifold always admits an η parallel Ricci tensor. In section 4, we study three dimensional Locally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold and it is shown that the manifold is locally ϕ-pseudo quasi-conformally symmetric if and only if the scalar curvature is constant. In the next section we have shown that a 3-dimensional pseudo-quasi conformally ϕ-recurrent $N(k)$-contact metric manifold is pseudo-quasi conformally ϕ-symmetric if and only if the manifold is of constant scalar curvature. Finally, we gave an example of a 3-dimensional locally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold.

2. Preliminaries

An odd dimensional manifold M is said to conceded an almost contact structure if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times R$ characterized by

$$J(U, f\frac{d}{dt}) = (\phi U - f\xi, \eta(U)\frac{d}{dt})$$

is integrable, where U is tangent to M, t is the coordinate of R and f is a differentiable function on $M \times R$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), i.e.,

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$g(V, \phi W) = -g(\phi V, W), \quad g(V, \xi) = \eta(V),$$

for all vector fields V, W. An almost contact metric structure becomes a contact metric structure if

$$g(V, \phi W) = d\eta(V, W),$$

for all vector fields V, W. Then the 1-form η is reduces to contact form and corresponding ξ is its characteristic vector field. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds

$$\nabla_V \xi = -\phi V - \phi h V. \tag{2.2}$$

Blair et al. [4] have developed the structure of (k, μ)-nullity distribution of a Contact metric manifold M given by

$$N(k, \mu) : p \to N_p(k, \mu)$$
$$N_p(k, \mu) = \{W \in T_pM / R(U, V)W = (kI + \mu h)[g(V, W)U - g(U, W)V]\},$$

for all $U, V \in \chi(M)$, where $(k, \mu) \in R^2$. A Contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $\mu = 0$, the (k, μ)-nullity distribution weakened to k-nullity distribution. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [6]

$$N(k) : p \to N_p(k) = \{W \in T_pM/R(U,V)W = k[g(V,W)U - g(U,W)V]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold [3]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [2]. In an $N(k)$-contact metric manifold, the following relations hold:

$$h^2 = (k-1)\phi^2, \tag{2.3}$$

$$R(\xi,U)V = k[g(U,V)\xi - \eta(V)U], \tag{2.4}$$

$$\begin{aligned} S(U,V) &= 2(n-1)g(U,V) + 2(n-1)g(hU,V), \\ &\quad +[2nk - 2(n-1)]\eta(U)\eta(V), n \geq 1, \end{aligned} \tag{2.5}$$

$$r = 2n(2n-2+k), \tag{2.6}$$

$$S(\phi U, \phi V) = S(U,V) - 2nk\eta(U)\eta(V) - 4(n-1)g(hU,V), \tag{2.7}$$

$$S(U,\xi) = 2nk\eta(U), \tag{2.8}$$

$$(\nabla_U \eta)(V) = g(U + hU, \phi V). \tag{2.9}$$

Also in a 3-dimensional $N(k)$-contact metric manifold, the Riemannian curvature tensor R and Ricci tensor S satisfies the following relations:

$$\begin{aligned} R(U,V)W &= \left(\frac{r}{2} - 2k\right)[g(V,W)U - g(U,W)V] \\ &\quad + \left(3k - \frac{r}{2}\right)\big[\eta(V)\eta(W)U - \eta(U)\eta(W)V \\ &\quad + g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\big], \end{aligned} \tag{2.10}$$

$$S(U,V) = \left(\frac{r}{2} - k\right)g(U,V) + \left(3k - \frac{r}{2}\right)\eta(U)\eta(V). \tag{2.11}$$

Definition 2.1. *An $N(k)$-contact metric manifold is said to be globally ϕ-pseudo-quasi-conformally symmetric if the pseudo-quasi-conformal curvature tensor $\tilde{C}$ satisfies the condition*

$$\phi^2((\nabla_X \tilde{C})(U,V)W) = 0 \tag{2.12}$$

for all $U, V, W, X \in \chi(M)$. In particular, if we take $U, V, W, X \in \chi(M)$ orthogonal to ξ then the manifold becomes locally ϕ-pseudo-quasi-conformally symmetric.

3. GLOBALLY ϕ-PSEUDO-QUASI-CONFORMALLY SYMMETRIC $N(k)$-CONTACT METRIC MANIFOLDS

Let M be an globally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold, then it follows from (2.12) and (2.1) that

$$-(\nabla_X \tilde{C})(U,V)W + \eta((\nabla_X \tilde{C})(U,V)W)\xi = 0. \tag{3.1}$$

In view of (1.1) in (3.1) and then by taking inner product with Y, we have

$$\begin{aligned}
&-(p+d)(\nabla_X R)(U,V,W,Y) - \left(q - \frac{d}{2n}\right)\big[(\nabla_X S)(V,W)g(U,Y) \\
&-\nabla_X S)(U,W)g(V,Y)\big] - q\big[g(V,W)g((\nabla_X Q)U,Y) \\
&-g(U,W)g((\nabla_X Q)V,Y)\big] + \frac{dr(X)(p+4nq)}{2n(2n+1)}\big[g(V,W)g(U,Y) \\
&-g(U,W)g(V,Y)\big] + (p+d)\eta\big((\nabla_X R)(U,V)W\big)\eta(Y) \\
&+\left(q - \frac{d}{2n}\right)\big[(\nabla_X S)(V,W)\eta(U) - (\nabla_X S)(U,W)\eta(U)\big]\eta(Y) \\
&+q\big[g(V,W)\eta\big((\nabla_X Q)U\big) - g(U,W)\eta\big((\nabla_X Q)V\big)\eta(Y) \\
&-\frac{dr(X)(p+4nq)}{2n(2n+1)}\big[g(V,W)\eta(U) - g(,W)\eta(V)\big]\eta(Y) = 0.
\end{aligned} \tag{3.2}$$

By considering $U = Y = e_i$ in (3.2), where $\{e_i, i = 1, 2, 3, 4, \cdots, 2n+1\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, we get

$$\begin{aligned}
&\left\{q - p - d - (2n-1)\left(q - \frac{d}{2n}\right)\right\}(\nabla_X S)(V,W) + \big\{-qg\big((\nabla_X Q)e_i, e_i\big) \\
&+\frac{2n-1}{2n(2n+1)}(p+4nq)dr(X) + q\eta\big((\nabla_X Q)e_i\big)\eta(e_i)\big\}g(V,W) \\
&+(p+d)\eta\big((\nabla_X R)(e_i,V)W\big)\eta(e_i) - \left(q - \frac{d}{2n}\right)(\nabla_X S)(\xi,W)\eta(V) \\
&-q\eta\big((\nabla_X Q)V\big)\eta(W) + \frac{dr(X)}{2n(2n+1)}(p+4nq)\eta(V)\eta(W) = 0.
\end{aligned} \tag{3.3}$$

On plugging $W = \xi$ in (3.3) and then by using (2.1), we obtain

$$\left(p + (2n-1)q + \frac{d}{2n}\right)(\nabla_X S)(V,\xi) = \left(\frac{p+(2n-1)q}{2n+1}\right) dr(X)\eta(V). \tag{3.4}$$

Again plugging $V = \xi$ in (3.4), we get $dr(X) = 0$ i.e., r is constat provided $p + (2n-1)q \neq 0$.

Thus we can able to state the following result:

Theorem 3.1. *A globally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold is a space of constant scalar curvature provided the pseudo-quasi conformal curvature tensor always reduces to either concircular curvature tensor or quasi-conformal curvature tensor* $(p + (2n-1)q \neq 0)$.

Next we consider the manifold of constant scalar curvature r, then from (3.4) it follows that

$$(\nabla_X S)(V,\xi) = 0, \text{ provided } \left(p + (2n-1)q + \frac{d}{2n}\right) \neq 0. \tag{3.5}$$

In view of (2.2), (2.8) and (2.9), equation (3.5) turns into

$$k\big[g(X,\phi V) + g(hX,\phi V)\big] = 0. \tag{3.6}$$

Which implies either $k = 0$ or

$$g(X,\phi V) + g(hX,\phi V) = 0. \tag{3.7}$$

Replacing X by ϕX in (3.6) and then by using (2.5), we obtain

$$S(\phi X,\phi V) = o.$$

Now taking covariant derivative above expression with respect to Z gives

$$(\nabla_Z S)(\phi X,\phi V) = o.$$

This leads us to the following result:

Theorem 3.2. *An $N(k)$-contact metric manifold is globally ϕ-pseudo quasi-conformally symmetric, then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ [2] or the manifold admits an η parallel Ricci tensor provided the pseudo-quasi conformal curvature tensor never reduces to conformal curvature tensor, that is:* $\left(\left(p + (2n-1)q + \frac{d}{2n}\right) \neq 0\right)$.

Finally, we describe a globally ϕ-pseudo-quasi-conformally symmetric Einstein $N(k)$-contact metric manifold, that is,

$$S(U,V) = \alpha g(U,V),$$

for all $U,V \in \chi(M)$, where α is constant and $QU = \alpha U$. Then it follows from (1.1) that

$$\begin{aligned}\tilde{C}(U,V)W &= (p+d)R(U,V)W \\ &+\left\{\left(2q-\frac{d}{2n}\right)\alpha-\frac{r(p+4nq)}{2n(2n+1)}\right\}[g(V,W)U-g(U,W)V].\end{aligned} \tag{3.8}$$

Now taking covariant differentiation of (3.8) along X, we get

$$\begin{aligned}(\nabla_X\tilde{C})(U,V)W &= (p+d)(\nabla_X R)(U,V)W \\ &\quad -dr(X)\left[\frac{p+4nq}{2n(2n+1)}\right][g(V,W)U-g(U,W)V].\end{aligned} \tag{3.9}$$

BY employing ϕ^2 on both sides of (3.9), we obtain

$$\begin{aligned}\phi^2\big((\nabla_X\tilde{C})(U,V)W\big) &= (p+d)\phi^2\big((\nabla_X R)(U,V)W\big) \\ &-\frac{(p+4nq)dr(X)}{2n(2n+1)}[g(V,W)\phi^2U-g(U,W)\phi^2V].\end{aligned} \tag{3.10}$$

Since the manifold is Einstein, the scalar curvature r is always constant and so $dr(W)=0$. Hence the equation (3.10) turns into

$$\phi^2\big((\nabla_X\tilde{C})(U,V)W\big) = (p+d)\phi^2\big((\nabla_X R)(U,V)W\big). \tag{3.11}$$

This leads us to the following result:

Theorem 3.3. *An Einstein $N(k)$-contact metric manifold is globally ϕ-pseudo-quasi-conformally symmetric if and only if it is ϕ-symmetric provided $p+d \neq 0$.*

4. 3-Dimensional Locally ϕ-pseudo-Quasi-Conformally Symmetric $N(k)$-Contact Metric Manifold

Let us consider a 3-dimensional locally ϕ-pseudo-quasi-conformally symmetric $N(k)$-contact metric manifold, that is: $\phi^2\big((\nabla_X\tilde{C})(U,V)W\big) = 0$, where $U,V,W,X \in \chi(M)$ and orthogonal ξ.

By considering (2.11) and (2.10) in (1.1), we get

$$\begin{aligned}
\tilde{C}(U,V)W &= \left(\frac{r}{6}-k\right)\left(2p+2q+\frac{3d}{2}\right)[g(V,W)U-g(U,W)V] \\
&+\left(3k-\frac{r}{2}\right)(p+q+d)[g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi] \\
&+\left(3k-\frac{r}{2}\right)\left(p+2q-\frac{d}{2}\right)[\eta(V)\eta(W)U-\eta(U)\eta(W)V].
\end{aligned} \tag{4.1}$$

Now taking covariant differentiation of (4.1) along X, yields

$$\begin{aligned}
(\nabla_X\tilde{C})(U,V)W &= \frac{dr(X)}{6}\left(2p+2q+\frac{3d}{2}\right)[g(V,W)U-g(U,W)V] \\
&-\frac{dr(X)(p+q+d)}{2}[g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi] \\
&-\frac{dr(X)}{2}\left(p+2q-\frac{d}{2}\right)[\eta(V)\eta(W)U-\eta(U)\eta(W)V] \\
&+\left(3k-\frac{r}{2}\right)(p+q+d)\big[g(V,W)(\nabla_X\eta)(U)\xi \\
&+g(V,W)\eta(U)\nabla_X\xi-g(U,W)(\nabla_X\eta)(V)\xi \\
&-g(U,W)\eta(V)\nabla_X\xi\big] \\
&+\left(3k-\frac{r}{2}\right)\left(p+2q-\frac{d}{2}\right)\big[(\nabla_X\eta)(V)\eta(W)U \\
&+(\nabla_X\eta)(W)\eta(V)U-(\nabla_X\eta)(U)\eta(W)V \\
&-(\nabla_X\eta)(W)\eta(U)V\big].
\end{aligned} \tag{4.2}$$

Since X,Y, Z and W orthogonal to ξ, we have from (4.2) that

$$\begin{aligned}
(\nabla_X\tilde{C})(U,V)W &= \frac{dr(X)}{6}\left(2p+2q+\frac{3d}{2}\right)[g(V,W)U-g(U,W)V] \\
&+\left(3k-\frac{r}{2}\right)(p+q+d)\big[g(V,W)(\nabla_X\eta)(U)\xi \\
&-g(U,W)(\nabla_X\eta)(V)\xi\big].
\end{aligned} \tag{4.3}$$

Operating ϕ^2 on both sides of (4.3), gives

$$\begin{aligned}
\phi^2\big((\nabla_X\tilde{C})(U,V)W\big) &= \frac{dr(X)(4p+4q+3d)}{12}[g(V,W)\phi^2U \\
&-g(U,W)\phi^2V\big].
\end{aligned} \tag{4.4}$$

Since $\phi^2\big((\nabla_X\tilde{C})(U,V)W\big)=0$, we have $dr(W)=0$ provided $\big(4p+4q+3d\big)\neq 0$ and hence the scalar curvature r is constant.

Conversely, if the scalar curvature r is constant i.e., $dr(W)=0$ then from (4.4), we get $\phi^2\big((\nabla_X\tilde{C})(U,V)W\big)=0$.
Thus, we can state the following theorem:

Theorem 4.1. *A 3-dimensional $N(k)$-contact metric manifold is locally ϕ-pseudo quasi-conformally symmetric if and only if the scalar curvature is constant.*

5. 3-Dimensional Pseudo-Quasi-Conformally ϕ-recurrent $N(k)$-contact metric Manifold

Definition 5.1. *An $N(k)$-contact metric manifold is said to be pseudo-quasi-conformally ϕ-recurrent if*

$$\phi^2\big((\nabla_W\tilde{C})(X,Y)Z\big)=A(W)\tilde{C}(X,Y)Z, \tag{5.1}$$

holds for all $X,Y,Z,W\in\chi(M)$.

In particular if $A(W)=0$, then pseudo-quasi-conformally ϕ-recurrent $N(k)$-contact metric manifold reduces to pseudo-quasi-conformally ϕ-symmetric.
Let us consider an pseudo-quasi conformally ϕ-recurrent $N(k)$-contact metric manifold, then it follows from (5.1) that

$$-(\nabla_X\tilde{C})(U,V)W+\eta\big((\nabla_X\tilde{C})(U,V)W\big)\xi=A(X)\tilde{C}(U,V)W, \tag{5.2}$$

from which we can easily seen that

$$\begin{aligned}&-g\big((\nabla_X\tilde{C})(U,V)W,Y\big)+\eta\big((\nabla_X\tilde{C})(U,V)W\big)\eta(Y)\\&=A(X)g\big(\tilde{C}(U,V)W,Y\big).\end{aligned} \tag{5.3}$$

By considering the expressions (1.1), (2.8) and (2.10) in (5.3) and then contracting over U and Y, we get

$$\begin{aligned}&-(4p+4q+3d)\bigg(\frac{dr(X)}{6}\bigg)g(V,W)\\&+\bigg(3k-\frac{r}{2}\bigg)\big(-p-2q+\frac{d}{2}\big)(\nabla_X\eta)(V)\eta(W)\\&+\bigg(3k-\frac{r}{2}\bigg)(-p-3q+2d)(\nabla_X\eta)(W)\eta(V)\end{aligned} \tag{5.4}$$

$$+\left(\frac{dr(X)}{6}\right)\big(2p+2q+\frac{3d}{2}\big)[g(V,W)-\eta(V)\eta(W)]$$
$$=A(X)\Big\{\big(\frac{r}{2}-k\big)(4p+4q+3d)g(V,W)$$
$$+\big(3k-\frac{r}{2}\big)(2p+4q-d)\eta(V)\eta(W)$$
$$+\left(3k-\frac{r}{2}\right)(p+q+d)[g(V,W)-\eta(V)\eta(W)]\Big\}.$$

On plugging $W=\xi$, above equation turns into

$$-(4p+4q+3d)\left(\frac{dr(X)}{6}\right)\eta(V)+\left(3k-\frac{r}{2}\right)\big(-p-2q+\frac{d}{2}\big)(\nabla_X\eta)(V)$$
$$=A(X)\Big\{\big(\frac{r}{2}-k\big)(4p+4q+3d)+\big(3k-\frac{r}{2}\big)(2p+4q-d)\Big\}\eta(V). \tag{5.5}$$

Again plugging $V=\xi$ in (5.5), we obtain

$$A(X)=\frac{-(4p+4q+3d)}{6[r(p+2d)+2k(p+4q-3d)]}dr(X). \tag{5.6}$$

This leads us to the following result:

Theorem 5.1. *In a 3-dimensional pseudo-quasi conformally ϕ-recurrent $N(k)$-contact metric manifold, the 1-form A is given by the expression (5.6).*

If we consider the manifold of constant scalar curvature r, then $dr(W)=0$. Hence from equation (5.6), we have

$$A(W)=0. \tag{5.7}$$

By using (5.7) in (5.1), we get

$$\phi^2\big((\nabla_W\tilde{C})(X,Y)Z\big)=0. \tag{5.8}$$

Thus we can state the following theorem:

Theorem 5.2. *A 3-dimensional pseudo-quasi conformally ϕ-recurrent $N(k)$-contact metric manifold with constant scalar curvature always turns into an pseudo-quasi conformally ϕ-symmetric manifold.*

Conversely, we assume that if the pseudo-quasi conformally ϕ-recurrent $N(k)$-contact metric manifold becomes pseudo-quasi conformally ϕ-symmetric manifold i.e., $A(W)=0$, then we have obtained from the expression (5.6) that

$dr(X) = 0$ which implies that r is constant.
This leads us to the following theorem:

Theorem 5.3. *If a 3-dimensional pseudo-quasi conformally ϕ-recurrent $N(k)$-contact metric manifold is pseudo-quasi conformally ϕ-symmetric, then the manifold is of constant scalar curvature r.*

Now from Theorem 5.3. and Theorem 5.4., we can able to conclude that:

Theorem 5.4. *A 3-dimensional pseudo-quasi conformally ϕ-recurrent $N(k)$- contact metric manifold is pseudo-quasi conformally ϕ-symmetric if and only if the manifold is of constant scalar curvature r.*

6. EXAMPLES

Example 1. *Let us consider 3-dimensional Riemannian manifold $M = \{(x_1, x_2, x_3) \in R^3\}$, where (x_1, x_2, x_3) is the standard coordinate in R^3. Let E_1, E_2, E_3 are the vector fields in R^3 satisfying the expressions*

(6.1) $$[E_1, E_2] = (1-\alpha)E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = (1+\alpha)E_2,$$

where $\alpha \neq \pm 1$ is a real number.

Let g be the Riemannian metric described by

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$
$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0.$$

Let η be the 1-form associating with the vector field $\xi = E_1$ is given by

(6.2) $$\eta(W) = g(W, E_1),$$

for any $W \in \chi(M)$. Let ϕ be the (1,1)-tensor field given by

(6.3) $$\phi(E_1) = 0, \qquad \phi(E_2) = E_3, \qquad \phi(E_3) = -E_2.$$

Moreover, we also have the following conditions:

$$hE_1 = 0, \qquad hE_2 = \alpha E_2, \qquad hE_3 = -\alpha E_3$$

By using Koszul's formula, we can easily find the following:

$$\begin{array}{ccc} \nabla_{E_1} E_1 = 0, & \nabla_{E_1} E_2 = 0, & \nabla_{E_1} E_3 = 0, \\ \nabla_{E_2} E_1 = -(1+\alpha)E_3, & \nabla_{E_2} E_2 = 0, & \nabla_{E_2} E_3 = (1+\alpha)E_1, \\ \nabla_{E_3} E_1 = (1-\alpha)E_1, & \nabla_{E_3} E_2 = -(1-\alpha)E_1, & \nabla_{E_3} E_3 = 0. \end{array}$$

By considering above expressions, we found that

$$\nabla_X \xi = -\phi X - \phi h X, \quad \textit{for} \quad E_1 = \xi.$$

Therefore, the manifold is said to be a contact metric manifold with the contact metric structure (ϕ, ξ, η, g).

Now the components of Riemannian Curvature tensor are given by

$$R(E_1, E_2)E_2 = (1-\alpha^2)E_1, \qquad R(E_1, E_3)E_3 = (1-\alpha^2)E_1,$$
$$R(E_1, E_2)E_1 = -(1-\alpha^2)E_2, \qquad R(E_2, E_3)E_3 = -(1-\alpha^2)E_2,$$
$$R(E_3, E_1)E_1 = (1-\alpha^2)E_3, \qquad R(E_3, E_2)E_2 = -(1-\alpha^2)E_3.$$

By virtue of above relations of the Riemannian curvature tensor, we conclude that the manifold is a $N(1-\alpha^2)$*-contact metric manifold. Hence the Ricci tensor* S *is given by*

$$S(E_1, E_1) = 2(1-\alpha^2), \qquad S(E_2, E_2) = 0, \qquad S(E_3, E_3) = 0.$$

Hence the scalar curvature of the manifold is obtained as

$$r = S(E_1, E_1) + S(E_2, E_2) + S(E_3, E_3) = 2(1-\alpha^2),$$

which is always constant. Thus from the Theorem 4.1. and Theorem 5.5., we conclude that the given three dimensional $N(k)$*-contact metric manifold is Locally* ϕ*-pseudo-Quasi-Conformally symmetric and pseudo-quasi conformally* ϕ*-recurrent.*

Example 2. *Next we consider 3-dimensional Riemannian manifold* $M = \{(x_1, x_2, x_3) \in R^3\}$*, where* (x_1, x_2, x_3) *is the standard coordinate in* R^3*. Let* E_1*,* E_2*,* E_3 *are the vector fields in* R^3 *satisfying the expressions*

$$[E_1, E_2] = 2E_3 + \frac{2}{x_1}E_1, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 0. \tag{6.4}$$

Let g be the Riemannian metric described by

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$
$$g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0.$$

Let η *be the 1-form associating with the vector field* $\xi = E_3$ *is given by*

$$\eta(W) = g(W, E_3), \tag{6.5}$$

for any $W \in \chi(M)$*. Let* ϕ *be the (1,1)-tensor field given by*

$$\phi(E_1) = E_2, \qquad \phi(E_2) = -E_1, \qquad \phi(E_3) = 0. \tag{6.6}$$

Moreover, we also have the following conditions:

$$hE_1 = -E_1, \qquad hE_2 = E_2, \qquad hE_3 = 0.$$

By using Koszul's formula, we can easily find the following:

$$\begin{array}{ccc} \nabla_{E_1} E_1 = -\frac{2}{x_1} E_2, & \nabla_{E_1} E_2 = \frac{2}{x_1} E_1, & \nabla_{E_1} E_3 = 0, \\ \nabla_{E_2} E_1 = -2E_3, & \nabla_{E_2} E_2 = 0, & \nabla_{E_2} E_3 = 2E_1, \\ \nabla_{E_3} E_1 = 0, & \nabla_{E_3} E_2 = 0, & \nabla_{E_3} E_3 = 0. \end{array}$$

By considering above expressions, we found that

$$\nabla_X \xi = -\phi X - \phi h X, \quad \textit{for} \quad E_3 = \xi.$$

Therefore, the manifold is said to be a contact metric manifold with the contact metric structure (ϕ, ξ, η, g).

Now the components of Riemannian Curvature tensor are given by

$$\begin{array}{ll} R(E_1, E_2)E_2 = \dfrac{4}{x_1} E_3 - \dfrac{4}{x_1^2} E_1, & R(E_1, E_3)E_3 = 0, \\ R(E_1, E_2)E_1 = \dfrac{4}{x_1^2} E_2, & R(E_2, E_3)E_3 = 0, \\ R(E_3, E_1)E_1 = 0, & R(E_3, E_2)E_2 = \dfrac{4}{x_1} E_1. \end{array}$$

By virtue of above relations of the Riemannian curvature tensor, we conclude that the manifold is a $N\left(-\frac{4}{x_1}\right)$-contact metric manifold. Hence the Ricci tensor S is given by

$$S(E_1, E_1) = -\frac{4}{x_1^2}, \qquad S(E_2, E_2) = -\frac{4}{x_1^2}, \qquad S(E_3, E_3) = 0.$$

Hence the scalar curvature of the manifold is obtained as

$$r = S(E_1, E_1) + S(E_2, E_2) + S(E_3, E_3) = -\frac{8}{x_1^2},$$

which is always constant. Thus from the Theorem 4.1. and Theorem 5.5., we conclude that the given three dimensional $N(k)$-contact metric manifold is locally ϕ-pseudo-quasi-conformally symmetric and pseudo-quasi conformally ϕ-recurrent.

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