

ITERATIVE AGGREGATION-DISAGGREGATION METHOD FOR A TRANSIENT HEAT CONDUCTION EQUATION OF COPPER

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ABSTRACT. In this paper we study the behavior of iterative aggregation-disaggregation method for a system of differential equations resulting from discretisation of one dimensional transient heat conduction equation of copper using finite difference method. For that, we apply the Backward-Euler method to the system previously obtained and associate a fixed point application. We define the iterative aggregation-disaggregation method and study its behavior. The Gauss-Seidel and SOR variants are presented. A numerical study of these methods is given at the end to complete this work.

1. INTRODUCTION

Mathematical models arising in many branches of science and engineering very often are expressed in terms of partial differential equations (PDEs). These are classified as elliptic, hyperbolic, and parabolic. The heat transfer problem is a class of PDE which plays a very important role in many interpretation of physical phenomena and which is among the most studied problems in research and

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teaching (see [2] and a references cited). In this article, we propose a numerical technique to obtain the solution of the transient heat conduction equation of copper. The choice of copper is due to its many characteristics such as its ability to conduct heat as well as electrical conductivity. It is a metal that is flexible and often used in our society [6], [11]. Another candidate for the proposed technique is the bio-heat transfer problem, which describes the exchange magnitude of heat transfer between tissue and blood. This problem is initially proposed by Pennes [15] and widely used to solve the temperature distribution for thermal therapy [13, 19, 20]. The human body involves multiple internal physical and physiological phenomena but also interactions with the environment by preserving or transmitting heat. Often this transmission is between blood and tissue.

The majority of heat transfer problem in engineering practice are transient in nature, and we are often led to make a discretization in time and space, which gives solutions at different times. In this work, we limit our interest to a heat conduction equation of copper using finite difference method for space discretization and the Backward-Euler method for time discretization. Therefore, we have to solve an algebraic system at each time step, either with direct or iterative methods.

The iterative aggregation-disaggregation (IAD) method is an efficient tool for solving linear systems, computing the stationary distribution of a finite Markov chain and eigenvalue problems [3, 4, 8]. The basic idea is that at each step of these methods, the linear system is replaced with a smaller system which called restriction or aggregation step, this smaller linear system is solved and its solution is used to improve the current iterate in the original system which called prolongation or disaggregation step. It is a method in which the principle is the exchange of information from distant parts of the system in a single step instead of propagating it through the iterations. The idea of aggregation appeared naturally in input-output economic models [17], and was extended to other linear systems and the problem of finding the stationary distribution of Markov chains; see [10]. In [18], the authors give a relationship between the IAD method and the two-stage multigrid method, and study their convergences. And in [1], to give the mathematical modelling of safety related systems respecting some European Standards for the railway transport, the authors use Markov's models to meet these standards. In their modeling, they use the aggregation/disaggregation method to compute some characteristics of Markov

chains. Our aim is to extend the application of the IAD method to the class of heat transfer problem.

This paper is organized as follows. In Section 2 we give a problem statement for solving a simple heat transfer equation by using finite differences and backward-Euler discretization methods in space and time. In Section 3 we define the IAD method for this problem type and study there convergence and some variants are also considered. In Section 4, to compare the different methods we give some numerical results to solve the heat conduction equation of copper. Finally, the findings are summarized in the conclusion.

2. PROBLEM STATEMENT

For the transient problem, the one dimensional time dependent governing differential equation is used as the basic mathematical model for the heat transfer which is given by

$$(2.1) \quad \rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + Q,$$

where T is a temperature, ρ the density, c the heat capacity, k thermal conductivity, x distance and t the time. If the thermal conductivity, density and heat capacity are constant over the model domain, the equation can be simplified to

$$(2.2) \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

where $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity and for the copper metal $\alpha = 1.14$. The boundary and initial conditions are expressed as:

$$(2.3) \quad \begin{cases} T(0, t) = T_a & x = 0, \forall t > 0, \\ T(L, 0) = T_b & x = L, \forall t > 0, \\ T(0, t) = f(x) & 0 < x < L, t = 0. \end{cases}$$

The first step in the discretization procedure is to replace the domain $[0, L] \times [0, t_f]$ by a set of mesh points. Here we apply equally spaced mesh points

$$x_i = i\Delta x = ih, \quad i = 0, \dots, N,$$

and

$$t_k = k\Delta t = k\tau, \quad k = 0, \dots, n.$$

Moreover, T_i^k denotes the mesh function that approximates $T(x_i, t_k)$ for $i = 0, \dots, N$ and $k = 0, \dots, n$. Requiring the PDE (2.1) to be fulfilled at a mesh point (x_i, t_k) leads to the equation.

Using (2.3) and the Backward-Euler method, the problem (2.2) can be written in the following form : for $k = 0, \dots, n-1$,

$$(2.4) \quad \frac{T^{k+1} - T^k}{\tau} = BT^{k+1},$$

where B is the corresponding matrix to a finite difference tree point discretization applied to problem (2.1). In this case, the matrix B have the form

$$(2.5) \quad B = \frac{\alpha}{h^2} \begin{pmatrix} -2 & 1 & & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}.$$

Let decompose B as:

$$B = D + L + U,$$

where D , L and U are the diagonal, upper and lower matrices of B . Then, the problem (2.4) can be written as:

$$(2.6) \quad (I - \tau D)T^{k+1} = \tau(L + U)T^{k+1} + T^k.$$

We associate a fixed point mapping F to the previous equation, defined by: for $k = 0, \dots, n-1$

$$(2.7) \quad \begin{cases} T^{p+1,k+1} = F(T^{p,k+1}) \\ T^{0,k+1} = T^k \end{cases},$$

such that

$$(I - \tau D)T^{p+1,k+1} = \tau(L + U)T^{p,k+1} + T^k,$$

then

$$\begin{aligned} T^{p+1,k+1} &= (I - \tau D)^{-1} \tau(L + U)T^{p,k+1} + (I - \tau D)^{-1} T^k \\ &= AT^{p,k+1} + b = F(T^{p,k+1}), \end{aligned}$$

with

$$(2.8) \quad \begin{cases} A = (I - \tau D)^{-1} \tau(L + U) \\ b = b(T^k) = (I - \tau D)^{-1} T^k \end{cases}.$$

Let:

- T^* is the unique solution of (2.2) upon the time interval $[0, T]$,
- $\bar{T}$ be an approximation of $T^*(t)$, $\forall t \in [0, t_f]$.

3. AGGREGATION-DISAGGREGATION METHOD

The idea of the aggregation-disaggregation method is as follow: consider a partition $g = \{\dots, g_j, \dots\}$, where $j \in \{1, \dots, m\}$ of $\{1, \dots, N\}$ ($m < N$); the subsets g_j being disjoint and each one being not empty. We introduce the linear or affine mappings:

$$(3.1) \quad \begin{cases} R \in \mathcal{L}(\mathbb{R}^N; \mathbb{R}^m) \text{ is a restriction or aggregation mapping.} \\ P_{\bar{u}} \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^N) \text{ or a } (\mathbb{R}^m; \mathbb{R}^N) \\ \quad \text{(the set of affine mapping from } \mathbb{R}^m \text{ to } \mathbb{R}^N) \text{ is a prolongation} \\ \quad \text{or disaggregation mapping depending on } \bar{T} \end{cases}$$

satisfying the following two properties:

- (1) $R \circ P_{\bar{T}} = I$ (the identity matrix).
- (2) $\Pi_{\bar{T}} = P_{\bar{T}} \circ R$ is a projection ($\Pi_{\bar{T}}^2 = \Pi_{\bar{T}}$).

The discretization matrix B is such that $B_{i,i} < 0$ and $B_{i,j} \geq 0$ for $i, j = 1, \dots, N$; then A defined in (2.8) is a nonnegative operator. Let $e = (1, \dots, 1)^t \in \mathbb{R}^N$ (where $(\cdot)^t$ design the transpose vector), and for $v \in \mathbb{R}^N$, we define the operator R by $Rv = e^t v$ and the operator $P_{\bar{T}} = \frac{\bar{T}}{e^t \bar{T}}$. And for $v > 0$, we define the vectorial norm by:

$$\|B\|_v = \inf \{ \beta / v^t |B| \leq \beta v^t \},$$

$|B|$ absolute value of components of B . ($\|x\|_v = v^t |x|$ is a norm on L^1 with weight vector v).

Algorithm 1 (Aggregation-disaggregation). For $k = 0, \dots, n-1$, given an initial condition $u^{0,k+1}$ and a convergence parameter ε ,

- (1) Solve the aggregated equation

$$(3.2) \quad z^p - RAP_{T^p, k+1} z^p = Rb(T^k).$$

- (2) Disaggregatte and iterate according to the formula

$$(3.3) \quad T^{p+1, k+1} = AP_{T^p, k+1} z^p + b(T^k).$$

(3) Test if $\|u^{p+1,k+1} - u^{p,k+1}\| < \varepsilon$ (or other stopping criteria). If yes, break; otherwise $k := k + 1$ and go to step (1).

In order to study the algorithm convergence, we can write the original problem (2.7), for $k = 0, \dots, n$ and $p = 1, 2, 3, \dots$, in the form

$$(3.4) \quad v^{p+1} = Av^p + b,$$

where $\forall p > 0$, $u^{p,k+1} = v^p$, and the aggregation-disaggregation system in the form

$$(3.5) \quad v^{P+1} = AS(v^p) + b,$$

where

$$S(v^P) = P_{v^p}(I - RAP_{v^p})^{-1}Rb.$$

Using the properties of operators R , P , and as $T^{*,k+1}$ solution of problem then $T^{*,k+1} - AT^{*,k+1} = b$ and $S(T^{*,k+1}) = P_{T^{*,k+1}}(I - RAP_{T^{*,k+1}})^{-1}Rb = T^{*,k+1}$.

Theorem 3.1. For $k = 0, \dots, n - 1$, starting with $v^0 = T^{0,k+1}$, for $p = 1, 2, 3, \dots$, the fixed point iteration: $F(v^p) = v^{p+1}$ such that

$$v^{p+1} = AS(v^p) + b$$

is locally convergent ie $\exists \mathcal{U}_{T^{*,k+1}}$ such that $\forall v^0 \in \mathcal{U}_{T^{*,k+1}}$, the sequence (v^p) converge to $T^{*,k+1}$.

Proof. The matrix A can be expressed as follow:

$$A = \frac{\tau\alpha}{h^2 + 2\tau\alpha} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & & 1 & 0 \end{pmatrix}.$$

Let consider $m = 1$, then

$$\begin{cases} A \geq 0 \\ e^t A = \|A\|_e = \frac{\tau\alpha}{h^2 + 2\tau\alpha} = \beta < 1 \end{cases}.$$

As the operators R defined by e^t and P_{v^p} by $\frac{v^p}{e^t v^p}$, then

$$RP_{v^p} = 1,$$

and

$$\Pi_{v^p} = P_{v^p} R = \frac{v^p e^t}{e^t v^p} ; \quad \Pi_{v^p}^2 = \Pi_{v^p}.$$

Therefore, the aggregate equation is given by

$$R(I - A)P_{v^p} z^p = Rb = e^t b = e^t (I - A) \frac{v^p}{e^t v^p} z^p,$$

then

$$z^p = \frac{e^t b}{1 - \frac{e^t A v^p}{e^t v^p}},$$

and

$$S(v^p) = P_{v^p} z^p = \frac{v^p}{e^t v^p} z^p.$$

Substitute

$$z^p = Rv^p - y$$

$$R(I - A)P_{v^p}(Rv^p - y) = Rb$$

and

$$\begin{aligned} S(v^p) &= v^p - P_{v^p} y \\ &= v^p - P_{v^p} ((R(I - A)P_{v^p})^{-1} R(I - A)P_{v^p} R\bar{v}^{k+1} - (R(I - A)P_{v^p})^{-1} Rb) \\ &= v^p - \frac{\Pi_{v^p}(I - A)\Pi_{v^p}}{1 - \frac{e^t A v^p}{e^t v^p}} v^p + \frac{\Pi_{v^p} b}{1 - \frac{e^t A v^p}{e^t v^p}} \\ &= v^p - \frac{\Pi_{v^p}(I - A)v^p}{1 - \frac{e^t A v^p}{e^t v^p}} + \frac{\Pi_{v^p} b}{1 - \frac{e^t A v^p}{e^t v^p}} \\ &= \left(I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right) \bar{v}^{k+1} + \frac{\Pi_{v^p} b}{1 - \frac{e^t A v^p}{e^t v^p}}. \end{aligned}$$

We know that A verify

$$\left[\begin{array}{l} \frac{e^t A v^p}{e^t v^p} \leq \beta \quad \text{ie } \frac{\|A v^p\|_e}{\|v^p\|_e} \leq \beta \\ 1 - \frac{e^t A v^p}{e^t b} \leq 1 - \beta \end{array} \right],$$

also

$$S(v^p) = \left(I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right) v^p + \frac{\Pi_{v^p} b}{1 - \frac{e^t A v^p}{e^t v^p}},$$

then

$$AS(v^p) + b = J(v^p)v^p + N(v^p)b,$$

with

$$\begin{aligned} J(v^p) &= A \left(I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right), \\ N(v^p) &= I + \frac{A \Pi_{v^p}}{1 - \frac{e^t A v^p}{e^t v^p}}. \end{aligned}$$

Let

$$\begin{aligned} T^{*,k+1} &= J(v^p)T^{*,k+1} + N(v^p)b \\ J(v^p)u^{*,k+1} + N(v^p)b &= A \left(I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right) u^{*,k+1} + \frac{A \Pi_{v^p}}{1 - \frac{e^t A v^p}{e^t v^p}} b + b \\ &= A u^{*,k+1} - \frac{A \Pi_{v^p}}{1 - \frac{e^t A v^p}{e^t v^p}} b + \frac{A \Pi_{v^p}}{1 - \frac{e^t A v^p}{e^t v^p}} b + b \\ &= u^{*,k+1}, \end{aligned}$$

then

$$F(v^p) - F(T^{*,k+1}) = AS(v^p) + b - T^{*,k+1} = J(v^p)(v^p - T^{*})$$

$$\begin{aligned} \|J(v^p)\|_e &= \left\| A \left(I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right) \right\|_e \\ &\leq \|A\|_e \left\| I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right\|_e \end{aligned}$$

$$\begin{aligned} e^t \left(I - \frac{P_{v^p} e^t (I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right) &= e^t - \frac{e^t (I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} = e^t - \frac{e^t - e^t A}{1 - \frac{e^t A v^p}{e^t v^p}} \\ &\leq e^t - \frac{e^t - e^t A}{1 - \beta}. \end{aligned}$$

However,

$$e^t A \leq e^t,$$

then

$$e^t - e^t A \geq 0,$$

and therefore

$$\frac{e^t - e^t A}{1 - \beta} \geq 0.$$

Then

$$e^t - \frac{e^t - e^t A}{1 - \beta} \leq e^t,$$

and we can then deduct

$$\left\| I - \frac{\Pi_{v^p}(I - A)}{1 - \frac{e^t A v^p}{e^t v^p}} \right\|_e \leq 1,$$

and finally

$$\|J(v^p)\|_e \leq \|A\|_e \leq \beta < 1$$

and

$$\|F(v^p) - F(T^{*,k+1})\|_e \leq \beta \|v^p - T^{*,k+1}\|_e.$$

For $m > 1$, we follow the same previous steps, so we conclude, for $k = 0, 1, 2, \dots \exists B(T^{*,k+1}, r)$, $r > 0$ such that

$$\forall v^0 \in B(T^{*,k+1}, r), (v^p) \text{ converge to th solution } T^{*,k+1}.$$

□

3.1. Some IAD variants.

In this section, we give some variants of the IAD applied to heat transfer problem:

3.1.1. Gauss-Seidel IAD variant. For solving (2.4), we use an other decomposition as follow: For $k = 0, \dots, n - 1$,

$$(3.6) \quad (I - \tau(D + L))T^{k+1} = \tau U T^{k+1} + T^k + \tau K_3.$$

We associate a fixed point mapping T^{GS} to the previous equation, defined by: for $k = 0, \dots, n - 1$,

$$(3.7) \quad \begin{cases} T^{p+1,k+1} &= F^{GS}(T^{p,k+1}) \\ T^{0,k+1} &= T^k \end{cases},$$

where

$$F^{GS}(T^{p,k+1}) = AT^{p,k+1} + b(T^k),$$

with

$$(3.8) \quad \begin{cases} A = \tau(I - \tau(D + L))^{-1}U \\ b(T^k) = (I - \tau(D + L))^{-1}(T^k + \tau K_3) \end{cases}.$$

Lemma 3.1. For $k = 0, \dots, n - 1$, for $p = 1, 2, 3, \dots$, the fixed point iteration $F^{GS}(v^p) = v^{p+1}$, starting with v^0 , is locally convergent.

Proof. We use the same reasoning as that of the previous theorem, we easily show that the fixed point T^{GS} is locally convergent. $\square$

3.1.2. SOR IAD variant. Let $\omega \in]0, 1[$, and define the fixed point mapping F^{SOR} corresponding to the fixed point mapping T by:

$$(3.9) \quad \forall v \in \mathbb{R}^N, \quad F^{SOR}(v) = \omega v + (1 - \omega)F(v).$$

Lemma 3.2. *If T is locally convergent with respect to the norm $\|\cdot\|_e$, then T^{SOR} is also locally convergent with respect to the norm $\|\cdot\|_e$.*

Proof. By Theorem 3.1, $\exists B(T^{*,k+1}, r)$ such that

$$\|F(v^p) - F(T^{*,k+1})\|_e \leq \alpha \|v^p - T^{*,k+1}\|_e,$$

$\forall v^0 \in B(T^{*,k+1}, r)$. Then

$$\begin{aligned} & \|F^{SOR}(v^p) - F^{SOR}(T^{*,k+1})\|_e \\ &= \|\omega(v^p - T^{*,k+1}) + (1 - \omega)(F(v^p) - F(T^{*,k+1}))\|_e \\ &\leq \omega \|v^p - T^{*,k+1}\|_e + (1 - \omega) \|F(v^p) - F(T^{*,k+1})\|_e \\ &\leq \omega \|v^p - T^{*,k+1}\|_e + (1 - \omega)\beta \|v^p - T^{*,k+1}\|_e \\ &\leq (\omega + (1 - \omega)\beta) \|v^p - T^{*,k+1}\|_e \end{aligned}$$

then $(\omega + (1 - \omega)\beta) < 1$ and F^{SOR} is locally convergent. $\square$

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we give a numerical comparison of two examples of one dimensional heat conduction equation of copper with and without iterative aggregation-disaggregation method (IAD method). We use implicit finite difference method and backward-Euler discretization schemes developed as in Eq.(2.4), Eq.(3.7) and Eq.(3.9).

Example 1. *Let consider the heat conduction equation of the Copper:*

$$(4.1) \quad \begin{cases} \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} & 0 \leq x \leq 1, t \geq 0 \\ T(0, t) = T(1, t) = 0 & \forall t > 0, \\ T(x, 0) = 40 - 3x & 0 < x < 1, t = 0. \end{cases}$$

Example 2. Let consider the heat conduction equation of the Copper:

$$(4.2) \quad \begin{cases} \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} & 0 \leq x \leq 1, t \geq 0 \\ T(0, t) = T(1, t) = 0 & \forall t > 0, \\ T(x, 0) = -\sin(3\pi x) + \frac{1}{4}\sin(6\pi x) & 0 < x < 1, t = 0. \end{cases}$$

Using Matlab and applying the IAD method, then the result show as follows.

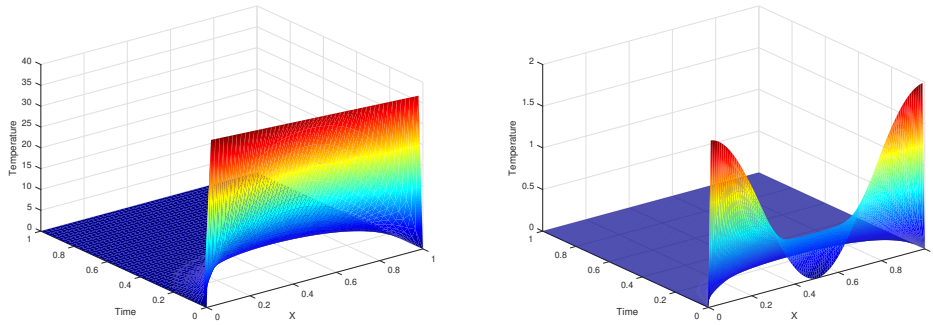


FIGURE 1. Temperature distributions at several times: left example1, right example2

We have noticed that all the figures associated with the Jacobi, Gauss-Seidel, and SOR variants coincide for the two examples treated here Figure1.

Figure2, gathers the variations of the iterations over time for the different methods proposed in this paper with and without the IAD method.

5. CONCLUSION

All the simulation part of this paper is carried out on Matlab. We have dealt with a one-dimensional transient heat conduction model. For the resolution of the thermal conduction equation of copper, we applied a finite difference discretization as well as an backward-Euler method. Next, we applied the iterative aggregation-disaggregation (IAD) method with Gauss-Seidel and SOR variants. The numerical examples used in the simulation showed the efficiency of the proposed method. We can therefore conclude, that the application of the IAD method makes it possible to accelerate the convergence, not only by a reduction in the number of iterations but also by reduction in execution time. Finally, we

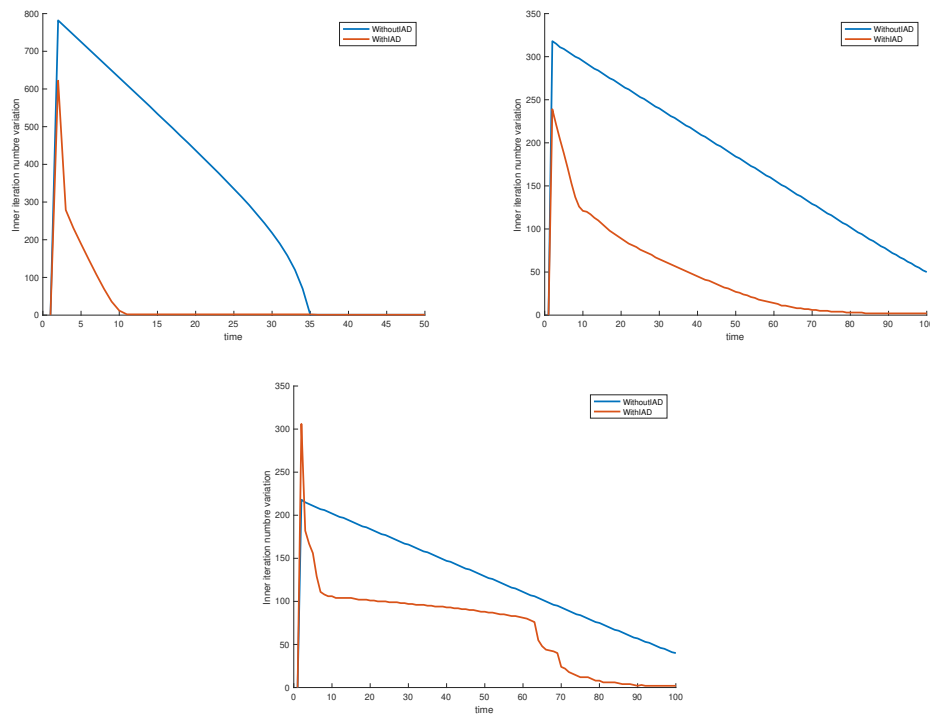


FIGURE 2. Comparison with and without IAD for example1; Variations of iterations over times: Jacobi, Gauss-Seidel and SOR ($w=1.5$).

note that this method is easy to implement and allows an acceleration of convergence and can be applied to a set of class of elliptic and parabolic problems.

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