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APPLICATION OF ELZAKI TRANSFORM TO VIBRATIONS IN MECHANICAL SYSTEM

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ABSTRACT. This paper is aimed to focus on the applicability of Elzaki Transform to system of Mechanical Vibrations. The applicability of Elzaki Transform to Undamped vibrations, Damped vibrations and Forced vibrations is demonstrated and verified by presenting some examples.

1. INTRODUCTION AND PRELIMINARIES

Many physical problems in science, engineering and technology are described by ordinary differential equations and partial differential equations and can be solved by using Integral transforms. There are many integral transforms like Laplace, Fourier, Hankel, Mellin etc. The choice of transform is decided by the nature of boundary conditions and the facility with which the transform can be inverted. Elzaki transform is one such powerful technique to solve differential equations with initial conditions. This transform has been effectively used in solving the liner differential equations [1], [2] and [3]. Elzaki transform is applicable to many fields of science and technology and many researchers [4], [5] [6], [7] and [8] are using this transform to solve various problems.

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Definition 1.1. *The Elzaki Transform of a function $f(t)$ is defined as $E[f(t)] = T(u) = u\int_0^\infty f(t)e^{-(\frac{t}{u})}$, $t > 0$, $u \in (-k_1, k_2)$ provided the integral exists where E is the Elzaki transform operator. If $E[f(t)] = T(u)$ then $f(t)$ is the inverse Elzaki transform of $T(u)$. The Elzaki transform of some of the standard functions are given in the following table.*

$f(t)$	$E[f(t)] = T(u)$
1	u^2
t	u^3
t^n	$n!u^{n+2}$
e^{at}	$\frac{u^2}{(1-au)}$
$sinat$	$\frac{au^3}{1+a^2u^2}$
$cosat$	$\frac{u^2}{1+a^2u^2}$
$sinhat$	$\frac{au^3}{1-a^2u^2}$
$coshat$	$\frac{u^2}{1-a^2u^2}$
$e^{at}sinbt$	$\frac{bu^3}{(1-au)^2+b^2u^2}$
$e^{at}cosbt$	$\frac{(1-au)u^2}{(1-au)^2+b^2u^2}$
$tsinat$	$\frac{2au^4}{(1+a^2u^2)^2}$

Application to Mechanical Vibrations: Vibrations occur when a physical system in stable equilibrium is disturbed. There are three types of vibrations. They are undamped or natural, damped and forced vibrations. These three types will be discussed in detail and solved using Elzaki transforms in the following sections.

2. Undamped Vibration:

When no external force acts on the body, the vibration is said to be natural or undamped. Let a cart of mass M is attached to a nearby wall by means of a spring. When the mass is attached to the spring and released, the spring will settle at some height. This position is the equilibrium position of the mass. If the mass pulled down or pushed up a little, the restoring force due to spring is $F_s = -kx$ where x is the displacement of the mass from its equilibrium position and $k > 0$ is the measure of stiffness of spring. By Newtons second law we get the equation of motion in the form of a differential equation using [9] given by

$$M\frac{d^2x}{dt^2} = F_s. \tag{2.1}$$

Then we get

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0. \tag{2.2}$$

It will be convenient to write this equation in the form

$$\frac{d^2x}{dt^2} + a^2x = 0, where \ a = \sqrt{\frac{k}{M}}. \tag{2.3}$$

If the cart is pulled to the position $x = x_0$ and released without any initial velocity, then we get the initial conditions in the form

$$x = x_0 \ and \ \frac{dx}{dt} = 0 \ when \ t = 0. \tag{2.4}$$

Applying Elzaki transform to equation (2.3) we get

$$\frac{T(u)}{u^2} - x_0 - u\frac{dx}{dt} + a^2T(u) = 0.$$

Using (2.4) we get $T(u)(\frac{1}{u^2} + a^2) - x_0 = 0$,

$$x = x_0E^{-1}\{\frac{u^2}{1+a^2u^2}\} = x_0cosat, \tag{2.5}$$

where E^{-1} is the inverse Elzaki transform operator. Equation (2.5) shows that x_0 is the amplitude of this simple harmonic vibration and its period is $T = 2\pi\sqrt{\frac{M}{k}}$.

3. Damped Vibrations

When the motion of a vibration is reduced by an external force, the vibration and its motion is said to be Damped. The damping force depends on the nature of the surrounding medium. If the cart is immersed in liquid, the magnitude of damping will be much greater and rate of dissipation of energy is much faster. The damping force is generally proportional to velocity and acts opposite to the direction of velocity. It is given by $F_d = -c\frac{dx}{dt}$ where $c > 0$ measures the resistance of the medium.Then equation (2.1) becomes

$$M\frac{d^2x}{dt^2} = F_s + F_d,$$
$$\frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = 0, \tag{3.1}$$

which can be conveniently written in the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = 0, \tag{3.2}$$

where $a = \sqrt{\frac{k}{M}}, b = \frac{c}{2M}$.

Applying Elzaki transform to equation (3.2) we get

$$\frac{T(u)}{u^2} - x_0 - u\frac{dx}{dt} + 2b[\frac{T(u)}{u} - ux_0] + a^2T(u) = 0.$$

Using (2.4) we get $T(u)(\frac{1}{u^2} + \frac{2b}{u} + a^2) - (1 + 2bu)x_0 = 0$,

$$x(t) = x_0E^{-1}\left\{\frac{(1 + 2bu)u^2}{(1 + 2bu + a^2u^2)}\right\}. \tag{3.3}$$

The solution of (3.2) is obtained by the nature of roots of the equation

$$1 + 2bu + a^2u^2 = 0. \tag{3.4}$$

At this stage we shall get three types of damping namely over damping, critically damping and under damping depending on the discriminant of (3.4) or depending on the values of a and b of (3.4). We shall discuss these three cases as follows.

3(a) Over damped case

Let $b > a$. This amounts to assuming that the frictional force due to viscosity is large compared to the stiffness of the spring. Let m_1, m_2 be the distinct roots of the equation (3.4). Then equation (3.3) becomes

$$x(t) = \frac{x_0}{m_1 - m_2}E^{-1}\{\frac{u^2}{u - m_1} - \frac{u^2}{u - m_2}\} = \frac{x_0}{m_1 - m_2}[\frac{-1}{m_1}e^{\frac{t}{m_1}} + \frac{1}{m_2}e^{\frac{t}{m_2}}] \tag{3.5}$$

Example 1. *Solve the differential equation* $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+4x = 0$ *with initial conditions* $x = 4$ *and* $\frac{dx}{dt} = -2$ *when* $t = 0$.

Solution: We shall find the solution of this initial value problem using Elzaki transform. Taking Elzaki transform on both sides of the given differential equation we get

$$\frac{T(u)}{u^2} - x_0 - u\frac{dx}{dt} + 5[\frac{T(u)}{u} - ux_0] + 4T(u) = 0.$$

Using the given initial conditions it becomes $T(u) = \frac{(4-22u)u^2}{(1+5u+4u^2)} = \frac{u^2}{3}[\frac{16}{(u+1)} - \frac{4}{(4u+1)}]$.

By taking inverse Elzaki transform we get $x(t) = \frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t}$.

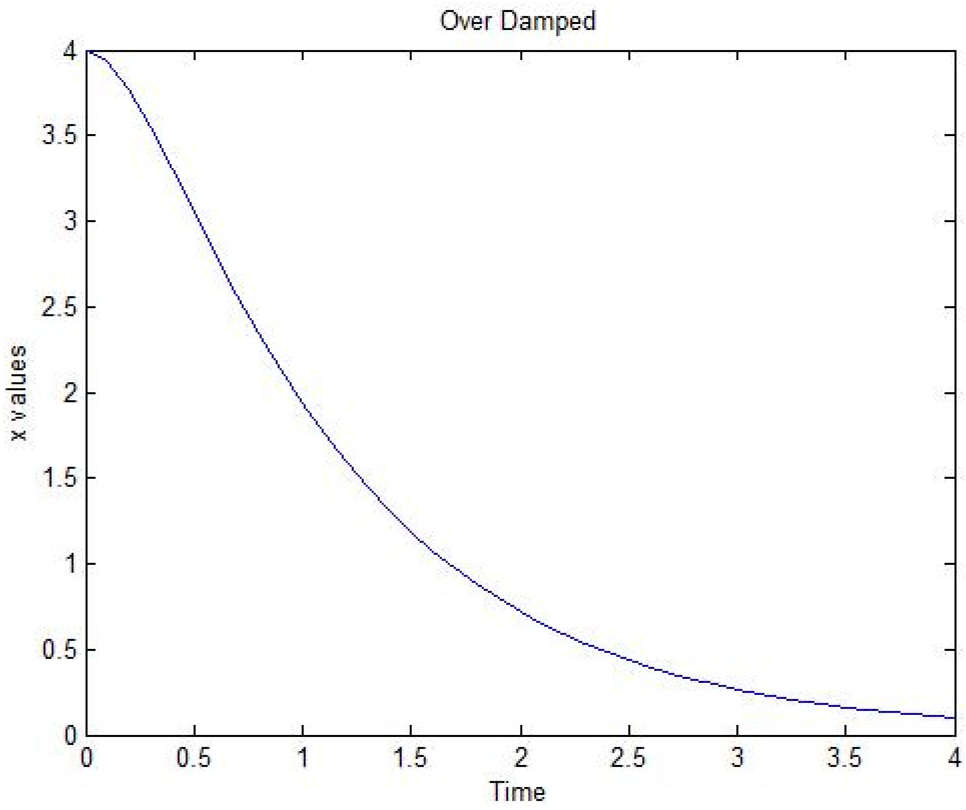

FIGURE 1.

From fig. 1 we can observe that in the case of over damping the body takes longer time to reach the equilibrium position $x = 0$.

3(b) Critically damped case

Let $b = a$ i.e., when viscosity is decreased until we reach $b = a$. Let m be the double root of the equation (3.4). Then equation (3.3) becomes

$$x(t) = \frac{x_0}{m^2}E^{-1}\{\frac{-1}{u-m} + \frac{(1+m)u}{(u-m)^2}\} = \frac{x_0}{m^2}[e^{\frac{t}{m}} - (1+m)te^{\frac{t}{m}}]. \tag{3.6}$$

Example 2. *Solve the differential equation* $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x = 0$ *with initial conditions* $x = 5$ *and* $\frac{dx}{dt} = -3$ *when* $t = 0$.

Solution: We shall find the solution of this initial value problem using Elzaki transform. Applying Elzaki transform to the given differential equation we get

$$\frac{T(u)}{u^2} - x_0 - u\frac{dx}{dt} + 6[\frac{T(u)}{u} - ux_0] + 9T(u) = 0.$$

Using the initial conditions it becomes $T(u) = \frac{(5+27u)u^2}{(1+6u+9u^2)} = u^2[\frac{5}{(1+3u)} + \frac{12u}{(1+3u)^2}]$. By taking inverse Elzaki transform we get $x(t) = 5e^{-3t} + 12te^{-3t}$.

From fig. 2 it is clear that in the case of critically damping the body reaches the equilibrium position very fast.

III(c) Underdamped case

Let $b < a$. The viscosity is now decreased by any amount, however small, then motion becomes vibratory and motion is underdamped. Let $\beta \pm i\alpha$ be the roots of the equation (3.4). Then equation (3.3) becomes

$$\begin{aligned} x(t) &= \frac{x_0}{2i\alpha}E^{-1}\{\frac{[1+\beta+i\alpha]u^2}{(u-(\beta+i\alpha))} - \frac{([1+\beta-i\alpha]u^2)}{(u-(\beta-i\alpha))}\} \\ &= \frac{x_0}{2i\alpha}\{\frac{[1+\beta+i\alpha]}{(-(\beta+i\alpha))}e^{\frac{t}{\beta+i\alpha}} + \frac{[1+\beta-i\alpha]}{(\beta+i\alpha)}e^{\frac{t}{(\beta-i\alpha)}}\}. \end{aligned}$$

By simplifying we get

$$x(t) = \frac{x_0e^{\beta t}}{(\beta^2-\alpha^2)^2}\{\frac{2i\alpha}{(\beta^2-\alpha^2)}cos(\frac{\alpha t}{(\beta^2-\alpha^2)}) + i(2+\frac{\beta}{\beta^2-\alpha^2})sin(\frac{\alpha t}{\beta^2-\alpha^2})\}.$$

Example 3. *Solve the differential equation* $\frac{d^2x}{dt^2} + 4x = 0$ *with initial conditions* $x = 1$ *and* $\frac{dx}{dt} = 1$ *when* $t = 0$.

Solution: We shall find the solution of this initial value problem using Elzaki transform. Applying Elzaki transform to the given differential equation we get

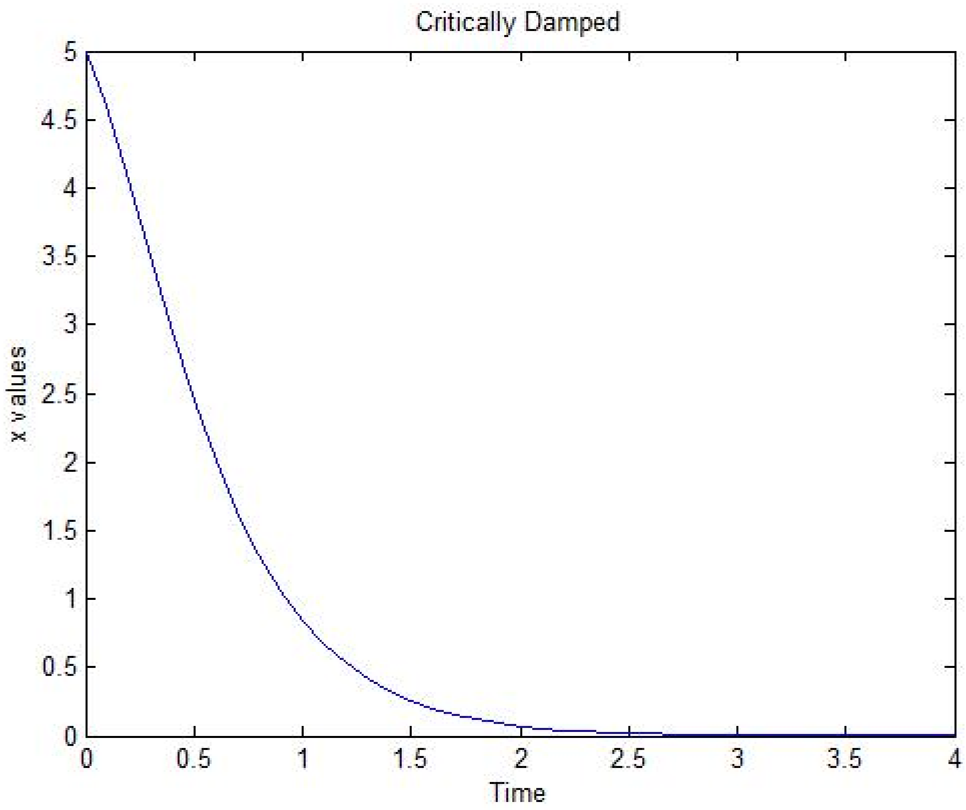

FIGURE 2.

$\frac{T(u)}{u^2} - x - u\frac{dx}{dt} + 4T(u) = 0$. Using the initial conditions it becomes

$$T(u) = \frac{((1+u)u^2)}{(1+4u^2)} = \frac{u^2}{4i}[\frac{(2i-1)}{(1+2iu)} + \frac{(2i+1)}{(1-2iu)}].$$

By taking inverse Elzaki transform we get $x(t) = \frac{1}{4i}\{(2i-1)e^{-2it} + (2i+1)e^{2it}\}$. By simplifying we get $x(t) = cos2t + \frac{1}{2}sin2t$.

From fig. 3 it is observed that for the under damping case the graph crosses the equilibrium position x=0 at regular intervals.

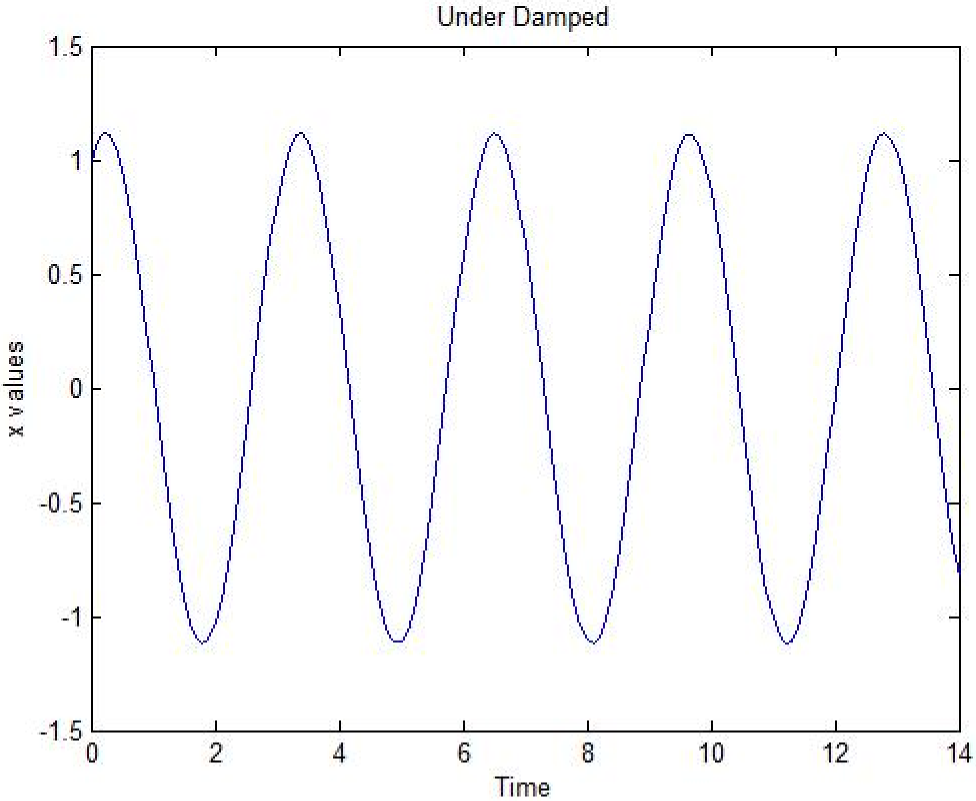

FIGURE 3.

4. FORCED VIBRATIONS

Consider an impressed force $F_e = f(t)$ acts on cart. Then equation (2.1) becomes

$$
\begin{aligned}
M\frac{d^2x}{dt^2} &= F_s + F_d + F_e \\
\frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x &= f(t),
\end{aligned}
\tag{4.1}
$$

which can be conveniently written as

$$
\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = g(t),
\tag{4.2}
$$

where $a = \sqrt{\frac{k}{m}}, b = \sqrt{\frac{c}{2M}}, g(t) = \frac{f(t)}{M}$. Applying Elzaki transform to equation (4.2) we get

$$\frac{T(u)}{u^2} - x_0 - u\frac{dx}{dt} + 2b[\frac{T(u)}{u} - ux_0] + a^2T(u) = E[g(t)].$$

Using (2.4) we get $T(u)(\frac{1}{u^2} + a^2) - (1+u)x_0 = E[g(t)]$,

$$x(t) = x_0E^{-1}\{\frac{(1+u)u^2}{1+2bu+a^2u^2}\} + E^{-1}\{\frac{E[g(t)]}{1+2b+a^2u^2}\}. \tag{4.3}$$

The first term on RHS of equation (4.3) can be obtained by using appropriate case of section III. By assigning particular value for g(t) we can obtain the second term of RHS of (4.3). Let us prove this by considering the following problem.

Example 4. *Solve the differential equation* $\frac{d^2y}{dt^2} + n^2y = kcosnt$ *with initial conditions* $y = a$ *and* $\frac{dy}{dt} = b$ *when* $t = 0$.

Solution: We shall find the solution of this initial value problem using Elzaki transform. Applying Elzaki transform to the given differential equation we get

$$\frac{T(u)}{u^2} - y_0 - u\frac{dy}{dt} + n^2T(u) = k\frac{u^2}{(1+n^2u^2)}.$$

Using the initial conditions it becomes $T(u) = \frac{(a+bu)u^2}{(1+n^2u^2)} + \frac{(ku^4)}{(1+n^2u^2)^2}$. By taking inverse Elzaki transform we get $y(t) = acosnt + \frac{b}{n}sinnt + \frac{kt}{2n}$ cosnt

5. CONCLUSION

Elzaki transform is a modern mathematical tool for solving the differential equations. It is easy to solve the initial value problems of engineering applications using this transform. Here in this work it is shown that problems of all varieties of mechanical vibrations can be easily solved by using Elzaki transform.

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