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THE CONTINUOUS WAVELET TRANSFORM FOR A FOURIER-JACOBI TYPE OPERATOR

C.P. Pandey[1] and Jyoti Saikia

ABSTRACT. The Jacobi operator is generalized by considering a singular differential difference operator Λ on $(0,\infty)$ and harmonic analysis corresponding to generalized Fourier transform is also investigated. To construct and investigate Fourier-Jacobi wavelet transform on half line, tools of harmonic analysis related to Λ is used.

1. INTRODUCTION

The wavelet transform of a function $f \in L^2(R)$ of the wavelet $\phi \in L^2(R)$ is defined by

$$(Wa_\phi f)(k,h) = \int_{-\infty}^{\infty} f(p)\bar{\phi}_{k,h}(p)dp, k \in \mathbb{R}, h > 0. \tag{1.1}$$

where

$$\phi_{h,k}(p) = h^{-1/2}\phi(\frac{p-k}{h}). \tag{1.2}$$

In terms of translation τ_b defined by

$$\tau_k\phi(p) = \phi(p-k), k \in \mathbb{R}$$

[1]*corresponding author*

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and dilation D_h defined by

$$D_h\phi(p) = h^{-1/2}\phi(\frac{p}{k}), h > 0,$$

we can write

$$\phi_{h,k}(p) = \tau_k D_k \phi(p). \tag{1.3}$$

It is known from (1.1),(1.2) and (1.3) that wavelet transform for a function is an integral transform and its kernel is dilated translate of wavelet ϕ.

The wavelet transform (1.1) can also express in convolution:

$$(Wa_\phi f)(k,h) = (f * g_{0,h})(k), \tag{1.4}$$

where

$$g(p) = \bar{\phi}(-p).$$

2. Preliminaries

The generalized Legendre function $P_\gamma^{(\sigma_1,\sigma_2)}(y)$ defined by

$$\begin{aligned} P_\gamma^{(\sigma_1,\sigma_2)}(y) = & \frac{(1+|y|)^{\sigma_2/2}}{\Gamma(1-\sigma_1)(|y|-1)^{\sigma_1/2}} \\ & \cdot F[\gamma + \frac{\sigma_2-\sigma_1}{2} + 1, -\gamma + \frac{\sigma_2-\sigma_1}{2}; 1-\sigma_1; \frac{1-|y|}{2}], \quad y \in R^n, \end{aligned} \tag{2.1}$$

where F[u,v;w;z] denotes the Gauss hypergeometric function is a generalization of the Jacobi polynomial [7,p.343]. It reduces to the Jacobi polynomial $P_\gamma^{(\sigma_1,\sigma_2)}(y)$ for $\gamma = n$, a non-negative integer. Integral transforms along with generalized Legendre functions as kernels have been investigated by Braaksma and Meulenbeld [1]. Theory and application of these transforms can also be found in [2–8]. The convolution theory developed by Flensted-Jensen and Koornwinder [5] is basis for the present work. The following normalized form will be used in the sequel

$$R_\gamma^{(\sigma_1,\sigma_2)}(y) = P_\gamma^{(\sigma_1,\sigma_2)}(y)/P_\gamma^{(\sigma_1,\sigma_2)}(1), y \in R^n. \tag{2.2}$$

Let ch(x) denote cosh(x) and sh(x) denote sinh(x). Then set

$$\phi_\chi(x) = R_{1/2(i\lambda-\rho)}^{(\sigma_1,\sigma_2)}(\sigma_1,\sigma_2)(ch2x). \tag{2.3}$$

Also, from [8] we know that $\phi_\lambda(t)$ is a solution of the IVP

$$\frac{1}{\Lambda(x)}\frac{d}{dx}(\Lambda(x)\frac{d}{dx}u(x)) = \Lambda u(x) = -(\chi^2+\rho^2)u(x) \tag{2.4}$$

$$u(0)=1, u^{'}(0)=0,$$

where

$$\Delta(x) = (e^x+e^{-x})^{2\sigma_2+1}(e^x+e^{-x})^{2\sigma_1+1} = 2^{2\rho}(shx)^{2\sigma_1+1}(chx)^{2\sigma_2+1},$$

$\rho = \sigma_1+\sigma_2+1>0$. Let $\phi_\chi(x)$ be the second kind Jacobi function is a solution of (2.1) such that

$$\Phi_\chi(x) = e^{(i\chi-\rho)x}[1+o(1)]asx\to\infty.$$

Thus

$$\Phi_\chi(x) = (e^x+e^{-x})^{(i\chi-\rho)}F(\frac{\sigma_2-\sigma_1+1-i\chi}{2}, \frac{\rho-i\chi}{2}; 1-i\chi; -\frac{1}{(shx)^2}). \tag{2.5}$$

We know that

$$\phi_\chi(x) = c(\chi)\Phi_\chi(x) + c(-\chi)\Phi_\chi(x). \tag{2.6}$$

Let us define L^q_μ, $1\le q\le\infty$, as the class of measurable functions f on the half line for which $\|f\|_{q,\sigma_1}<\infty$, where

$$\|f\|_{q,\sigma_1} = (\int_0^\infty |f(x)|^q d\mu(y))^{1/q}, ifq<\infty,$$

and

$$\|f\|_{\infty,\sigma_1} = \|f\|_\infty = esssup_{x\ge 0}|f(y)|.$$

The Fourier-Jacobi transform defined for a function $f\in L^1_{\sigma_1}$ is given by

$$F_j(f)(\chi) = \widehat{f}(\chi) = \int_0^\infty f(y)\phi_\chi(y)d\mu(y) = (2\pi)^{-1/2}\Lambda(y)dy, \tag{2.7}$$

and the inverse mapping is given by

$$g(y) = (2\pi)^{-1/2}\int_0^\infty \widehat{g}(\chi)\varphi_\chi(y)|c(\chi)|^2 d\chi = \int_0^\infty \widehat{g}(\chi)\varphi_\chi(y)dv(\chi), \tag{2.8}$$

where

$$dv(\chi) = (2\pi)^{-1/2}|c(\chi)|^2 d\chi$$

and

$$c(\chi) = \frac{2^{\rho-i\chi}\Gamma(i\chi)\Gamma(\sigma_1+1)}{\Gamma((\rho+i\chi)/2)\Gamma((\sigma_1+\sigma_2+1+i\chi)/2)}. \tag{2.9}$$

As in [5] he convolution is defined by

$$(f_1 * f_2)(y) = \int_0^\infty \int_0^\infty f_1(x) f_2(s) k(y,s,x) d\mu(x) d\mu(s), \tag{2.10}$$

where

$$K(x_1,x_2,x_3) = \frac{2^{(1/2)-2\rho}\Gamma(\sigma_1+1)(chx_1chx_2chx_3)^{\sigma_1-\sigma_2-1}}{\Gamma(\sigma_1+(1/2))(shx_1shx_2shx_3)^{2\sigma_1}} \times F(\sigma_1+\sigma_2, \sigma_1-\sigma_2; \sigma_1+1/2; \frac{1-B}{2}),$$

with

$$B = \begin{cases} \frac{(chx_1)^2+(chx_2)^2+(chx_3)^2-1}{2}, |x_1-x_2| < x_3 < x_1+x_2 \\ 0, \text{otherwise.} \end{cases}$$

Then $K(x_1,x_2,x_3)$ satisfies the following properties:

(i) In all the variables $K(x_1,x_2,x_3)$ is symmetric;
(ii) $K(x_1,x_2,x_3) \geq 0$;
(iii) $\int_0^\infty K(x_1,x_2,x_3) d\mu(x_3) = 1$.

Also it has been shown that in [5] that

$$\varphi_\chi(x_1)\varphi_\chi(x_2) = \int_0^\infty \varphi_\chi(x_3) K(x_1,x_2,x_3) d\mu(x_3). \tag{2.11}$$

Appling (1.2) and (1.3), we have

$$K(x_1,x_2,x_3) = \int_0^\infty \varphi_\chi(x_1)\varphi_\chi(x_2)\varphi_\chi(x_3) dv(\chi). \tag{2.12}$$

An inner product on $L^2(\mu)$ is defined by

$$< f_1, f_2 >= \int_0^\infty f_1(x)\overline{f_2(x)} d\mu(x).$$

Similar definition is given to $L^q(\mu)$. From [5] we have the following

Lemma 2.1. *Let* $1 \leq q < 2, \frac{1}{q} + \frac{1}{s} = 1$ *and* $f \in L^q(\mu)$. *Then*

$$|\widehat{f}(\chi)| \leq \|f\|_q \|\varphi_\chi\|_s. \tag{2.13}$$

If $f \in L^1(\mu), \widehat{f}(\mu)$, *is continuous in* $\overline{D_1}$ *and for all* $\chi \in \overline{D_1}$

$$|\widehat{f}(\chi)| \leq \|f\|_1. \tag{2.14}$$

Theorem 2.1. *Let q, s, r satisfy $\frac{1}{q} + \frac{1}{s} = 1 + \frac{1}{r}; 1 \leq q, s, r \leq \infty$ for $f_1 \in L^q(\mu)$ and $f_2 \in L^s(\mu)$, $f_1 * f_2 \in L^r(\mu)$ and $\|f_1 * f_2\|_r \leq \|f_1\|_q \|f_2\|_s$. Moreover, for $f_1, f_2 \in L^1(\mu)$ we have*

$$(f_1 * f_2)\widehat{\ }(\chi) = \widehat{f_1}(\chi)\widehat{f_1}(\chi). \tag{2.15}$$

For any $f_1 \in L^2(\mu)$, the below Parseval identity holds for the Fourier- Jacobi transform:

$$\int_0^\infty |f_1(x)|^2 d\mu(x) = \int_0^\infty |\widehat{f_1}(x)|^2 d\nu(x).$$

The Fourier- Jacobi translation τ_b of $\varphi \in L^q(\mu)$ defined by

$$\tau_b \varphi(y) = \varphi(y, b) = \int_0^\infty \varphi(z) K(y, b, z) d\mu(z), 0 < y, b < \infty, \tag{2.16}$$

maps $\tau_b(y)$ defined on the positive half of the real axis into the function $\varphi(y, b)$ defined on the upper half of the positive half plane. τ_b is also called generalized translation. Using Höder's inequality it can be shown that

$$\|\tau_b f_1\|_{L^q(\mu)} \leq \|f_1\|_{L^q(\mu)}$$

and the map $y \to \tau_b f_1$ is continuous for all $f_1 \in L^q(\mu), q \in [1, \infty)$.

Definition 2.1. *A function $\omega \in L^q(\mu)$ is a Fourier-Jacobi wavelet, satisfies the condition of admissibility*

$$0 < C_\omega^\chi = \int_0^\infty |F_j(\omega)(\chi)|^2 \frac{d\chi}{\chi} < \infty. \tag{2.17}$$

Definition 2.2. *Let $\omega \in L^2(\mu)$ be a Jacoi wavelet, then for a suitable function f on $L^2(\mu)$ the continuous Fourier-Jacobi wavelet transform is defined by*

$$J_\omega^\chi(f)(\sigma_1, \sigma_2) = \int_0^\infty f(y) \overline{\omega_{\sigma_1,\sigma_2}^\chi(y)} d\mu(y), \tag{2.18}$$

where $\sigma_1 > 0, \sigma_2 \geq 0$,

$$\omega_{\sigma_1,\sigma_2}^\mu(y) = \int_0^\infty K(\sigma_2, y, z)\omega(\sigma_1, z) d\mu(z), \tag{2.19}$$

and $\omega_{\sigma_1}(y) = \omega(\sigma_1, y)$.

Theorem 2.2. *Let a Fourier-Jacobi wavelet is $\omega \in L^2(\mu)$. Then*

(i) *For all $f \in L^2(\mu)$ then Plancherel formula we have*

$$\int_0^\infty |f(y)|^2 d\mu(y) = \frac{1}{C_\omega} \int_0^\infty \int_0^\infty |J_\omega^\mu(f)(\sigma_1, \sigma_2)|^2 d\mu(\sigma_2) d\mu(\sigma_1).$$

(ii) *Assume that* $\|F_j(\omega)\|_\infty < \infty$. *For* $f \in L^2(\mu)$ *and* $0 < \varepsilon_1 < \varepsilon_2 < \infty$, *the function*

$$f^{\varepsilon_1,\varepsilon_2}(y) = \frac{1}{C_\omega}\int_0^\infty\int_0^\infty J^\mu_\omega(f)\omega^\mu_{\varepsilon_1,\varepsilon_2}(y)d\mu(\sigma_2)d\mu(\sigma_1),$$

belongs to $L^2(\mu)$ *and satisfies* $\lim_{\varepsilon_1\to 0,\varepsilon_2\to\infty}\|f^{\varepsilon_1,\varepsilon_2} - f\|_{2,\mu} = 0$.

(iii) *For* $f \in L^1(\mu)$ *such that* $F_\chi(f) \in L^1(\mu)$, *we have*

$$f(y) = \frac{1}{C^\chi_\omega \int_0^\infty}(\int_0^\infty J^\mu_\omega(f)\omega^\mu_{\varepsilon_1,\varepsilon_2}(y)d\mu(\sigma_2))d\mu(\sigma_1),$$

for almost all $y \geq 0$.

3. Harmonic analysis related to Fourier-Jacobi operator Λ

Let the map N be defined by $Nf(y) = \Lambda(y)f(y)$. Let $L^q(\mu), 1 \leq q \leq \infty$, be the class of measurable function f on the half line for which $\|f\|_{q,\mu} = \|M^{-1}f\|_{q,\mu} < \infty$.

Generalized Fourier transform

For $\chi \in \mathbb{C}$ and $y \in \mathbb{R}$,

$$\phi_\chi(y) = \Lambda(y)\varphi_\chi(y). \tag{3.1}$$

The generalized Fourier transform defined for a function $f_1 \in L^1(\mu)$ is given by

$$F_\Lambda(f_1)(\chi) = \int_0^\infty f_1(y)\phi_\chi(y)d\mu(y). \tag{3.2}$$

Theorem 3.1. *Let* $f_1 \in L^1(\mu)$ *such that* $F_\Lambda(f_1) \in L^1(\mu)$. *Then for almost all* $y > 0$,

$$f_1(y) = \int_0^\infty F_\Lambda(f_1)(\chi)\phi_\chi(y)d\nu(\chi).$$

Proof. By (3.1),(3.2) and Proposition 2.1(ii) we have

$$\begin{aligned}\int_0^\infty (f_1)(\chi)\phi_\chi(y)d\nu(\chi) &= \Lambda(y)\int_0^\infty F_{\sigma_1+2n}(M^{-1}f_1)(\chi)\varphi_\chi(y)dv(\chi)\\ &= \Lambda(y)M^{-1}f_1(y) = f_1(y).\end{aligned}$$

□

Theorem 3.2.

(i) *For every* $f_1 \in L^1(\mu) \bigcap L^1(\mu)$ *the Plancherel formula we have*

$$\int_0^\infty |f_1(y)|^2 d\mu(y) = \int_0^\infty |F_\Lambda(f_1)(\chi)|^2 d\nu(\chi).$$

(ii) *Unique isometric isomorphism from* $L^2(\mu)$ *onto* $L^2(\mu)$ *is extend by generalized Fourier transform* F_Λ*. And its inverse transform is given by*

$$F_\Lambda^{-1}(f_2)(y) = \int_0^\infty f_2(\chi)\phi_\chi(y) d\nu(\chi),$$

where the integral is converges in $L^2(\mu)$*.*

Proof. Let $f_1 \in L^1(\mu) \bigcap L^1(\mu)$. By (3.1) we have

$$\int_0^\infty |F_\Lambda(f_1)(\chi)|^2 d\nu(\chi) = \int_0^\infty |F_\chi(M^{-1}f_1)(\chi)|^2 d\nu(\chi)$$

$$= \int_0^\infty |M^{-1}f_1(y)|^2 d\mu(y) = \int_0^\infty |f_1(y)|^2 d\mu(y)$$

which concludes that (i)and (ii) can be proved in standard manner. □

4. Generalized convolution product

Definition 4.1. *Define the generalized translation operator* $T^y, 0 \leq y$*, by the relation*

$$T^y f_1(b) = \tau_\chi^y(M^{-1}f_1)(b), 0 \leq b, \tag{4.1}$$

where τ_χ^y *is the Jacobi translation operator.*

Definition 4.2. *The generalized convolution product of two functions* f_1 *and* f_2 *on half line is defined by*

$$f_1 * f_2(y) = \int_0^\infty T^y f_1(b) f_2(b) d\mu(b), 0 \leq y. \tag{4.2}$$

Proposition 4.1.

(i) *Let* f *be in* $L^q(\mu), 1 \leq q \leq \infty$*. Then* $\forall 0 \leq y$*, the function* $T^y f_1 \in L^q(\mu)$*, and* $\|T^y f_1\|_{q,\mu} \leq \Lambda \|f_1\|_{q,\mu}$.

(ii) *For* $f_1 \in L^q(\mu), q = 1 or 2$*, we have*

$$F_\Lambda(T^y f_1)(\chi) = \phi_\chi(y) F_\Lambda(f_1)(\chi).$$

(iii) *Let* $q, s \in [1, \infty]$ *such that* $\frac{1}{q} + \frac{1}{s} = 1$. *If* $f_1 \in L^q(\mu)$ *and* $f_2 \in L^s(\mu)$ *then*

$$\int_0^\infty T^y f_1(b) f_2(b) d(b) = int_0^\infty f_1(b) T^y f_2(b) d(b).$$

(iv) *Let* $q, s, r \in [1, \infty]$ *such that* $\frac{1}{q} + \frac{1}{s} - 1 = \frac{1}{r}$. *If* $f_1 \in L^q(\mu)$ *and* $f_2 \in L^s(\mu)$ *then* $f_1 \sharp f_2 \in L^r(\mu)$ *and* $\|f_1 \sharp f_2\|_{r,\mu} \leq \|f_1\|_{q,\mu} \|f_2\|_{s,\mu}$.

(v) *For* $f_1 \in L^1(\mu)$ *and* $f_2 \in L^q(\mu), q = 1$ *or* 2, *we have*

$$F_\Lambda(f_1 \sharp f_2) = F_\Lambda(f_1) F_\Lambda(f_2).$$

5. Generalized wavelets

Definition 5.1. *A generalized wavelet is a function* $\phi \in L^q(\mu)$ *satisfying the condition off admissibility*

$$0 < C_\phi = \int_0^\infty |F_\Lambda(f_2)(\chi)|^2 \frac{d\nu(\chi)}{\chi} < \infty. \tag{5.1}$$

For $f_2 \in L^2(\mu)$ *and* $(h, k) \in (0, \infty) \times (0, \infty)$ *put*

$$\phi_{h,k}(y) = \int_0^\infty \phi(hz) K(k, y, z) d\mu(z). \tag{5.2}$$

Proposition 5.1. *For all* $h > 0$ *and* $0 \leq k$ *we have*

$$\phi_{h,k}(y) = k\Lambda(y)(M^{-1}\phi)^\mu_{h,k}(y). \tag{5.3}$$

Proof. Using (2.13), (2.14) and (3.1) we can easily prove that

$$\phi_{h,k}(y) = k\Lambda(y)(M^{-1}\phi)^\mu_{h,k}(y).$$

□

Definition 5.2. *Let a generalized wavelet be* $f_2 \in L^2(\mu)$. *We define for regular functions* f *on the half line, the generalized continuous Fourier-Jacobi wavelet transform is given by*

$$L_\phi(f_1)(h, k) = \int_0^\infty f_1(y) \overline{\phi_{h,k}(y)} d\mu(y), \tag{5.4}$$

or

$$L_\phi(f_1)(h, k) = f_1 * \overline{\phi_h}(k),$$

where the generalized convolution product $*$ *is given* (4.2).

Proposition 5.2. *We have*

$$L_\phi(f_1)(h,k) = S^\mu_{M^{-1}f_2}(M^{-1}f_1)(h,k). \tag{5.5}$$

Proof. From (2.6), (3.1) and (5.4) we deduce that

$$\begin{aligned} L_\phi(f_1)(h,k) &= \int_0^\infty f_1(y)\overline{\phi_{h,k}(y)}d\mu(y) \\ &= \int_0^\infty (M^{-1}f_1)(y)\overline{(M^{-1}\phi)^\mu_{h,k}(y)}d\mu(y) = S^\mu_{M^{-1}f_2}(N^{-1}f_1)(h,k), \end{aligned}$$

which concludes the proof. □

Theorem 5.1. *(Plancherel formula) Let $\phi \in L^2(\mu)$. be a generalized wavelet. For every $f_1 \in L^2(\mu)$. we have Plancherel formula*

$$\int_0^\infty |f_1(y)|^2 d\mu(y) = \frac{1}{C_\phi}\int_0^\infty \int_0^\infty |L_\phi(f_1)(h,k)|^2 d\mu(k)\frac{d\mu(h)}{h}.$$

Proof. By (2.19) and (5.4) we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty |L_\phi(f_1)(h,k)|^2 d\mu(k)\frac{d\mu(h)}{h} \\ &= \int_0^\infty \int_0^\infty |S^\mu_{M^{-1}f_2}(M^{-1}f_1)(h,k)|^2 d\mu(k)\frac{d\mu(h)}{h} \\ &= C^\mu_{M^{-1}f_2}\int_0^\infty |M^{-1}f_1(y)|^2 d\mu(y) = C_\phi \int_0^\infty |f_1(y|^2 d\mu(y). \end{aligned}$$

Which concludes the proof. □

Theorem 5.2. *(Calderon's formula) Let a generalized wavelet be $\phi \in L^2(\mu)$ such that $\|F_\Lambda(\phi)\|_\infty < \infty$. Then for $f_1 \in L^2(\mu)$ and $0 < \varepsilon_1 < \varepsilon_1 < \infty$, then the function*

$$f^{\varepsilon_1,\varepsilon_2} = \frac{1}{C_\phi}\int_{\varepsilon_1}^\infty \int_0^\infty L_\phi(f_1)(h,k)\phi_{\varepsilon_1,\varepsilon_2}(y)d\mu(k)\frac{d\mu(h)}{h}$$

belongs to $L^2_{\sigma_1,n}$ and satisfies $\lim_{\varepsilon_1\to 0,\varepsilon_2\to\infty}\|f_1^{\varepsilon_1,\varepsilon_2} - f_1\|_{2,\mu} = 0$.

Proof. By (2.19), (3.2)and (5.4) we have

$$f_1^{\varepsilon_1,\varepsilon_2}(y) = \frac{\Lambda(y)}{C^\mu_{M^{-1}f_2}}\int_{\varepsilon_1}^\infty \int_0^\infty S^\chi_{M^{-1}f_2}(M^{-1}f_1)(h,k)(M^{-1}f_2)^\mu_{h,k}(y)d\mu(k)\frac{d\mu(h)}{h}.$$

□

Theorem 5.3. *Let a generalized wavelet be $\phi \in L^2(\mu)$. If $f_1 \in L^1(\mu)$ and $F_\Lambda(f_1) \in L^2(\mu)$ then we have*

$$f_1(y) = \frac{1}{C_\omega}\int_0^\infty (\int_0^\infty L_\omega(f_1)(h,k)\phi_{h,k} d\mu(k))\frac{d\mu(h)}{h}$$

for almost all $0 \leq y$.

Proof. By (2.16), (3.1)and (5.4) we have

$$\frac{1}{C_\omega}\int_0^\infty (\int_0^\infty L_\omega(f_1)(h,k)\phi_{h,k} d\mu(k))\frac{d\mu(h)}{h}$$
$$= \frac{\Delta(y)}{C^\mu_{M^{-1}f_2}}\int_0^\infty (\int_0^\infty S^\mu_{N^{-1}\phi}(M^{-1}f_1)(h,k)(M^{-1}f\phi)^\mu_{h,k}(y) d\mu(k))\frac{d\mu(h)}{h},$$

and the result follows from Theorem 2.2. □

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Department of Mathematics
North Eastern Regional Institute of science and Technology
Nirjuli, 791109, Arunachal Pradesh
India.
Email address: drcppandey@gmail.com

Department of Mathematics
North Eastern Regional Institute of science and Technology
Nirjuli, 791109, Arunachal Pradesh
India.
Email address: jyotisaikia885@gmail.com