

DETOUR PEBBLING ON PATH RELATED GRAPHS

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ABSTRACT. Given a distribution of pebbles on the vertices of a connected graph G , a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles on an adjacent vertex. The *pebbling number of a vertex v* in a graph G is the smallest number $f(G, v)$ such that for every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. The *pebbling number of G* is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G, v)$ over all vertices v . The detour pebbling number of G is denoted by $f^*(G)$ and is obtained by the detour path. In this paper, we compute the detour pebbling number for some path related graphs.

1. INTRODUCTION

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [8].

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Graph pebbling is an optimization model of the network for the transport of resources consumed in transit. As it moves from one place to another, electricity, heat, or other resources may dissipate, oil tankers may use some of the oil it transports, information may be lost as it passes through its medium, or military forces may be lost when travelling through a area.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Consider a connected graph with fixed number of pebbles distributed on its vertices. A *pebbling move* consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The *pebbling number of a vertex v* in a graph G is the smallest number $f(G, v)$ such that for every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the *pebbling number of G* is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G, v)$ over all vertices v . The pebbling number of a graph was extended to t -pebbling number of a graph and there are so many articles with regard to t -pebbling numbers [2], [?], [4], [5] and [6]. The detour pebbling was introduced by Lourdusamy et. al in [10].

In this paper, we compute the detour pebbling number for some path related graphs.

2. PRELIMINARY

We begin with some theorems and definitions which are useful for the subsequent sections from [9], [10]. For basic graph theoretic terminologies the reader can refer [7].

Definition 2.1. Let G be a connected graph. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the distance between u and v in G . The p^{th} power of G , denoted by G^p , is the graph obtained from G by adding edge uv to G , whenever $2 \leq d_G(u, v) \leq p$. That is, $E(G^p) = \{uv : 1 \leq d_G(u, v) \leq p\}$.

The square of paths is denoted by P_n^2 and the vertex set is notated by $\{v_1, v_2, \dots, v_n\}$.

Definition 2.2. The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining the edges of those pair of these new vertices which lie on the adjacent edges of G .

The middle graph of a path with n vertices is denoted by $M(P_n)$.

Definition 2.3. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G , say G_1 and G_2 and joining each vertex u in G_1 to the neighbours of the corresponding vertex v in G_2 .

The shadow graph of a path with n vertices is denoted by $D_2(P_n)$. Let us denote the vertices of the first copy of P_n by $u_1, u_2, \dots, u_n$ and the second copy of P_n by $v_1, v_2, \dots, v_n$.

Definition 2.4. [10] A detour pebbling number of a vertex v of a graph G is the smallest number $f^*(G, v)$ such that for any placement of $f^*(G, v)$ pebbles on the vertices of G it is possible to move a pebble to v using a detour path by a sequence of pebbling moves. The detour pebbling number of a graph is denoted by $f^*(G)$, is the maximum $f^*(G, v)$ over all the vertices of G .

Theorem 2.1. [10] For any path P_n with n vertices, the detour pebbling number is $f^*(P_n) = 2^{n-1}$.

Theorem 2.2. [10] Let $K_{1,n}$ be an n -star where $n > 1$. The detour pebbling number for the n -star graph is $f^*(K_{1,n}) = n + 2$.

3. MAIN RESULTS

In this section, we compute the detour pebbling number for square of paths, middle graphs of paths and shadow graph of paths.

Theorem 3.1. Let P_n^2 be the square of path with n vertices. The detour pebbling number $f^*(P_n^2) = 2^{n-1}$.

Proof. Placing $2^{n-1} - 1$ pebbles on the vertex v_n , we cannot move a pebble to v_1 , as the length of the detour path from v_n to v_1 is $n - 1$. Thus $f^*(P_n^2) \geq 2^{n-1}$. For the sufficient part, we consider any distribution D with 2^{n-1} pebbles on the graph P_n^2 .

Case 1: n is odd.

Let v_i be the target vertex. Assume $p(v_i) = 0$. Suppose i is odd. A spanning path $P : v_{i+2}, v_{i+4}, \dots, v_n, v_{n-1}, v_{n-3}, v_{n-5}, \dots, v_2, v_1, v_3, v_5, \dots, v_i$ of length $n - 1$ is a detour path from v_i consisting of the 2^{n-1} pebbles. Then by Theorem 2.1, using 2^{n-1} pebbles we can move a pebble to v_i . Suppose i is even. Then consider the spanning path $P : v_{i+2}, v_{i+4}, \dots, v_{n-1}, v_n, v_{n-2}, v_{n-4}, \dots, v_3, v_1, v_2, v_4, \dots, v_i$ of length $n - 1$. Since this path consists of all the vertices, 2^{n-1} pebbles are distributed on this detour path and hence by Theorem 2.1, using 2^{n-1} pebbles we can move a pebble to v_i .

Case 2: n is even.

Let v_i be the target vertex. Assume $p(v_i) = 0$. Suppose i is odd. Then there exists a spanning path $P : v_{i+2}, v_{i+4}, \dots, v_{n-1}, v_n, v_{n-2}, \dots, v_2, v_1, v_3, v_5, \dots, v_i$ of length $n - 1$ which is a detour path from v_i . Then by Theorem 2.1, using 2^{n-1} pebbles we can move a pebble to v_i . Suppose i is even. Then a spanning path $P : v_{i+2}, v_{i+4}, \dots, v_n, v_{n-1}, v_{n-3}, v_{n-5}, \dots, v_3, v_1, v_2, v_4, \dots, v_i$ of length $n - 1$ is a detour path containing 2^{n-1} pebbles on it. Hence we can move a pebble to the target vertex using the detour path. $\square$

Theorem 3.2. *Let $M(P_n)$ be the middle graph path with $2n - 1$ vertices. Then the detour pebbling number $f^*(M(P_n)) = 2^{2n-2}$.*

Proof. Label the vertices of the path P_n as $u_1, u_2, u_3, \dots, u_n$ and let us label the $n - 1$ vertices which we have inserted to form a middle graph of path by $v_1, v_2, \dots, v_{n-1}$.

Placing $2^{2n-2} - 1$ pebbles on the vertex u_n , we cannot reach u_1 as the detour distance from u_n to u_1 is $2n - 2$. Thus $f^*(M(P_n)) \geq 2^{2n-2}$. We now prove the sufficient part. Let us consider any distribution D with 2^{2n-2} pebbles on the middle graph of the path $M(P_n)$.

Case 1: Let u_1 be the target vertex.

Since $d^*(u_1, u_i) \leq 2n - 2$, for any i , we can reach the vertex u_1 . By symmetry we can reach u_n . Let u_i , where $1 < i < n$ be the target vertex. Then either the path $P_1 : u_i, v_i, u_{i+1}, v_{i+1}, \dots, u_n$ of length $2n - 2i$ contains at least $2^{2(n-i)}$ pebbles or the path $P_2 : u_1, v_1, u_2, v_2, \dots, u_i$ of length $2i - 2$ contains at least 2^{2i-2} pebbles. Clearly these paths are detour, we can reach the target. Otherwise the total number of pebbles distributed is $p(P_1) + p(P_2) < 2^{2(n-i)} + 2^{2i-2} = 2^{2n-2}$ which is a contrary. Hence $f^*(M(P_n)) = 2^{2n-2}$.

Case 2: Suppose v_i , where $i = 1$ to $n - 1$ be the target vertex.

Then either the path $P_3 : v_i, u_{i+1}, v_{i+1}, u_{i+2}, \dots, u_n$ of length $2n - 2i - 1$ contains at least $2^{2n-2i-1}$ pebbles or the path P_4 of length $2i - 1$ contains at least 2^{2i-1} pebbles. Since these paths are detour, we can use Theorem 2.1 and hence we reach the target. Otherwise the total number of pebbles distributed is $p(P_3) + p(P_4) < 2^{2n-2}$, which is a contradiction. Therefore $f^*(M(P_n)) = 2^{2n-2}$. $\square$

First we prove the following theorems, which are useful for determining the detour pebbling number for shadow graph of paths.

Theorem 3.3. *Let G be the shadow graph of path $D_2(P_3)$, the detour pebbling number $f^*(G) = 17$.*

Proof. Placing 15 pebbles on v_3 and a pebble on v_1 , we cant reach the vertex u_1 . Thus $f^*(D_2(P_3)) \geq 17$. For sufficiency, let us consider any distribution D of 17 pebbles on the vertices of the graph $G = D_2(P_2)$.

Case 1: Let u_1 be the target vertex.

Suppose $\langle G - \{u_3, v_3\} \rangle$ contains at least 8 pebbles then we can reach the target using the detour path of length 3. Otherwise assume that $p(\langle G - \{u_3, v_3\} \rangle) = i$, where $0 \leq i \leq 7$. Therefore $p(u_3) + p(v_3)$ contains $17 - i$ pebbles. Anyhow we can move $8 - i$ pebbles from the vertices u_3 and v_3 to $\langle G - \{u_3, v_3\} \rangle$. Thus using the detour path of length 3 and by Theorem 2.1, we can reach the target vertex.

Case 2: Let u_2 be the target vertex.

Since $\langle \{u_1, u_2, u_3, v_1, v_3\} \rangle$ is isomorphic to $K_{1,4}$, if $p(u_1) + p(u_3) + p(v_1) + p(v_3)$ is at least 6, by Theorem 2.2, we can move a pebble to the target. Otherwise v_2 contains at least 4 pebbles and hence we can reach the target, as the detour distance from v_2 to the target is two. By symmetry, we can reach all the vertices of the graph. Hence $f^*(G) = 17$. $\square$

Theorem 3.4. *Let G be the shadow graph of path $D_2(P_4)$. Then the detour pebbling number $f^*(G) = 128$.*

Proof. Placing 127 pebbles on the vertex v_4 , we cannot reach the vertex u_1 as the detour distance from v_4 to u_1 is 7. Thus $f^*(G) \geq 128$. For sufficient part, let us consider any distribution D with 128 pebbles on the graph G .

Case 1: Let u_1 be the target vertex.

Consider the spanning path $P_1 : v_4, u_3, u_4, v_3, u_2, v_1, v_2, u_1$ of length 7. Since it contains all the vertices of the graph G , this is also a detour path of length

7. Thus by Theorem 2.1, we can move a pebble using the detour path. By symmetry, if v_4 is the target vertex, we are done.

Case 2: Let v_1 be the target vertex.

Now considering the spanning path $P_2 : v_4, u_3, u_4, v_3, u_2, u_1, v_2, v_1$, which is a longest path in G containig all vertices and of length 7. Since 128 pebbles on distributed and by Theorem 2.1, we can move a pebble using the detour path. By symmetry, if u_4 is the target vertex, we are done.

Case 3: Let any vertices other than u_1, u_n, v_1, v_n be the target vertex.

Without loss of generality, let us assume that u_2 be the target vertex. Since $\langle \{u_1, u_2, v_1, v_2\} \rangle$ is isomorphic to P_4 and if $p(\langle \{u_1, u_2, v_1, v_2\} \rangle) \geq 8$, then by Theorem 2.1, we can reach the target vertex. Otherwise the remaining number of pebbles is at least $128 - 7 = 121$ are distributed on the vertices $\{u_3, u_4, v_3, v_4\}$. But since $\langle \{u_2, u_3, u_4, v_2, v_3, v_4\} \rangle$ is isomorphic to $D_2(P_3)$ and by Theorem 3.3, we can move a pebble to the target. By symmetry we can reach all the vertices of the graph. Hence $f^*(G) = 128$. $\square$

Now we find the detour pebbling number for shadow graph of even paths and then we compute for the shadow graphs of odd paths.

Theorem 3.5. *Let G be the shadow graph of path $D_2(P_n)$, where n is even. Then the detour pebbling number $f^*(G) = 2^{2n-1}$.*

Proof. Placing $2^{2n-1} - 1$ pebbles on the vertex v_n , we cannot reach the vertex u_1 as the detour distance from v_n to u_1 is $2n - 1$. Thus $f^*(G) \geq 2^{2n-1}$. We prove the sufficient part by induction on n . Since $D_2(P_2)$ is isomorphic to P_4 and from Theorem 3.4, we conclude that the result is true for $n = 2$ and for $n = 4$. Assume that the result is true for $4 \leq n' < n$. Let us consider any distribution D with 2^{2n-1} pebbles on the graph G .

Without loss of generality, let x be any target vertex, other than $\{u_{2n-1}, u_n, v_{n-1}, v_n\}$. Let $G_1 = \langle G - \{u_{n-1}, u_n, v_{n-1}, v_n\} \rangle$. Suppose $p(G_1) \geq 2^{2(n-2)-1}$. Since G_1 is isomorphic to $D_2(P_{n-2})$, by induction we are done. Otherwise $p(G_1) \leq 2^{2(n-2)-1} - 1$.

Case 1: Suppose $2^{2n-4(i+1)-1} \leq p(G_1) \leq 2^{2n-4i-1} - 1$, for each $i = 1$ to $n/2 - 2$. Since $\langle \{u_{2n-1}, u_{2n}, v_{2n-1}, v_{2n}\} \rangle$ is isomorphic to P_4 and by Theorem 2.1, $\frac{2^{2n-1} - [2^{2n-4i-1} - 1] - 1}{16}$ pebbles can be moved to either u_{n-2} or v_{n-2} . Also since,

$$\frac{2^{2n-1} - [2^{2n-4i-1} - 1]}{16} + 2^{2n-4(i+1)-1} \geq 2^{2n-5} = 2^{2(n-2)-1} = f^*(D_2(P_{n-2})),$$

we can move a pebble to any vertex in G_1 , by induction.

Case 2: Suppose $1 \leq p(G_1) \leq 7$.

Then remaining pebbles are distributed in $\{u_{n-1}, u_n, v_{n-1}, v_n\}$. Thus moving as many as possible from these to G_1 we note that either u_{n-2} or v_{n-2} contains $2^{2(n-2)-1} - 1$ pebbles. Thus totally the subgraph G_1 contains at least $2^{2(n-2)-1}$ pebbles. Since $f^*(D_2(P_{n-2})) = 2^{2(n-2)-1}$ and we are done by induction.

Case 3: Suppose $p(G_1) = 0$.

Then all the pebbles are distributed on the vertices $\{u_{n-1}, u_n, v_{n-1}, v_n\}$. Thus we can move $2^{2(n-2)-1}$ pebbles from here to G_1 and thus by induction, we can reach the target vertex.

By symmetry, we can reach all the vertices of the graph $D_2(P_n)$, where n is even. Thus $f^*(G) = 2^{2n-1}$. $\square$

Theorem 3.6. *Let G be the shadow graph of path $D_2(P_n)$, where n is odd. Then the detour pebbling number $f^*(G) = 2^{2n-2} + 1$.*

Proof. Placing $2^{2n-2} - 1$ pebbles on the vertex v_n and a pebble on v_1 , we cannot reach the vertex u_1 . Thus $f^*(G) \geq 2^{2n-2} + 1$.

We prove the sufficient part by induction on n . The result is true for $n = 3$, by Theorem 3.3. Assume that the result is true for $5 \leq n' < n$. Let us consider any distribution D with $2^{2n-2} + 1$ pebbles on the vertices of the graph G .

Without loss of generality, let x be any target vertex, other than $\{u_n, v_n\}$. Let $G_1 = \langle G - \{u_n, v_n\} \rangle$. Suppose $p(G_1) \geq 2^{2n-3}$. Since G_1 is isomorphic to $D_2(P_{n-1})$, by Theorem 3.5 we can reach the target. Otherwise $p(G_1) \leq 2^{2n-3} - 1$.

Case 1: Suppose $2^{2n-2-(i+1)} \leq p(G_1) \leq 2^{2n-2-i}$, for each $i = 1$ to $2n - 5$.

Then the remaining pebbles are distributed in the vertices u_{2n} and v_{2n} . Thus we can move at least $\frac{2^{2n-2} + 1 - 2^{2n-2-i} - 1}{2}$ pebbles to G_1 from these two vertices. Also since

$$\frac{2^{2n-2} + 1 - 2^{2n-2-i} - 1}{2} + 2^{2n-2-(i+1)} \geq 2^{2n-3} = 2^{2(n-1)-1} = f^*(D_2(P_{n-1})),$$

by induction we can move a pebble to the target.

Case 2: Suppose $1 \leq p(G_1) \leq 3$.

Then there are at least $2^{2n-2} + 1 - 3$ pebbles are distributed on the remaining two vertices. After moving as much as possible to G_1 we can see that G_1 contains 2^{2n-1} pebbles. Hence by induction we can reach the target.

Case 3: Suppose $p(G_1) = 0$.

Then the pebbles will be distributed only on the vertices u_n and v_n . Also we can move 2^{2n-1} pebbles to G_1 , which is equal to its detour pebbling number. Thus by induction we can reach the target.

By symmetry, we can reach any vertex in the graph G . Hence $f^*(G) = 2^{2n-2} + 1$. $\square$

Thus the detour pebbling number for certain path related graphs were computed.

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