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RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF PRODUCT OF HYPERGEOMETRIC FUNCTION AND $\bar{H}$- FUNCTION

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ABSTRACT. The subject of fractional calculus deals with investigation of integrals and derivatives of any arbitrary real and complex order. In this research paper to evaluate some theorem for Riemann-Liouville fractional Integrals, applied on the product of hypergeometric function and $\bar{H}$- function in the literature of special functions. Further we point out their special cases also have been discussed.

1. INTRODUCTION

Recently, for modeling of relevant systems in various fields of sciences and engineering, such as electromagnetic, fluid mechanics, signals processing, stochastic dynamical system, plasma physics, earth sciences,astrophysics nonlinear biological system, relaxation and diffusion processes in complex systems, propagation of seismic waves, anomalous diffusion and turbulence, etc. see, Glöckle and Nonnenmacher [3], Mainardi et al. [6], Metzler and Klafter [8] and others.

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The $\bar{H}$- function of one variable defined by Buschman and Srivastva [2] and we will represent here the following manner:

$$\bar{H}^{M,N}_{P,Q}[Z] = \bar{H}^{M,N}_{P,Q}\left[Z\Big|^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}\right] = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \bar{\Phi}(\xi)\, z^{\xi}\, d\xi, \tag{1.1}$$

where $i = \sqrt{(-1)}$, $(z \neq 0)$ is a complex variable and in (1.1) $z^{\xi} = \exp[\xi\{\log|z| + i\arg z\}]$. In which $\log|z|$ represent the natural logarithm of $|z|$ and $\arg|z|$ is not necessarily the principle value.

Also,

$$\bar{\Phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\ \prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\ \prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

and throughout the paper $a_j, (j = 1, ..., P)$ and $b_j, (j = 1, ..., Q)$ are complex parameter $\alpha_j \geq 0, (j = 1, ..., P), \beta_j \geq 0, (j = 1, ..., Q)$and exponents $A_j, (j = 1, ..., N)$ and $B_j(j = N+1, ..., Q)$ are non- negative integer values. Integral is convergent, where

$$\Omega = \sum_{j=1}^{M}|\beta_j| - \sum_{j=n+1}^{N}|A_j\alpha_j| - \sum_{j=M+1}^{Q}|\beta_j B_j| - \sum_{j=N+1}^{P}|\alpha_j| \geq 0. \tag{1.3}$$

2. Riemann-Liouville Fractional Integrals

We define the Riemann- Liouville integrals of a function $f(x)$ as follows.

Let $L(a,b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a,b]$:

$$L(a,b) = \{f : ||f||_1 = \int_a^b |f(t)|dt < \infty\}. \tag{2.1}$$

Let $f(x) \in L(a,b), \alpha \in \mathbb{C}\ and\ \Re(\alpha) > 0$ then

$${}_aI^{\alpha}_x f(x) = {}_a D^{-\alpha}_x f(x) = I^{\alpha}_{+a} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, x > a \tag{2.2}$$

is called Riemann- Liouville left-sided fractional integral of order α.

Let $f(x) \in L(a,b), \alpha \in \mathbb{C}, \Re(\alpha) > 0$ then

$$ {}_xI_b^{\alpha} f(x) = {}_x D_b^{-\alpha} f(x) = I_{-b}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_b^x (t-x)^{\alpha-1} f(t)dt, x < b \tag{2.3} $$

is called Riemann- Liouville right-sided fractional integral of order α.

The definition of Gauss's hypergeometric function in terms of Pochhamer symbol:

$$ {}_2F_1(a,b;c;x) = \sum_{r=0}^{\infty} \frac{(a)_r \ (b)_r}{(c)_r} \frac{x^r}{r!} \tag{2.4} $$

From Rainville [11], we have

$$ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k) \tag{2.5} $$

3. Main Result

In this section, we establish two theorems of fractional integral involving product of hypergeometric function and $\bar{H}$- function.

Theorem 3.1. *Let M, N, P and Q be non-negative integers such that* $0 \leq N \leq P, 0 \leq M \leq Q$ *and* $\sum_{j=1}^{N} A_j\alpha_j - \sum_{j=N+1}^{P} \alpha_j + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q} \beta_j B_j > 0$. *Then we have the following result:*

$$ \begin{aligned} &{}_aI_x^{\alpha}\{x^{\rho-1} (a-x)^{\sigma-1} e^{-xt} \, {}_2F_1(\alpha, \beta; \gamma; bx^{\zeta}) \\ &\times \bar{H}_{P,Q}^{M,N} \left[yx^{\mu}(x-a)^{\nu} \Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}} \right] \} = (-1)^{\alpha+\sigma+n-k-1} \\ &\cdot \sum_{l,n=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{t^{n-k}}{(n-k)!\, l!} (a-x)^{l+\alpha+\sigma-n-k-1} a^{\rho-l+\zeta k-1} \bar{H}_{P+3,Q+1}^{M+2,N+2} \\ &\cdot \left[a^{\mu} (a-x)^{\nu} y \Big|_{y_1}^{y_2} \right] \end{aligned} \tag{3.1} $$

where

$$ \begin{aligned} y_1 &= (1-\sigma+l-\zeta k, \mu), (\alpha-\sigma+n-k, \nu), (b_j, \beta_j)_{1,M}, \\ &\quad (b_j, \beta_j; B_j)_{M+1,Q}, (1-\sigma+n+k, \nu), \\ y_2 &= (1-\sigma+n+k, \nu), (1-l-\sigma+n+k, \nu), (a_j, \alpha_j; A_j)_{1,N}, \ (a_j, \alpha_j)_{N+1,P}, \\ &\quad (\alpha+\sigma-n-k, \nu), (1-\rho+\zeta k, \mu), (\alpha-\sigma+n-k-l, \nu), \end{aligned} $$

$f(k) = \frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k}\frac{b^k}{k!}$. *Also, satisfy the following conditions:*

(i) ρ, σ *are complex numbers and* μ, ν *are positive real numbers;*

(ii) $|\arg z| < \frac{1}{2}A\pi, A$ *defined as* $A = \sum\limits_{j=N+1}^{P} \alpha_j - \sum\limits_{j=M+1}^{Q} \beta_j B_j > 0$*;*

(iii) $\min[Re(\frac{1-\rho-\sigma}{\mu-\nu}), \min 1 \leq j \leq M[Re(\frac{b_j}{\beta_j})]] > \max[-Re(\frac{(\sigma)}{\nu}), \max 1 \leq j \leq N[Re(\frac{a_j-1}{\alpha_j A_j})]]$.

Proof. The exponential function change into summation series , express the hypergeometric function with the help of equation (2.4), using the equations (2.5) and (2.2), interchanging the order of integration, applying Euler's integral formula of hypergeometric function, we obtain the form after little simplification (say I_1):

$$I_1 = (-1)^{\alpha+\sigma+n-k-1} e^{-at} \sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k) \frac{t^{n-k}}{(n-k)!}\frac{1}{\Gamma\alpha}\,(a-x)^{\alpha+\sigma-n-k-1}\,a^{\rho+\zeta k-1}$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Gamma(\sigma-n-k+\nu\xi)\Gamma(\alpha)}{\Gamma(\alpha+\sigma-n-k-\nu\xi)}\,{}_2F_1(\sigma+n-k+\nu\xi, 1-\rho+\mu\xi-\zeta k; \tag{3.2}$$

$$\alpha-\sigma+n+k+\nu\xi; \frac{b-x}{b})\,b^{\mu\xi}\,(x-b)^{\nu\xi}\,y^{\xi}\,\bar{\phi}(\xi)\,d\xi.$$

Finally, re-interpreting the Mellin-Barnes counter integral, we obtain the right hand side of the equation(3.1). This completes proof of theorem 3.1 . □

Theorem 3.2. *Let M, N, P and Q be non-negative integers such that* $0 \leq N \leq P, 0 \leq M \leq Q$ *and* $\sum\limits_{j=1}^{N} A_j\alpha_j - \sum\limits_{j=N+1}^{P} \alpha_j + \sum\limits_{j=1}^{M} \beta_j - \sum\limits_{j=M+1}^{Q} \beta_j B_j > 0$*. Then we have the following result:*

$$\begin{aligned} &{}_xI_b^{\alpha}\{x^{\rho-1}\,(b-x)^{\sigma-1}\,e^{-xt}\,{}_2F_1(\alpha,\beta;\gamma;bx^{\zeta}) \\ &\times\ \bar{H}_{P,Q}^{M,N}\left[yx^{\mu}(x-b)^{\nu}\Big|^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}\right]\} \\ &= (-1)^{\alpha+n-k-1}e^{-bt}\sum_{l,n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!\,l!}\,(x-b)^{l+\alpha+\sigma-n-k-1} \\ &\quad b^{\rho-\zeta k-1}\bar{H}_{P+2,Q+2}^{M+1,N+3}\left[b^{\mu}\,(x-b)^{\nu}y\Big|^{y_2}_{y_1}\right], \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} y_2 &= (1-\sigma+n+k,\nu),(1-l-\sigma-n+k,\nu),(1-\alpha-n-k+\sigma),\\ &\quad (a_j,\,\alpha_j;A_j)_{1,N},(a_j,\,\alpha_j)_{N+1,P},(\alpha-\sigma-n-k,\nu),(1-\rho+\zeta k,\mu)\\ y_1 &= (1-\rho+l-\zeta k,\nu),(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q},\\ &\quad (1-\sigma-n+k,\nu),(1-l-\alpha+\sigma-n-k,\nu), \end{aligned}$$

$f(k)$ given in equation (3.1) and condition (i)-(iii) in Theorem 3.1 are also satisfied.

Proof. The exponential function change into summation series, express the hypergeometric function with the help of equation (2.4), using the equations (2.5) and (2.3), interchanging the order of integration, applying Euler's integral formula of hypergeometric function, we obtain the form after little simplification (say I_2):

(3.4)

$$\begin{aligned} I_2 &= (-1)^{\alpha+n-k-1}e^{-bt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{t^{n-k}}{(n-k)!}\frac{1}{\Gamma\alpha}\,(x-b)^{\sigma-n-k-\nu\xi-1}\,b^{\rho-\zeta k-1}\\ &\times\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Gamma(\sigma-n-k+\nu\xi)\Gamma(\alpha)}{\Gamma(\alpha+\sigma-n-k-\nu\xi)}\,{}_2F_1[(\sigma+n-k+\nu\xi),1-\rho+\mu\xi-\zeta k;\\ &\quad(\alpha-\sigma+n+k+\nu\xi);\frac{b-x}{b})]\,b^{\mu\xi}\,(x-b)^{\nu\xi}\,y^{\xi}\,\bar{\phi}(\xi)\,d\xi. \end{aligned}$$

Finally, re-interpreting the Mellin-Barnes counter integral, we obtain the right hand side of the equation(3.3). This completes proof of theorem 3.2. □

4. Special Cases

On specializing the parameter of the hypergeometric function and $\bar{H}$- function the result in equation (3.1) leads the following results:

(i) Putting $t = 0$ and $b = 0$ in equation (3.1), the exponential function and hypergeometric function reduces to unity and same time replacingρ by $\lambda + 1$

and σ by 1, equation (3.1) reduces to the result.

(4.1)
$$ {}_aI_x^{\alpha}\{x^{\lambda}\,\bar{H}_{P,Q}^{M,N}\left[yx^{\mu}(x-a)^{\nu}\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]\} = a^{\lambda}(x-a)^{\alpha} \sum_{l=0}^{\infty}\frac{\left(\frac{a-x}{a}\right)^l}{l!}\bar{H}_{P+1,Q+1}^{M+1,N+1}\left[a^{\mu}(x-a)^{\nu}y\Big|_{(l-\lambda,\mu),(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q},(-l-\alpha,\nu)}^{(-l,\nu),(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P},(-\lambda,\mu)}\right]$$

(ii) Replacing ν by $-\nu$ in equation (4.1) reduce to the result

(4.2)
$$ {}_aI_x^{\alpha}\{x^{\lambda}\,\bar{H}_{P,Q}^{M,N}\left[yx^{\mu}(x-a)^{-\nu}\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]\} = a^{\lambda}(x-a)^{\alpha} \sum_{l=0}^{\infty}\frac{\left(\frac{a-x}{a}\right)^l}{l!}\bar{H}_{P+2,Q}^{M+2,N}\left[a^{\mu}(x-a)^{-\nu}y\Big|_{(1+l,\mu),(l-\lambda,\mu),(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(-l,\mu),(1+l+\alpha,\nu),(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P},(-\lambda,\mu)}\right]$$

(iii) Next replacing μ by $-\mu$ in equation (4.1) reduce to the result

(4.3)
$$ {}_aI_x^{\alpha}\{x^{\lambda}\,\bar{H}_{P,Q}^{M,N}\left[yx^{-\mu}(x-a)^{\nu}\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]\} = a^{\lambda}(x-a)^{\alpha} \sum_{l=0}^{\infty}\frac{\left(\frac{a-x}{a}\right)^l}{l!}\bar{H}_{P+1,Q+1}^{M+1,N+1}\left[a^{\mu}(x-a)^{\nu}y\Big|_{(l-\lambda,\mu),(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q},(-l-\alpha,\nu)}^{(-l,\nu),(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P},(-\lambda,\mu)}\right]$$

(iv) Putting $t = 0$ and $b = 0$ in equation (3.3), the exponential function and hypergeometric function reduces to unity and same time replacing ρ by $\lambda + 1$ and σ by 1, equation (3.3) reduces to the result:

(4.4)
$$ {}_xI_b^{\alpha}\{x^{\lambda}\,\bar{H}_{P,Q}^{M,N}\left[yx^{\mu}(x-b)^{\nu}\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]\} = (-1)^{\alpha}\,b^{\lambda}(x-b)^{\alpha} \sum_{l=0}^{\infty}\frac{\left(\frac{b-x}{b}\right)^l}{l!}\bar{H}_{P+1,Q+1}^{M+1,N+2}\left[b^{\mu}(x-b)^{\nu}y\Big|_{(l-\lambda,\mu),(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q},(-l-\alpha,\nu)}^{(-\lambda,\mu),(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P},(-l,\nu),(-\alpha,\nu)}\right]$$

(v) Replacing ν by $-\nu$ in equation (4.4) reduce to the result.

(4.5)
$$ {}_xI_b^{\alpha}\{x^{\lambda}\,\bar{H}_{P,Q}^{M,N}\left[yx^{\mu}(x-b)^{-\nu}\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]\} = (-1)^{\alpha}\,b^{\lambda}(x-b)^{\alpha} \sum_{l=0}^{\infty}\frac{\left(\frac{b-x}{b}\right)^l}{l!}\bar{H}_{P+2,Q}^{M+3,N}\left[b^{\mu}(x-b)^{\nu}y\Big|_{(1+l,\nu),(l-\lambda,\mu)(1+\alpha,\nu),(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(-\lambda,\mu),(1+l+\alpha,\nu),(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]$$

(vi) Next replacing μ by $-\mu$ in equation (4.4) reduces to the result.

(4.6)
$$ {}_xI_b^{\alpha}\{x^{\lambda}\,\bar{H}_{P,Q}^{M,N}\left[yx^{-\mu}(x-b)^{\nu}\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}}\right]\} = (-1)^{\alpha}\,b^{\lambda}(x-b)^{\alpha} \sum_{l=0}^{\infty}\frac{\left(\frac{b-x}{b}\right)^l}{l!}\bar{H}_{P,Q+1}^{M,N+3}\left[b^{-\mu}(x-b)^{\nu}y\Big|_{(b_j,\beta_j)_{1,M},\,(b_j,\beta_j;B_j)_{M+1,Q},(1+\lambda,\mu)}^{(a_j,\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P},(-l,\nu),(-\alpha,\nu),(1-l+\lambda,\mu)}\right]$$

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