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WAVELET FRAMES AND TIME-FREQUENCY LOCALIZATION IN LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. We construct a wavelet frame system on locally compact abelian (LCA) group G associated with the multiresolution analysis and Haar measures. We show the characterization of the wavelet frame set and the scaling sequence on $L^2(G)$. The dilation and translation of wavelet frame sets for time-frequency localization in LCA groups have been set up. We obtain an orthonormal wavelet basis for $L^2(G)$ using the scaling sequence. We also establish the relationship between multiresolution analysis and wavelet functions. Finally, we obtain periodization for the multiresolution analysis using time-frequency localization on a periodic wavelet frame. The periodization holds wavelets' regular properties and decay conditions.

1. INTRODUCTION

Mallat [10] introduced the classical multiresolution analysis which was a sequence of increasing function on closed subspace $\{U_i\}_{i\in N_0}$ of $L^2(G)$ such that $\bigcap_{i\in N_0} U_i = \{0\}$, $\bigcup_{i\in N_0} U_i$ dense in $L^2(R)$; which satisfies $h(z) \in U_i$ and $h(\alpha z) \in U_{i+3k}, k = 1, 2, 3, \cdots, i \in N_0$, where α is a scalar and $N_0 = N \cup \{0\}$. Again, there

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exists an element $\Phi \in U_0$ such that the collection of translation of a wavelet function $\Phi_i\{\Phi(z-3k) : k = 1, 2, 3, \cdots\}$ represents a compact space of orthonormal basis of increasing sequence for U_0; the function Φ is represented the wavelet scaling function. In recent years, the work of multiresolution analysis and wavelet functions has been generalized in much different literature Azarmi [1] and Daubechies [6]. Mallat [10] has constructed an orthonormal wavelet basis for compact space $L^2(R)$. Shah et al. [12] constructed wavelet frame packets associated with multiresolution analysis, and wavelet frames form an orthonormal basis for $L^2(G)$. Chen [5] have constructed wavelet frame packets using multiresolution analysis, corresponding to an orthonormal wavelet basis for $L^2(G)$. Long et al. [9] has given the concept of bi-orthogonal multiple wavelets generated from transformation and provided the method for establishing compactly supported bi-orthogonal multiple wavelets by the same function. The aim of the paper is to obtain wavelet frames from the splitting trick of multiresolution analysis on $L^2(G)$ and constructing wavelet on an orthonormal basis of $L^2(G)$.

The primary motivation of the work starts from the work of Benedetto and Benedetto [2] who introduced a wavelet theory for local fields and related groups. Bownik and Jahan [3] introduced the concept of characterization of scaling sequence of a multiresolution analysis on $L^p(G)$, $1 \leq p \leq \infty$. Also, constructed an orthonormal wavelet basis of $L^2(G)$ using a scaling sequence. Gol and Tousi [8] have generalized a shift-invariant space involving the spectral function from R^n to the locally compact abelian (LCA) group. The spectral function and scaling sequence are characterized in terms of all conditions of multiresolution analysis. More results in this direction can be found in [4, 11].

We characterize the theory of a multiresolution analysis of $\{U_i\}_{i\in N_0}$ on an LCA group. We define the scaling sequence concept to the structure of wavelet basis in any space U_i to U_{i+1}. We establish a multiresolution analysis transformation on $L^2(G)$, which holds the periodic wavelet frame regular properties and decay conditions. We find dilations and translations of wavelet frame sets for time-frequency localization in LCA groups. Then we obtain an orthonormal wavelet basis for $L^2(G)$. We establish the relationship between multiresolution analysis and wavelet functions. We obtain periodization for the multiresolution analysis using time-frequency localization on a periodic wavelet frame.

2. PRELIMINARIES

Let G be an LCA group and $\widehat{G}$ its dual group. Shah et al. [12] defined multiresolution analysis and orthonormal basis properties by taking some choice of LCA groups. Let q be the natural number > 1 and the sequence of the form $z = (z_i) = (\cdots z_{3m-2}, z_{3m-1}, z_{3m}, z_{3m+1}, \cdots)$, where $z_i \in \{0, 1, 2, 3, \cdots (q-1)\}$ for $i \in N_0$ and $z_i = 0$ for $i < m = m(z)$; the group operation on G is defined as the component-wise addition. The LCA topological group on G is determined by the system of neighborhoods of complete inner product space as $V_l = \{(z_i) \in G : z_i = 0 \text{ for } i \leq l\}$, $l \in Z$. Clearly, each neighborhood of complete space V_l is a compact abelian subgroup of G, $V_{l+1} \subset V_l$ for $l \in Z$ and $\bigcap V_l = 0$. If set $V = V_0$, then the group under operation is multiplication. and if $V \neq V_0$, then the group operation is addition. If LCA G is having an additive inverse element, then G exists compact abelian subgroups.

For $1< p < \infty$, we take $L^p(G)$ as the Lebesgue spaces of Borel's subgroup of G defined by the Haar measure ρ with $\rho(V_0) = \rho(V) \leq 1$. Let $\widehat{G}$ be the compact abelian dual group of an abelian group G and the group G of all sequences of the form $\xi = (\xi_i) = (\cdots, \ \xi_{3k-2}, \ \xi_{3k-1}, \ \xi_{3k}, \ \xi_{3k+1}, \ \xi_{3k+2}, \ \cdots)$, where $\xi_i \in \{0, 1, 2, 3, 4, \cdots (q-1)\}$ for $i \in N_0$ and $\xi_i = 0$ for $i < m = m(z)$. Then the group operation is co-ordinate wise addition. The neighborhood of compact support $\widehat{V_l}$ and the Haar measure $\widehat{\mu}$ for $\widehat{G}$ are imported as compact abelian subgroups for G. If S is disconnected compact abelian subgroup of G, then $S = \{z_i \in G : z_i = 1, i > 0\}$.

Folland [7] has given the concept of the quotient group $S/A(S)$ contains q elements and the orthogonal subgroup $S^{\perp}$ of S consists of all sequence ρ_i of $\widehat{G}$ holds the condition $\rho_i = 1$ for $i > 1$, where A is an abelian group automorphism of G. Let G be a compact abelian group and R set of real number, then the linear map $T : G \to \widehat{G}$ by $T(z) = \sum_{i \in I} z_i q^i$, $z \in G$, where T is a transformation of the compact abelian subgroup V onto the interval $(0, \infty)$ as defines an isomorphism of Banach spaces (G, ρ) and (R, χ), w is the lebesgue measure of the infinite measure on set of real number R and I be an indexing set. Also, the image of N under T is the set of positive integer $T(N) = Z^+$; thus, for every $\beta \in Z^+$. If $T(h_\beta) = \beta$, then T is called compact group automorphism where h_β is an element of N. The prestige hypothesis are needed to authorization that an multiresolution analysis $\{U_i\}_{i \in N_0}$ satisfies the density property $\overline{\bigcup_{i=0}^{\infty} U_i} = L^p(G)$

where $0 < p < \infty$ which are able to an epimorphism $E : G \to G$ with a finite kernel such that $\bigcup_{i\in N_0} \ker E^i$ dense in G, i.e. $\ker E^i = \{(Y, Z) \in G \times G : z_{i+1} = z_{i+2} = \cdots = 0, z \in Z\}$ with epimorphism E and $G = R/Z$. In this process, we define a transformation $\widehat{T} : \widehat{G} \to G$ the compact abelian subgroup automorphism $H \in \text{Aut}\widehat{G}$, the subgroup $\widehat{G}$ and the element ρ_β of $S^\perp$ for G, we character $\omega(Az, \rho) = T(z, H\nu)$ for all $z \in G, \rho \in \widehat{G}$.

Definition 2.1. *[1] Let G be a LCA group and $A \in Aut(G)$. The operator δ_A is defined on $L^2(G)$ by $\delta_A h(z) = |A|^{1/2} h(Az)$, for all $h \in L^2(G)$.*

Definition 2.2. *[2] Let G be a LCA group with compact open subgroup $S \subseteq G$, let Θ be a choice coset representatives in $\widehat{G}$ for $\widehat{S} = \widehat{G}/S^\perp$, let $A \in Aut(G)$, and consider $[s] \in G/S$. The dilated translate of $h \in L^2(G)$ is defined as*

$$h_{A,[s]}(z) = \delta_A \tau_{[s],\Theta} h(z) = |A|^{1/2} (h \times \widehat{w}_{[s],\Theta})(Az), \tag{2.1}$$

where $\widehat{w}_{[s],D}$ is the pseudo-measure.

3. Main Results

3.1. **Wavelet frames on LCA groups.** Let G be an LCA group and consider that G consists of a disconnected finite or countably infinite subgroup S such that the quotient group G/S is compact. Moreover, we suppose that an automorphism A of G such that $A(S) \subset S$. A sequence $\{U_i\}_{i\in N_0}$ of closed subspace of $L^2(G, \mu)$ is called a multiresolution analysis of $L^2(G, \mu)$ where μ stand for the Haar measure on G, if the following conditions are satisfied:

(i) $\overline{U_i \subseteq U_{i+1}}, i \in N_0$;
(ii) $\bigcup_{i\in N_0} U_i = L^2(G)$;
(iii) $\bigcap_{i\in N_0} U_i = \{0\}$;
(iv) $h \in U_i \iff \sigma h \in U_{i+1}$ i.e. $U_i = \sigma^i U_0, i \in N_0, \sigma \in G$;
(v) U_0 is left shift invariant i.e. if $h \in U_0$ then $L_\gamma h$ invariant subspace of U_0;
(vi) The collection $\{L_\gamma h : \gamma \in \Gamma$ is an orthonormal basis of U_0, Γ a discrete topological subgroup of G;
(vii) There exists a scaling function Φ such that the collection $\{\Phi(\cdot oj)\}_{j\in S}$ of translates of Φ are stable and U_0 is the closed linear span of $\Phi(\cdot oj)$ where o is the group operation of G.

Now a family of wavelets $\{\psi_1, \psi_2, \cdots \psi_N\}$ are called system of functions, the orthogonal complement W_0 of U_0 in U_1 is constructed by using the translation of $\{\psi_1, \psi_2, \cdots \psi_N\}$. Since the union of the space $\{U_i\}_{i\in N_0}$ is dense in $L^2(G, \mu)$ where their intersection is zero, we observe that the function $\{\psi_1, \psi_2, \cdots \psi_N\}$ is scaled and translated to span $L^2(G, \mu)$. Then we have $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\}$ is called a wavelet generator for $L^2(G, \mu)$. After that we are able to find a single wavelet for $L^2(G, \mu)$, if $\Psi = \{\psi\}$. Let $\{\Omega_1, \Omega_2, \cdots \Omega_N\}$ be a measurable subset of $\widehat{G}$, and let $\psi_i = 1_{\Omega_i}$, for each $i = 1, 2 \cdots N$. We call this $\{\Omega_1, \Omega_2, \cdots \Omega_N\}$ is a wavelet collection of sets if $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\}$ is a wavelet generator for $L^2(G, \mu)$. If $N = 1$, then $\Omega = \Omega_1$ is a wavelet set. Since the frame is a constant multiple of the wavelet set itself see [8], recovering functions from their frame coefficient does not require the frame's computation. Hereafter, we shall focus on wavelet frames.

Given $h \in L^2(G)$, let $h_{i,k}$ denote the scale and shift invariant function

$$h_{i,k} = D_{2_{-i}} T_k h, \quad 1 \le i \le N, k \in G/S, \tag{3.1}$$

where G/S is a quotient group, D and T are dilation and translation operator respectively. Let I be an index set. For given $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\} \subset L^2(G)$, let $\Theta \subseteq \widehat{G}$ a choice of coset acting in $\widehat{G}$ for $\widehat{S} = \widehat{G}/S^{\perp}$, $A \subseteq \text{Aut}(G)$ be a countable nonempty set of automorphisms of G and a coset $B \subseteq G/S$. We define the wavelet system

$$X(\Psi) = \{\psi_{i,j,k} : 1 \le i \le N, j \in A, k \in B\}, \tag{3.2}$$

where $\psi_{i,j,k} = 2^{j/2}\psi_i(2^j \cdot -k)$, and j and k encode certain dilation and translation information. The wavelet system $X(\Psi) \subset L^2(G)$ is called a wavelet frame with frame bound C if

$$h(z) = \frac{1}{C} \sum_{1\le i\le N} \sum_{j\in A} \sum_{k\in B} \langle h, \psi_{i,j,k}\rangle \psi_{i,j,k}(z), \quad \text{for all} \quad h \in L^2(G).$$

This is equivalent to saying that the wavelet system $X(\Psi) \subset L^2(G)$ is a wavelet frame with frame bound 1 and $\|\psi_i\|_{L^2(G)} = 1$ for $1 \le i \le N$. A function $\Phi \in L^2(G)$ is called scaling function, if it is satisfies a scaling sequence $\Phi_{j,k} = 2^{j/2}\Phi(2^j \cdot -k)$ and j and k encode certain dilation and translation information.

3.2. **Wavelet frames and (τ, Θ)-congruence.** Now we formulate wavelet frames on the LCA group G by appropriate dilation and translation operators.

Definition 3.1. *Let G be a LCA group with compact open subgroup $S \subseteq G$, let $\Theta \subseteq \widehat{G}$ be a choice of coset acting in $\widehat{G}$ for $\widehat{S} = \widehat{G}/S^{\perp}$ and let $A \subseteq Aut(G)$ be a collection of all group group automorphisms of G and $B \subseteq G/S$. Consider $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\} \subseteq L^2(G)$. The wavelet system $X(\Psi)$ is a wavelet frame for $L^2(G)$ corresponding to Θ and A if $\{\psi_{i,j,k} : 1 \leq i \leq N, j \in A, k \in B\}$ form an orthonormal basis for $L^2(G)$, where $\psi_{i,j,k}(z) = \delta_j \tau_k \Theta \psi_i(z) = 2^{j/2}\psi_i(2^j \cdot -k)(jz)$.*

Definition 3.2. *Let G be a LCA subgroup $S \subseteq G$, and $\Theta \subseteq \widehat{G}$ be a choice of co-set acting in $\widehat{G}$, and let U and $U^{'}$ be sub sets of $\widehat{G}$. We say U is (τ, Θ)-congruence to $U^{'}$, if there exist an indexing set $I \subseteq Z$ as well as $\{U_m : m \in I\}$ and $\{U^{'}_m : m \in I\}$ are multiresolution analysis of U and $U^{'}$ respectively, into Lebesgue measure subsets and sequence $\{\iota_m\}_{m\in I}$, $\{\iota^{'}_m\}_{m\in I} \subseteq \Theta$ such that for all $m \in I$, $U_m \subseteq \iota_m + S^{\perp}$ and $U_m = U^{'}_m - \iota^{'}_m + \iota_m$.*

Theorem 3.1. *Let G be a LCA group with compact open subgroup $S \subseteq G$, let $\Theta \subseteq \widehat{G}$ be a choice of co-set acting on $\widehat{G}$ for $\widehat{S} = \widehat{G}/S^{\perp}$, and let $A \subseteq Aut(G)$ be a countable infinite set of $Aut(G)$ and a coset $B \subseteq G/S$. Assuming that $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\}$ is a set of wavelet generators for $L^2(G)$. The wavelet system $X(\Psi)$ in Eq.* (3.2) *is a wavelet frame set if and only if following two points hold:*

(i) *$\{Q^*\psi_{i,j,k} : Q \in A, 1 \leq i \leq N, j \in A, k \in B\}$ design on $\widehat{G}$ up to sets of zero measure, where Q^* is an action of the adjoint automorphism on $\widehat{G}$;*

(ii) *For all $i = \{1, 2, 3, \cdots N\}$, $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\}$ is (τ, Θ)-congruence to $S^{\perp}$ up to set of zero measure.*

Proof. Let a finite sequence $\{\Omega_1, \Omega_2, \cdots \Omega_N\}$ with $\psi_i = \check{1}_{\Theta_i}$, where $\check{1}$ the inverse Fourier transform of an indicator function 1_{Θ}. Recall that $\psi_{i,j,k}(z) = \delta_j \tau_k \Theta \psi_i(z) = 2^{j/2}\psi_i(2^j \cdot -k)(jz)$ and also that $\iota^{'}_0$ is the unique element of $\Theta \bigcap S^{\perp}$. First we consider condition (ii), this condition implies that all $u(\psi_i) = 1$, because $\widehat{G}$ is ι-compact each $u(\psi_i) = 1$ and each $1_{\Theta_i} \in L^2(\widehat{G})$. Let $I_i \subseteq Z$ be the index set for (τ, Θ)-congruence and let $\{U_{i,m} : m \in I_i\}$ be the corresponding partition of ψ_i, since ψ_i is (τ, Θ)-congruence to $S^{\perp}$, we get

$$u(\psi_i) \;=\; \sum_{m\in I_i} u(U_{i,m}) = \sum_{m\in I_i} u(U_{i,m} + \iota^{'}_m - \iota_m) = \sum_{m\in I_i} u(U^{'}_{i,m}) = u(S^{\perp}) = 1,$$

where $\{U^{'}_{i,m} : m \in I_i\}$ is a partition of $S^{\perp}$ and $\iota^{'}_m = \iota\iota^{'}_0$ for all $m \in I_i$. When properties (ii) is consider, we have $\phi_i = 1_{\Omega_i} \in L^2(\widehat{G})$. Furthermore, $\|1_{\Omega_i}\| = 1$ so that $\|\phi_i\|_2 = 1$.

Conversely, if $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\}$ is called a wavelet generator and $\{\Omega_1, \Omega_2, \cdots \Omega_N\}$ be a measurable subset of $\widehat{G}$, then

$$1 = \|\phi_i\|_2 = \|1_{\Omega_i}\|_2 = v(\Theta_i).$$

We shall also need the fact that condition (i) implies that $\{\psi_1, \psi_2, \psi_3, \cdots \psi_N\}$ pairwise disjoint upto sets of measure zero, for any $Q \in A$ and we observe that for $i \neq k$

$$u(\psi_i \cap \psi_k) \;\; = \;\; |Q|^{-1} u(Q^*(\psi_i \cap \psi_k)) = |Q|^{-1} u(Q^m(\psi_i) \cap Q^m(\psi_k)) = 0,$$

by condition (i).

Next we consider conditions (i) and (ii) implies that $\{\psi_{i,j,k} : 1 \leq i \leq N, j \in A, k \in B\}$ form an orthonormal basis for $L^2(G)$. We observe that $\|\psi_{i,j,k}\|_2 = 1$ for any $j \in A$ and $k \in B$. Indeed we compute

$$\|\psi_{i,j,k}\|_2^2 = \int_{\widehat{G}} \left|\widehat{\psi_{i,j,k}}(\alpha)\right|^2 d\alpha = |Q|^{-1} \int_{\widehat{G}} \left|\widehat{\psi}_i\left((Q^*)^{-1}\alpha\right)\right|^2 d\alpha = u(\psi_i) = 1.$$

The system $X(\Psi)$ collection of wavelet frames, let $\Theta \subseteq \widehat{G}$ be co-set representatives in $\widehat{G}$ for $\widehat{S} = \widehat{G}/S^{\perp}$ implies ii. Let $\{\psi_1, \psi_2, \psi_3, \cdots \psi_N\}$ is generator of wavelet set, $Q \in A$, and $\{\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_N\}$ be a measurable sub set of $\widehat{G}$ $u(Q^m\psi_i) = |Q|u(\psi_i) = |Q| < \infty$. Thus, there are countable numbers of $\iota \in \Theta$ such that $u((\iota + S^{\perp}) \cap (Q^*\psi_i)) > 0$. Let $H = \bigcup_{Q\in A} Q^*\psi_i$, which countably infinite $\iota \in \Theta$ such that $u((\iota + S^{\perp}) \cap H) > 0$. Choose Θ are countably infinite, then there are for all $\iota \in \Theta$ such that $u((\iota + S^{\perp}) \cap H) = 0$. Let $M = 1_{\iota} + S^{\perp} \in L^2(\widehat{G})$, then for all $j \in A, 1 \leq i \leq N$ and $k \in G/S$, we get $\langle M, \psi_{i,j,k}\rangle = 0$, because $\psi_{i,j,k}$ zero of H. Therefore $1 = \|M\|_2^2 = \sum_{1\leq i\leq N} \sum_{j\in A} \sum_{k\in B} |\langle M, \psi_{i,j,k}\rangle|^2 = 0$, and so that for all $i = \{1, 2, 3, \cdots N\}$, $\Psi = \{\psi_1, \psi_2, \cdots \psi_N\}$ is (τ, Θ)-congruence to $S^{\perp}$ up to set of zero measure. Finally, we show that conditions (i) and (ii) are initial conditions for $\{Q^*\psi_{i,j,k} : Q \in A, 1 \leq i \leq N, j \in A, k \in B\}$ design on $\widehat{G}$ up to sets of zero measure. By the properties (ii) implies that $Q^*\psi_i$ rectangular shape $\widehat{G}$ upto set of Lebesgue nonzero measure, if $(i, X), (k, Y) \in \{1, 2, 3, \ldots, N\} \times \psi_{i,j,k}$ are the distinct pairs, then $\langle \psi_{i,X,[0]}, \psi_{i,Y,[0]}\rangle = 0$ by orthogonality. where $X, Y \in A$. Therefore,

$$u(X^m\psi_i \cap Y^m\psi_i) = \langle 1_{X^m\Omega_i}, 1_{Y^m\Omega_i}\rangle = \langle \psi_{i,X,[0]}, \psi_{i,Y,[0]}\rangle = 0.$$

For each ψ_i is (τ, Θ)-congruence to $S^{\perp}$, since $u(\psi_i) = 1 < \infty$, there are countably a many coset $\iota + S^{\perp}$ such that $\psi_i \cap (\iota + S^{\perp})$ has the zero measure. Let $\{\iota_m : m \in I_i\}$

be the set of some $\iota \in \Theta$. Let $U_m = Q^m \cap (\iota_m + S^{\perp})$ and $V_m^{'} = U_m - \iota_m + \iota_0^{'}$, where $\iota_0^{'} \in \Theta \cap S^{\perp}$. Clearly, $\{U_m\}$ is a partition of Q^m and $\{U_m^{'}\}$ is a partition of $S^{\perp}$. This is complete the proof. □

Remark 3.1. *If $U_m = U_m^{'} - \iota_m^{'} + \iota_m$, then $U_m \subseteq \iota_m + S^{\perp}$ is equivalent to $U_m^{'} \subseteq \iota_m^{'} + S^{\perp}$. Thus, in the case $U^{'} = S^{\perp}$, we take $\iota_m^{'} = \iota_0^{'}$ for all m where $\iota_0^{'}$ is the unique element of $\Theta \bigcap S^{\perp}$. Clearly (τ, Θ) - congruence is an equivalence relation and it preserves Haar measure.*

3.3. **Construction of wavelet functions.** We establish an orthonormal wavelet basis of $L^2(G)$. Given an multiresolution analysis $\{U_i\}_{i\in N_0}$ of closed subspace of $L^2(G)$, we define wavelet spaces as the orthogonal complements of spaces U_i in U_{i+1}. We Construct wavelet functions whose shifts form the basis in these orthogonal complement spaces.

Definition 3.3. *Let $U = \oplus_{j\in A, k\in B} U_{\Phi_{j,k}}$ be a shift invariant subspace of $L^2(G)$, where $U_{\Phi_{j,k}} = span\{T_l\Phi_{j,k} : l \in kerE^i\}$ and $\Phi_{j,k}$ is a scaling equation of $U_{\Phi_{j,k}}$. The function $T_l\Phi_{j,k}$ describe on $\widehat{G}$ by Fourier transform*

$$T_l\Phi_{j,k} = \sum_{1\le i\le N} \sum_{j\in A} \sum_{k\in B} \left|\widehat{\Phi}_{j,k}\widehat{\psi}_{i,j,k}\right|^2 \tag{3.3}$$

is called the wavelet function of U.

Definition 3.4. *Let G be an LCA group and set of all group automorphisms A of $A \subset Aut(G)$ with $kerE^i$. We define operators ω_κ^i on $L^2(G)$, for $i \in N_0$, $j \in A$ $\kappa \in \widehat{G}$, as follows*

$$\omega_\kappa^0 = h, \quad \omega_\kappa^i h(z) = \frac{2^j}{kerE^i} \sum_{\alpha\in kerE^i} \left(\overline{\kappa(\alpha)}h(z+\alpha)\right)\overline{j(\alpha)}. \tag{3.4}$$

Proposition 3.1. *Let $\{U_i\}_{i\in N_0}$ be an multiresolution analysis of $L^2(G)$ with scaling sequence $\{\Phi_{j,k}\}_{j\in A,k\in B}$. The following are equivalent:*

(i) *The structure $\{T_l\Phi_{j,k}\}_{l\in kerE^i, j\in A, k\in B}$ is orthonormal basis,*
(ii) *The structure $\{n^{i/2}w_\kappa^i\Phi_{j,k}\}_{\kappa\in D(E^i), j\in A, k\in B}$ is orthonormal basis, where the operators ω_κ^i are as in Definition* (3.4) *and $n = |kerE|$,*
(iii) *We get*

$$\left\langle \omega_\kappa^i\Phi_{j,k}, \omega_\kappa^i\Phi_{j,k}\right\rangle = n^{-i} \quad for\ all \quad \kappa \in D(E^i) \subseteq Aut(G). \tag{3.5}$$

Proof. By Lemma 3.6. [3] we have

$$T_l\Phi_{j,k} = \sum_{\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\omega_\kappa^i(T_l\Phi_{j,k}) = \sum_{\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\overline{\kappa(l)}\omega_\kappa^i\Phi_{j,k}. \tag{3.6}$$

By the Plancherel formula and Eq. (3.3) for any $h, g \in L^2(G)$ we have

$$\left\langle \omega_\kappa^i h, \omega_{\kappa'}^i g\right\rangle\left\langle \widehat{\omega_\kappa^i} h, \widehat{\omega_{\kappa'}^i} g\right\rangle = 0, \quad \text{for} \quad \kappa \neq \kappa' \in D(E^i). \tag{3.7}$$

Hence, for any $l, m \in \ker E^i$,

$$\begin{aligned}
&\left\langle T_l\Phi_{j,k}, T_m\Phi_{j,k}\right\rangle \\
&= \Big\langle \sum_{\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\overline{\kappa(l)}\omega_\kappa^i\Phi_{j,k}, \sum_{\kappa'\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\overline{\kappa'(m)}\kappa(m)\omega_{\kappa'}^i\Phi_{j,k}\Big\rangle \\
&= \sum_{\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\overline{\kappa(l)}\kappa(m)\left\langle \omega_\kappa^i\Phi_{j,k}, \omega_\kappa^i\Phi_{j,k}\right\rangle.
\end{aligned} \tag{3.8}$$

From Theorem 3.4. [12], we have

$$\sum_{\eta\in D(E^i)}\eta(n-k) = \begin{cases} |\ker E^i|, & \text{for} \quad n = k, \\ 0, & \text{otherwise.} \end{cases}$$

This provides the needed equation (3.5) and hence, conflict can converse. □

Taken away now, we consider that $\{\Phi_{j,k}\}_{j\in A,k\in B}$ is an orthonormal scaling sequence. Particularly, Proposition (3.1) and Eq. (3.5) influence for all $1 \leq i \leq N$. Recall that by Definition (3.4) we get

$$\omega_\kappa^i\Phi_{j,k} = \sum_{\pi,\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\mu^{i+1}_{\kappa+\widehat{E}^i\pi}\omega^{i+1}_{\kappa+\widehat{E}^i\pi}\Phi_{j+1,k+1},$$

where coefficient μ_κ^i are defined as in Lemma 3.8. [3]. Then, by Proposition (3.1) and Eq. (3.7) we have

$$\begin{aligned}
n^{1-i} &= \left\langle \omega_\kappa^{i-1}\Phi_{j-1,k-1}, \omega_\kappa^{i-1}\Phi_{j-1,k-1}\right\rangle \\
&= \Big\langle \sum_{\pi,\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\mu^i_{\kappa+\widehat{E}^{i-1}\pi}\omega^i_{\kappa+\widehat{E}^{i-1}\pi}\Phi_{j,k}, \\
&\qquad \sum_{\pi',\kappa\in D(E^i)}\sum_{j\in A}\sum_{k\in B}\mu^i_{\kappa+\widehat{E}^{i-1}\pi'}\omega^i_{\kappa+\widehat{E}^{i-1}\pi'}\Phi_{j,k}\Big\rangle \\
&= \sum_{\pi,\kappa\in A}\left|\mu^i_{\kappa+\widehat{E}^{i-1}\pi}\right|^2\left\langle \omega^i_{\kappa+\widehat{E}^{i-1}\pi}\Phi_{i,k}, \omega^i_{\kappa+\widehat{E}^{i-1}\pi}\Phi_{i,k}\right\rangle,
\end{aligned}$$

Thus, by Eq. (3.5) we get

$$\sum_{\pi,\kappa\in D(E^i)} \left|\mu^i_{\kappa+\widehat{E}^{i-1}\pi}\right|^2 = n. \tag{3.9}$$

Now in the present case, wavelet spaces and bases are obtained as our main focus. Let the orthonormal scaling sequence $\{\Phi_{j,k}\}$ and U_i be an multiresolution analysis of $L^2(G)$. We concentrate to obtain wavelet functions ψ_ρ, $\rho = 1, 2, \cdots n-1$ in the space U_{i+1} be an multiresolution analysis of $L^2(G)$ such that the construction $\{T_\gamma\psi_\rho\}_{\gamma\in \ker E^i}$ is orthogonal, orthonormal for various values of ρ, and mutually orthogonal to the space U_i. To design similar functions and scaling functions, we pursue the process illustrate below.

We rewrite $D(E) = \{\pi_0, \pi_1, \cdots \pi_{n-1}\}$, where $\pi_0 = 0$ and $\pi_t \in A \subset \text{Aut}(G), t = 0, 1, \cdots n-1$. We determine $b_{0k} = \mu_{\kappa+\widehat{E}^i\pi_j}/\sqrt{n}$, where $\kappa \in D(E^i), j = 0, 1, \cdots n-1$. By Eq. (3.3), we get $\sum_{j=0}^{n-1} |b_{0k}|^2 = 1$. We develop this row to an $n \times n$ unitary matrix $B = \{b^{n-1}_{\rho,j=0}\}$. We set $\nu^{\rho,j}_{\kappa+\widehat{E}^i\pi_j} = \sqrt{n}b_{\rho,j}$ for $\rho = 1, 2\cdots n-1, \kappa \in D(E^i), j = 0, \cdots n-1$. By Eq. (3.9), we get describe $\nu^{\rho,i}_\chi$ for all $\chi \in D(E^{i+1})$. Then we develop this sequence to $\widehat{G}$ by context $\nu^{\rho,i}_\chi = \nu^{\rho,i}_\kappa$ for $\chi \in (\ker E^{i+1})^\perp + \kappa, \eta \in D(E^{i+1})$. After that, we derive wavelet functions ψ^ρ_i for $\rho = 1, \cdots n-1$, in terms of Fourier transform through the equation $\widehat{\psi}_{i,\rho}(\chi) = \nu^{\rho,i}_\chi \widehat{\Phi}_{i+1}(\chi)$ for $\chi \in \widehat{G}$, and the wavelet spaces by

$$\omega^{(\rho)}_i = \text{span}\{T_l\psi_{i,\rho,k} : l \in \ker E^i, k \in B\}. \tag{3.10}$$

Theorem 3.2. *Suppose $\{U_i\}_{i\in N_0}$ is a multiresolution analysis of $L^2(G)$ and an orthonormal scaling sequence $\{\Phi_{j,k}\}_{j\in A,k\in B}$. Then, for every $i \in N_0$ we have*

$$U_{i+1} = U_i \oplus V^{(1)}_i \oplus \cdots \oplus V^{(n-1)}_i,$$

and an orthonormal basis of the system $\{T_l\psi_{i,\rho,k}\}_{l\in ker E^i,k\in B}$ of the space $\omega^{(\rho)}_i$ for $\rho = 1, \cdots n-1$. In the act of Proposition (3.1) *and Definition* (3.3) *the wavelet structure*

$$\{T_l\psi_{i,\rho,k} : l \in ker E^i, i \in N_0, k \in B, \rho = 1, \cdots n-1\},$$

well-organized with the regular function $\Phi_0 \equiv 1$ forms an orthonormal basis for $L^2(G)$.

Proof. For every settled $\kappa \in D(E^i)$ and $i \in N_0$, by Eq. 3.7, we have

$$\omega_\kappa^i \Phi_{j,k} = \sum_{r=0}^{n-1} \mu^{i+1}_{\kappa+\widehat{E}^i\pi_r} \omega^{i+1}_{\kappa+\widehat{E}^i\pi_r} \Phi_{j+1,k+1}. \tag{3.11}$$

Similarly, by Lemma 3.6. [3] and Proposition (3.1) we have

$$\omega_\kappa^i \psi_{i,\rho,k} = \sum_{r=0}^{n-1} \nu^{\rho,i}_{\kappa+\widehat{E}^i\pi_r} \omega^{i+1}_{\kappa+\widehat{E}^i\pi_r} \Phi_{\rho+1,k+1}. \tag{3.12}$$

In special case, for Eq. (3.11) implies that $\psi_{i,\rho,k} \in U_{i+1}$ and hence $\omega_i^{(\rho)} \subset U_{i+1}$ for all $\rho = 1, \cdots n-1$.

We need that:

(i) $\omega_i^{(\rho)} \perp U_i$ for all $\rho = 1, \cdots n-1$

(ii) $\omega_i^{(\rho)} \perp \omega_i^{(r)}$ for all $\rho \neq r, \rho, r = 1, \cdots n-1$.

For (i), first note that $\sum_{r=0}^{n-1} \alpha^{\rho,i}_{\kappa+\widehat{E}^i\pi_r} \overline{\mu^{i+1}_{\kappa+\widehat{E}^i\pi_r}} = 0$ by the case that the matrix E establish raised is orthogonal. Using Eq. (3.5), Eq. (3.7), Eq. (3.11) and Eq. (3.12) we get

$$\begin{aligned}
\left\langle \omega_\kappa^i \psi_{i,\rho,k}, \omega_\kappa^i \Phi_{\rho,k} \right\rangle &= \left\langle \sum_{r=0}^{n-1} \nu^{\rho,i}_{\kappa+\widehat{E}^i\pi_r} \omega^{i+1}_{\kappa+\widehat{E}^i\pi_j} \Phi_{\rho+1,k+1}, \sum_{r'=0}^{n-1} \mu^{i+1}_{\kappa+\widehat{E}^i\pi_{r'}} \omega^{i+1}_{\kappa+\widehat{E}^i\pi_{r'}} \Phi_{\rho+1,k+1} \right\rangle \\
&= \sum_{r=0}^{n-1} \nu^{\rho,r}_{\kappa+\widehat{E}^i\pi_r} \overline{\mu}_{\kappa+\widehat{E}^i\pi_r} \left\langle \omega^{i+1}_{\kappa+\widehat{E}^i\pi_{r'}} \Phi_{\rho+1,k+1}, \omega^{i+1}_{\kappa+\widehat{E}^i\pi_{r'}} \Phi_{\rho+1,k+1} \right\rangle = 0.
\end{aligned}$$

Using Eq. (3.7), the first part of the theorem is proved. Besides, since E is orthogonal, we get

$$\sum_{r=0}^{n-1} \nu^{\rho,i}_{\kappa+\widehat{E}^i\pi_r} \overline{\alpha}^{r,i}_{\kappa+\widehat{E}^i\pi_r} = n\delta_{\rho,r} \quad for \quad \rho, r = 1, \cdots n-1.$$

Hence,

$$\begin{aligned}
&\langle \omega_\kappa^i \psi_{i,\rho,k}, \omega_\kappa^i \psi_{i,j,k} \rangle \\
&= \sum_{r=0}^{n-1} \nu^{\rho,i}_{\kappa+\widehat{E}^i\pi_r} \nu^{r,i}_{\kappa+\widehat{E}^i\pi_r} \langle \omega^{i+1}_{\kappa+\widehat{E}^i\pi_{r'}} \Phi_{\rho+1,k+1}, \omega^{i+1}_{\kappa+\widehat{E}^i\pi_{r'}} \Phi_{i+1,k+1} \rangle = n^{-i}\delta_{\rho,r}.
\end{aligned}$$

This proves the second part of the theorem. Furthermore, by Proposition (3.1) $\{T_l \psi_{i,k,\rho}\}_{l \in \ker E^i, k \in [s]}$ is an orthonormal basis of $V_i^{(\rho)}$. Since $\dim U_i = \dim V_i^{(\rho)} = n^i$ and

$$U_i \oplus V_i^{(1)} \oplus \cdots \oplus V_i^{(n-1)} \subset U_{i+1}$$

the dimension result suggest the equality in the above formation. □

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