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THE WIENER INDEX OF DISJUNCTION AND SYMMETRIC DIFFERENCE OF TWO GRAPHS

S. Menaka[1], R.S. Manikandan, and R. Muruganandam

ABSTRACT. In this paper, the Wiener Index of disjunction and symmetric difference of two graphs will be presented.

1. INTRODUCTION

All graphs considered here are simple, connected and finite. Let $V(G)$, $E(G)$, $d_G(v)$ and $d_G(u,v)$ denote the vertex set, the edge set, the degree of a vertex and the distance between the vertices u and v of a graph G respectively.

A graph with n vertices and m edges is called a (n,m) graph. First we present the definitions and notations which are required throughout this paper.

A topological index of a graph G is a real number which is invariant under automorphism of G and it doesnot depend on the labeling or pictorial representation of a graph. A topological index related to distance is called a "distance based topological index". In 1947, H. Wiener [7] introduced the first distance - based topological index which is named as Wiener Index and it is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v).$$

[1]*corresponding author*

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The topological indices based on distances between vertices of a graph are widely used for characterising molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds [1, 7]

The Wiener Index of cartesian product and lexicographic product of graphs found in [5]. Iztok Peterin [2] et. al. found the formulae for the Wiener Index of the strong product of a connected graph G of constant eccentricity with a cycle. In [6] K. Pattabiraman et. al. found the Wiener Index of the tensor product of a path and a cycle. In [3] Eliasi. et. al determined the Wiener Index of graphs which are constructed by some operations such as Mycielski's construction, generalised hierarchical product and t^{th} subdivision of graphs.

In this work, we present general formula for the Wiener Index of disjunction and symmetric difference of two graphs.

First we discuss necessary definitions and Lemmas.

The disjunction of graphs G_1 and G_2 is denoted by $G_1 \vee G_2$, and it is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2) : u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2)\}.$$

The symmetric difference of graphs G_1 and G_2 is denoted by $G_1 \oplus G_2$, and it is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) : u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ but not both}\}.$$

2. Basic Lemmas

Let G_1 and G_2 be two simple connected graphs with n_1, n_2 vertices and e_1, e_2 edges respectively. Then

$$1.\ d_{G_1 \vee G_2}((u_1, v_1)(u_2, v_2)) = \begin{cases} 1, & u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2), \\ 2, & \text{otherwise;} \end{cases}$$

$$2.\ d_{G_1 \oplus G_2}((u_1, v_1)(u_2, v_2)) = \begin{cases} 1, & u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2) \text{ but not both}, \\ 2, & \text{otherwise.} \end{cases}$$

Remark 2.1. *[4] For a graph G, let $A(G) = \{(x, y) \in V(G) \times V(G) |\ x$ and y are adjacent in G$\}$ and let $B(G) = \{(x, y) \in V(G) \times V(G) |\ x$ and y are not adjacent* *in G*$\}$. *For each* $x \in V(G)$, $(x,x) \in B(G)$. *Clearly* $A(G) \cup B(G) = V(G) \times V(G)$ *Let* $C(G) = \{(x,y)|\ x \in V(G)\}$ *and* $D(G) = B(G) - C(G)$. *Clearly* $B(G) = C(G) \cup D(G)$, $C(G) \cap D(G) = \phi$. *The summation* $\sum_{(x,y)\in A(G)}$ *runs over the ordered pairs of* $A(G)$. *For simplicity, we write the summation* $\sum_{(x,y)\in A(G)}$ *as* $\sum_{xy\in G}$. *Similarly, we write the summation* $\sum_{(x,y)\in B(G)}$ *as* $\sum_{xy\notin G}$. *Also the summation* $\sum_{xy\in E(G)}$ *runs over the edges of* G. *We denote the summation* $\sum_{x,y\in V(G)}$ *by* $\sum_{x,y\in G}$.

Lemma 2.1. *[4] Let* G *be a graph with* $v(G)$ *vertices and* $e(G)$ *edges respectively. Then*

(1) $\sum_{xy\in G} 1 = 2e(G)$;
(2) $\sum_{xy\notin G} 1 = 2e(\overline{G}) + v(G)$;
(3) $\sum_{xy\notin G, x\neq y} 1 = 2e(\overline{G})$;
(4) $\sum_{xy\notin G, x=y} 1 = v(G)$.

3. THE WIENER INDEX OF DISJUNCTION OF GRAPHS

In this section we find the exact value of the Wiener Index of disjunction of graphs.

Theorem 3.1. *Let* $G_1,\ G_2$ *be two graphs with* $n_1,\ n_2$ *vertices and* $m_1,\ m_2$ *edges respectively. Then*

$$W(G_1 \vee G_2) = 4\overline{m_1}(m_2 + n_2) + 4\overline{m_2}(m_1 + n_1) + 2m_2(n_1 + 2m_1) + 2m_1n_2 + 8\overline{m}_1\overline{m}_2$$

Proof.

$$\begin{aligned} &2 \times W(G_1 \vee G_2) \\ &= \sum_{x,y\in V(G_1)} \sum_{u,v\in V(G_2)} d_{G_1\vee G_2}((x,u)(y,v)) \\ &= \sum_{x,y\in V(G_1)} \left[\sum_{uv\in G_2} d_{G_1\vee G_2}((x,u)(y,v)) + \sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v)) \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{x,y\in V(G_1)}\sum_{uv\in G_2} d_{G_1\vee G_2}((x,u)(y,v)) + \sum_{x,y\in V(G_1)}\sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v))\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1\vee G_2}((x,u)(y,v)) + \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1\vee G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v)) + \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v))\\
&= C_3 + C_1 + C_2 + C_4
\end{aligned}$$

where C_3, C_1, C_2 and C_4 are terms of the above sums taken in order.

$$\begin{aligned}
C_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1\vee G_2}((x,u)(y,v)) = \sum_{xy\notin G_1}\sum_{uv\in G_2} 1 = 4\overline{m}_1 m_2 + 2n_1 m_2\\
C_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v)) = \sum_{xy\in G_1}\sum_{uv\notin G_2} 1 = 4m_1\overline{m}_2 + 2m_1 n_2\\
C_3 &= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1\vee G_2}((x,u)(y,v)) = \sum_{xy\in G_1}\sum_{uv\in G_2} 1 = 4m_1 m_2\\
C_4 &= \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v))\\
&= \sum_{xy\notin G_1}\left[\sum_{uv\notin G_2, u\neq v} d_{G_1\vee G_2}((x,u)(y,v)) + \sum_{uv\notin G_2, u=v} d_{G_1\vee G_2}((x,u)(y,v))\right]\\
&= \sum_{xy\notin G_1}\sum_{uv\notin G_2, u\neq v} d_{G_1\vee G_2}((x,u)(y,v)) + \sum_{xy\notin G_1}\sum_{uv\notin G_2, u=v} d_{G_1\vee G_2}((x,u)(y,v))\\
&= \sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u\neq v} d_{G_1\vee G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u\neq v} d_{G_1\vee G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u=v} d_{G_1\vee G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{G_1\vee G_2}((x,u)(y,v))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} 2 + \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} 2 + \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} 2 \\
&= 2\Bigg(\sum_{xy\notin G_1, x\neq y} 1\Bigg)\Bigg(\sum_{uv\notin G_2, u\neq v} 1\Bigg) + 2\Bigg(\sum_{xy\notin G_1, x=y} 1\Bigg)\Bigg(\sum_{uv\notin G_2, u\neq v} 1\Bigg) \\
&\quad + 2\Bigg(\sum_{xy\notin G_1, x\neq y} 1\Bigg)\Bigg(\sum_{uv\notin G_2, u=v} 1\Bigg) \\
&= 8\overline{m}_1\overline{m}_2 + 4n_1\overline{m}_2 + 4\overline{m}_1 n_2
\end{aligned}
$$

Adding C_1, C_2, C_3, C_4 we get the desired result. □

4. The symmetric difference of two graphs

In this section, we find the exact value of the symmetric difference of two graphs.

Theorem 4.1. *Let $G_1,\ G_2$ be two graphs with $n_1,\ n_2$ vertices and $m_1,\ m_2$ edges respectively. Then*

$$
\begin{aligned}
W(G_1 \oplus G_2) &= 4\overline{m}_1(m_2+n_2) + 4\overline{m}_2(m_1+n_1) \\
&\quad + 2m_2(n_1+4m_1) + 2m_1n_2 + 8\overline{m}_1\overline{m}_2
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&2\times W(G_1\oplus G_2) \\
&= \sum_{x,y\in V(G_1)} \sum_{u,v\in V(G_2)} d_{G_1\oplus G_2}((x,u)(y,v)) \\
&= \sum_{x,y\in V(G_1)} \Bigg[\sum_{uv\in G_2} d_{G_1\oplus G_2}((x,u)(y,v)) + \sum_{uv\notin G_2} d_{G_1\oplus G_2}((x,u)(y,v))\Bigg] \\
&= \sum_{x,y\in V(G_1)} \sum_{uv\in G_2} d_{G_1\oplus G_2}((x,u)(y,v)) + \sum_{x,y\in V(G_1)} \sum_{uv\notin G_2} d_{G_1\oplus G_2}((x,u)(y,v)) \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1\oplus G_2}((x,u)(y,v)) + \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1\oplus G_2}((x,u)(y,v))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1\oplus G_2}((x,u)(y,v)) + \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{G_1\oplus G_2}((x,u)(y,v))\\
&= B_3 + B_1 + B_2 + B_4
\end{aligned}
$$

where B_3, B_1, B_2 and B_4 are terms of the above sums taken in order.

$$
\begin{aligned}
B_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1\oplus G_2}((x,u)(y,v)) = \sum_{xy\notin G_1}\sum_{uv\in G_2} 1 = 4\overline{m}_1 m_2 + 2n_1 m_2\\
B_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1\oplus G_2}((x,u)(y,v)) = \sum_{xy\in G_1}\sum_{uv\notin G_2} 1 = 4m_1\overline{m}_2 + 2m_1 n_2\\
B_3 &= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1\oplus G_2}((x,u)(y,v)) = \sum_{xy\in G_1}\sum_{uv\in G_2} 2 = 8m_1 m_2\\
B_4 &= \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{G_1\oplus G_2}((x,u)(y,v))\\
&= \sum_{xy\notin G_1}\left[\sum_{uv\notin G_2, u\neq v} d_{G_1\oplus G_2}((x,u)(y,v)) + \sum_{uv\notin G_2, u=v} d_{G_1\oplus G_2}((x,u)(y,v))\right]\\
&= \sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u\neq v} d_{G_1\oplus G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u\neq v} d_{G_1\oplus G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u=v} d_{G_1\oplus G_2}((x,u)(y,v))\\
&\quad + \sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u=v} d_{G_1\oplus G_2}((x,u)(y,v))\\
&= \sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u\neq v} 2 + \sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u\neq v} 2 + \sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u=v} 2\\
&= 2\Bigg(\sum_{xy\notin G_1, x\neq y} 1\Bigg)\Bigg(\sum_{uv\notin G_2, u\neq v} 1\Bigg) + 2\Bigg(\sum_{xy\notin G_1, x=y} 1\Bigg)\Bigg(\sum_{uv\notin G_2, u\neq v} 1\Bigg)\\
&\quad + 2\Bigg(\sum_{xy\notin G_1, x\neq y} 1\Bigg)\Bigg(\sum_{uv\notin G_2, u=v} 1\Bigg)\\
&= 8\overline{m}_1\overline{m}_2 + 4n_1\overline{m}_2 + 4\overline{m}_1 n_2
\end{aligned}
$$

Adding B_1, B_2, B_3, B_4 we get the desired result. □

REFERENCES

[1] A. A. DOBRYNIN, I. GUTMAN, S. KLAVZAR, P. ZIGERT: *Wiener Index of Hexagonal systems*, Acta. Appl. Math. 72 (2002), 247-294.

[2] I. PETERIN, P.Z. PLETERSEK: *Wiener Index of strong product of graphs*, Opuscula Mathematica, **38**(1) (2018), 81-94.

[3] M. ELIASI, G. RACISI, B. TAESI: *Wiener Index of some graph operations*, Discrete Applied Mathematics, **160**(9) (2012), 1333-1344.

[4] R. MURUGANANDAM, R. S. MANIKANDAN: *Gutman Index of some graph operations*, International Journal of Applied Graph Theory, **1**(2) (2017), 1-29.

[5] K. PATTABIRAMAN, P. PAULRAJA: *Wiener and vertex PI indices of the strong product of graphs*, Discuss. Math. Graph Theory, **32** (2012), 749-769.

[6] K. PATTABIRAMAN, P. PAULRAJA: *Wiener Index of the tensor product of a path and a cycle, Discuss*, Math. Graph Theory, **31** (2011), 737-751.

[7] H. WIENER: *Structural determination of paraffin boiling points*, J. Amer. Chem. Soc. **69** (1947), 17-20.

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE OF ENGINEERING-BIT CAMPUS,
ANNA UNIVERSITY, TIRUCHIRAPPALLI,
INDIA.
Email address: menaka@aubit.edu.in

DEPARTMENT OF MATHEMATICS
BHARATHIDASAN UNIVERSITY CONSTITUENT COLLEGE,
KUMULUR, LALGUDI, TIRUCHIRAPPALLI,
INDIA
Email address: manirs2004@yahoo.co.in

DEPARTMENT OF MATHEMATICS
PERIYAR E.V.R. COLLEGE
TIRUCHIRAPPALLI
INDIA.
Email address: muruganraju1980@gmail.com