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ON SOLUTIONS TO ITERATIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. We present existence and uniqueness of solution for iterative differential equation $y' = f(x, y, y(g(y)))$ using Picard's iteration method by imposing some conditions on f and g. These conditions are sufficient conditions (not necessary conditions) for existence of solution. Furthermore, examples are provided to clarify existence and uniqueness of solution.

1. INTRODUCTION

In this article we will investigate existence and uniqueness of solution of IVP(initial value problem)

$$\begin{cases} y'(x) &= f(x, y, y(g(y))), \quad \text{for } x \in [x_0 - a, x_0 + a] \subset \mathbb{R}, \\ & \quad \text{where } x_0 \in \mathbb{R}, a \in (0, \infty) \\ y(x_0) &= y_0, y_0 \in \mathbb{R} \end{cases} \tag{1.1}$$

by Picard's iteration method, where

$$f : [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]^2 \to \mathbb{R}$$

and

$$g : [y_0 - b, y_0 + b] \to [x_0 - a, x_0 + a], \; b \in (0, \infty)$$

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are continuous. Picard's iteration method is one of the method to show existence and uniqueness of solution for differential equations having initial condition [8, 11].

We can call equation (1.1) an iterative differential equation since iterates of the unknown function are involved. These equations are vital in the investigation of dynamical systems, infectious disease models, etc. Specific forms of such equations can, for instance, be found in [2].

There are related literature that discussed about existence and uniqueness of solution of IVP of type (1.1). We can mention, for instance, the existence of solution of the IVP

$$\begin{cases} y' & = & f(x, y, y(y)) \\ y(x_0) & = & y_0 \end{cases}$$

and

$$\begin{cases} y' & = & f(x, y(h(x) + y(g(y)))) \\ y(x_0) & = & x_0 \end{cases}$$

have been established in [9] and [8] respectively. Problem of type (1.1) has been occurred in physical problem, for instance in the study of electrodynamics [4]. In this manner several existence results have been appeared in [1, 3, 5–7, 10, 12] for iterative differential equation.

The article is organized as follows. In section 2 we discuss about existence of solution. In section 3 we present uniqueness of solution. Lastly, in section 4 we provide examples that illustrate section 2 and 3.

For later purpose we state the following lemma.

Lemma 1.1. *Suppose* $\alpha(x), \gamma(x), \beta(t)$ *and* $\eta(t)$ *are functions such that* $\alpha, \gamma \in C(I_1)$ *and* $\beta, \eta \in C(I_2)$, *where* I_1 *and* I_2 *are closed intervals in* $\mathbb{R}$. *Then* $|\alpha^2\beta^2 - \gamma^2\eta^2| \leq A|\alpha - \gamma| + B|\beta - \eta|$, *where* $A = \max\{\beta^2|\alpha + \gamma| : x \in I_1$ *and* $t \in I_2\}$ *and* $B = \max\{\gamma^2|\beta + \eta| : x \in I_1$ *and* $t \in I_2\}$.

2. Existence of Solution

Let $S = \{(x, y) : x \in [x_0 - a, x_0 + a]$ and $y \in [y_0 - b, y_0 + b]\}$. We consider the following conditions on the function f.

F-1 There exist $0 < M \leq 1$ such that

$$M = \max\{|f(x, y, y(g(y)))| : (x, y) \in S \text{ and } g(y) \in [x_0 - a, x_0 + a]\}.$$

F-2 $Ma \leq b$

F-3 There exist $L_1, L_2 > 0$ such that

$$|f(x,u,u(g(u))) - f(x,v,v(g(v)))| \leq L_1|u-v| + L_2|u(g(u)) - v(g(v))|$$

for $x, g(u), g(v) \in [x_0 - a, x_0 + a], u, v \in [y_0 - b, y_0 + b]$, where g satisfies the following conditions.

G-1 $g(y_0) = x_0$

G-2 There exist $0 < L_3 \leq 1$ such that $|g(u) - g(v)| \leq L_3|u - v|$ for $u, v \in [y_0 - b, y_0 + b]$.

G-3 $L_3 b \leq a$.

We notice that

$$|g(y) - x_0| = |g(y) - g(y_0)| \leq L_3|y - y_0| \leq L_3 b \leq a, \ \forall y \in [y_0 - b, y_0 + b].$$

We now choose a picard sequence of functions as

$$\begin{cases} y^n(x) &= y_0 + \int_{x_0}^{x} f(x, y^{n-1}, y^{n-1}(g(y^{n-1})))dx \\ y^0(x) &= y_0 \end{cases} \tag{2.1}$$

for $x \in [x_0 - a, x_0 + a]$.

Lemma 2.1. *Suppose f satisfies conditions **F-1,F-2** and **F-3**, and g satisfies conditions **G-1,G-2** and **G-3**. If $\{y^n\}$ satisfies* (2.1)*, the following holds true.*

(1) $|y^n - y_0| \leq Ma, |g(y^n) - x_0| \leq Ma \leq a, n = 1, 2, \cdots$ *for* $x \in [x_0 - a, x_0 + a]$.

(2) y^n *is continuous and differentiable on* $[x_0 - a, x_0 + a]$ *for* $n = 1, 2, \cdots$.

(3)

$$|y^n - y^{n-1}| \leq G_n \frac{|x - x_0|^n}{n!} \textit{ for } n \geq 1 \textit{ and } x \in [x_0 - a, x_0 + a], \tag{2.2}$$

where G_n is defined recursively as

$$\begin{cases} G_1 = M, G_2 = L_1 M + L_2 L_3 M^2, \\ G_n = G_{n-1}\left(L_1 + L_2 L_3 M + L_2 L_3^{n-1} M^{n-1}\right), n \geq 3 \end{cases} . \tag{2.3}$$

Proof. (1) Let $x \in [x_0 - a, x_0 + a]$. Then

$$\begin{aligned} |y^1 - y^0| = |\int_{x_0}^{x} f(x, y^0, y^0(g(y^0)))| &\leq \int_{x_0}^{x} |f(x, y_0, y_0)|dx \\ &\leq M|x - x_0| \leq Ma, \end{aligned}$$

$$|y^2 - y^0| = |\int_{x_0}^{x} f(x, y^1, y^1(g(y^1)))| \leq \int_{x_0}^{x} |f(x, y^1, y^1)|dx \leq M|x - x_0| \leq Ma$$

and

$$|y^3 - y^0| = |\int_{x_0}^{x} f(x, y^2, y^2(g(y^2)))| \leq \int_{x_0}^{x} |f(x, y^2, y^2)|dx \leq M|x - x_0| \leq Ma.$$

And so

$$|g(y^1) - x_0| \leq L_3|y^1 - y^0| \leq L_3Ma \leq a,$$
$$|g(y^2) - x_0| \leq L_3|y^2 - y^0| \leq L_3Ma \leq a$$

and

$$|g(y^3) - x_0| \leq L_3|y^3 - y^0| \leq L_3Ma \leq a.$$

Suppose $|y^{n-1} - y^0| \leq L_3Ma \leq a$ and $|g(y^{n-1}) - x_0| \leq Ma, x \in [x_0 - a, x_0 + a]$. Then

$$\begin{aligned} |y^n - y^0| &= |\int_{x_0}^{x} f(x, y^{n-1}, y^{n-1}(g(y^{n-1})))|dx \\ &\leq \int_{x_0}^{x} |f(x, y_{n-1}, y_{n-1}(g(y^{n-1})))|dx \\ &\leq M|x - x_0| \leq Ma \end{aligned}$$

and

$$|g(y^n) - x_0| \leq L_3|y^n - y^0| \leq L_3Ma \leq a.$$

(2) Clearly y^0 is continuous and differentiable on $[x_0 - a, x_0 + a]$.

$$\begin{aligned} |y^1(x) - y^1(t)| &\leq \int_{t}^{x} |f(s, y^0, y^0(g(y^0)))|ds, \ x \in [t - \delta, t + \delta] \\ &\leq M|x - t| \leq M\delta. \end{aligned}$$

So for a given $\epsilon > 0$ we find a $\delta < \dfrac{\epsilon}{M}$ for which y^1 is continuous. Using mathematical induction we can easily show that y^n is continuous on $[x_0 - a, x_0 + a]$. Next,

$$\frac{y^n(x + h) - y^n(x)}{h} = \frac{1}{h}\int_{x}^{x+h} f(s, y^{n-1}, y^{n-1}(g(y^{n-1})))ds,$$

$x \in [x_0 - a, x_0 + a]$. As $h \to 0$, $\dfrac{dy^n}{dx}$ exist for all n.

(3) Let $x \in [x_0 - a, x_0 + a]$. Then

$$\begin{aligned} |y^1 - y^0| &= |\int_{x_0}^{x} f(x, y^0, y^0(g(y^0)))| \le \int_{x_0}^{x} |f(x, y_0, y_0)|dx \\ &= G_1|x - x_0|, \text{ where}|; G_1 = M, \end{aligned}$$

and

$$\begin{aligned} |y^2 - y^1| &= |\int_{x_0}^{x} f(x, y^1, y^1(g(y^1))) - \int_{x_0}^{x} f(x, y^0, y^0(g(y^0)))| \\ &\le \int_{x_0}^{x} |f(x, y^1, y^1(g(y^1))) - f(x, y_0, y_0)|dx \\ &\le \int_{x_0}^{x} [L_1|y^1 - y_0| + L_2|y^1(g(y^1)) - y_0|]dx \\ &\le \int_{x_0}^{x} [L_1|y^1 - y_0|dx + \int_{x_0}^{x} L_2M|g(y^1)g(y^0)| \\ &= G_2\frac{|x - x_0|^2}{2!}, \text{ where } G_2 = L_1M + L_2L_3M^2. \end{aligned}$$

Since

$$\begin{aligned} & |y^2(g(y^2)) - y^1(g(y^1))| \\ = \; & \left|y^2(g(y^2)) - y^2(g(y^1)) + y^2(g(y^1)) - y^1(g(y^1))\right| \\ \le \; & \int_{g(y^1)}^{g(y^2)} |f(x, y^1, y^1(g(y^1))|dx + |y^2(g(y^1)) - y^1(g(y^1))| \\ \le \; & L_3M|y^2 - y^1| + \frac{1}{2}\left[L_1L_3^2M + L_2L_3^3M^2\right]|y^1 - y^0|^2 \\ \le \; & \frac{1}{2}\left[L_1L_3M^2 + L_2L_3^2M^3 + L_1L_3^2M^3 + L_2L_3^3M^4\right]|x - x_0|^2, \end{aligned}$$

we have

$$\begin{aligned} |y^3 - y^2| &\le \int_{x_0}^{x} |f(x, y^2, y^2(g(y^2))) - f(x, y^1, y^1(g(y^1)))| \\ &\le \int_{x_0}^{x} L_1|y^2 - y^1|dx + \int_{x_0}^{x} L_2|y^2(g(y^2)) - y^1(g(y^1))|dx \\ &\le \frac{1}{6}\left[L_1^2M + L_1L_2L_3M^2 + L_1L_2L_3M^2 + L_2^2L_3^2M^3 + \right. \\ &\quad \left. L_1L_2L_3^2M^3 + L_2^2L_3^3M^4\right]|x - x_0|^3 = G_3\frac{|x - x_0|^3}{3!}, \end{aligned}$$

where $G_3 = G_2[L_1 + L_2L_3M + L_2L_3^2M^2]$. Next,

$$\begin{aligned}
& |y^3(g(y^3)) - y^2(g(y^2))| \\
= \; & \left|y^3(g(y^3)) - y^3(g(y^2)) + y^3(g(y^2)) - y^2(g(y^2))\right| \\
\leq \; & \int_{g(y^2)}^{g(y^3)} |f(x, y^2, y^2(g(y^2))|dx + |y^3(g(y^2)) - y^2(g(y^2))| \\
\leq \; & L_3M|y^3 - y^2| + \frac{1}{6}\left[L_1^2M + L_1L_2L_3M^2 + L_1L_2L_3M^2 \right. \\
& \left. + L_2^2L_3^2M^3 + L_1L_2L_3^2M^3 + L_2^2L_3^3M^4\right] |g(y^2) - x_0|^3 \\
\leq \; & L_3M|y^3 - y^2| + \frac{1}{6}\left[L_1^2M + L_1L_2L_3M^2 + L_1L_2L_3M^2 \right. \\
& \left. + L_2^2L_3^2M^3 + L_1L_2L_3^2M^3 + L_2^2L_3^3M^4\right] L_3^3M^3|x - x_0|^3
\end{aligned}$$

and hence

$$\begin{aligned}
|y^4 - y^3| \; \leq \; & \int_{x_0}^{x} |f(x, y^3, y^3(g(y^3))) - f(x, y^2, y^2(g(y^2)))| \\
\leq \; & \int_{x_0}^{x} L_1|y^3 - y^2|dx + \int_{x_0}^{x} L_2|y^3(g(y^3)) - y^2(g(y^2))|dx \\
\leq \; & \frac{1}{24}\left[L_1^3M + L_1^2L_2L_3M^2 + L_1^2L_2L_3M^2 + \right. \\
& L_1L_2^2L_3^2M^3 + L_1^2L_2L_3^2M^3 + L_1L_2^2L_3^3M^4 + \\
& L_2L_3M\left[L_1^2M + L_1L_2L_3M^2 + L_1L_2L_3M^2 + L_2^2L_3^2M^3 \right. \\
& \left. + L_1L_2L_3^2M^3 + L_2^2L_3^3M^4\right] + L_2L_3^3M^3\left[L_1^2M + \right. \\
& L_1L_2L_3M^2 + L_1L_2L_3M^2 + L_2^2L_3^2M^3 + L_1L_2L_3^2M^3 + \\
& \left. \left. L_2^2L_3^3M^4\right]\right] |x - x_0|^4 \\
= \; & G_4\frac{|x - x_0|^4}{4!},
\end{aligned}$$

where $G_4 = G_3[L_1 + L_2L_3M + L_2L_3^3M^3]$.

We now suppose

$$|y^{n-1} - y^{n-2}| \leq G_{n-1}\frac{|x - x_0|^{n-1}}{(n-1)!}$$

for $n \geq 1$ and $x \in [x_0 - a, x_0 + a]$, where G_{n-1} is defined recursively as

$$\begin{cases} G_1 = M, G_2 = L_1M + L_2L_3M^2, \\ G_{n-1} = G_{n-2}\left(L_1 + L_2L_3M + L_2L_3^{n-2}M^{n-2}\right), n \geq 3 \end{cases}$$

Then

$$\begin{aligned}
& |y^{n-1}(g(y^{n-1})) - y^{n-2}(g(y^{n-2}))| \\
\leq\ & |y^{n-1}(g(y^{n-1})) - y^{n-1}(g(y^{n-2}))| + \\
& |y^{n-1}(g(y^{n-2})) - y^{n-2}(g(y^{n-2}))| \\
\leq\ & \int_{g(y^{n-2})}^{g(y^{n-1})} |f(x, y^{n-2}, y^{n-2}(g(y^{n-2}))|dx + \\
& |y^{n-1}(g(y^{n-2})) - y^{n-2}(g(y^{n-2}))| \\
\leq\ & L_3 M|y^{n-1} - y^{n-2}| + \frac{1}{(n-1)!} G_{n-1}|g(y^{n-2}) - g(y^0)|^{n-1} \\
\leq\ & L_3 M G_{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!} + \frac{1}{(n-1)!} G_{n-1} L_3^{n-1}|y^{n-2} - y^0|^{n-1} \\
\leq\ & L_3 M G_{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!} + G_{n-1} L_3^{n-1} M^{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!} \\
=\ & [L_3 M + L_3^{n-1} M^{n-1}] G_{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!}.
\end{aligned}$$

Hence

$$\begin{aligned}
& |y^n - y^{n-1}| \\
\leq\ & \int_{x_0}^{x} |f(x, y^{n-1}, y^{n-1}(g(y^{n-1}))) - f(x, y^{n-2}, y^{n-2}(g(y^{n-2})))|dx \\
\leq\ & \int_{x_0}^{x} L_1|y^{n-1} - y^{n-2}|dx + \\
& \int_{x_0}^{x} L_2|y^{n-1}(g(y^{n-1})) - y^{n-2}(g(y^{n-2}))|dx \\
\leq\ & \int_{x_0}^{x} L_1 G_{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!}dx + \\
& \int_{x_0}^{x} L_2|[L_3 M + L_3^{n-1} M^{n-1}] G_{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!}|dx \\
\leq\ & \int_{x_0}^{x} [L_1 + L_2 L_3 M + L_2 L_3^{n-1} M^{n-1}] G_{n-1}\frac{|x-x_0|^{n-1}}{(n-1)!}|dx \\
=\ & G_{n-1}[L_1 + L_2 L_3 M + L_2 L_3^{n-1} M^{n-1}]\frac{|x-x_0|^{n}}{n!}| \\
=\ & G_n\frac{|x-x_0|^{n}}{n!}|,
\end{aligned}$$

where $G_n = G_{n-1}[L_1 + L_2L_3M + L_2L_3^{n-1}M^{n-1}]$.

□

Remark 2.1. *The series* $\sum_{n=1}^{\infty} \frac{G_n}{n!}|x - x_0|^n$ *converges.*

Lemma 2.2. *Suppose f satisfies conditions **F-1,F-2** and **F-3**, and g satisfies conditions **G-1,G-2** and **G-3**. Let $\{y^n\}$ be sequence of functions satisfying* (2.1). *If* $y(x) = y^0(x) + [y^1(x) - y^0(x)] + [y^2(x) - y^1(x)] + [y^3(x) - y^2(x)] + \cdots$ *for* $x \in [x_0 - a, x_0 + a]$, *the following holds true:*

(1) $y(x) = \lim_{n\to\infty} y^n$, *for* $x \in [x_0 - a, x_0 + a]$;
(2) $y(x)$ *is the solution of the integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y, y(g(y)))dx, \ x \in [x_0 - a, x_0 + a];$$

(3) $|y(x) - y_0| \leq Ma$, *for* $x \in [x_0 - a, x_0 + a]$.

Proof. Let for $x \in [x_0 - a, x_0 + a]$.

(1) We see that

$$\begin{aligned} y(x) &= y^0(x) + [y^1(x) - y^0(x)] + [y^2(x) - y^1(x)] + \\ &\quad [y^3(x) - y^2(x)] + \cdots \\ &\leq y^0(x) + \sum_{n=1}^{\infty} G_n \frac{|x - x_0|}{n!} \\ &< \infty. \end{aligned}$$

Note that

$$\begin{aligned} y^n &= y^0(x) + [y^1(x) - y^0(x)] + [y^2(x) - y^1(x)] + \\ &\quad [y^3(x) - y^2(x)] + \cdots + \\ &\quad y^n(x) - y^{n-1}(x). \end{aligned}$$

Hence

$$y(x) = \lim_{n\to\infty} y^n.$$

(2) Since

$$\begin{aligned}
&|y^n(g(y^n)) - y(g(y))| \\
= \ &|y^n(g(y^n)) - y^n(g(y)) + y^n(g(y)) - y(g(y))| \\
\leq \ &|y^n(g(y^n)) - y^n(g(y))| + |y^n(g(y)) - y(g(y))| \\
\leq \ &\int_{g(y)}^{g(y^n)} |f(x, y^{n-1}, y^{n-1}(g(y^{n-1}))|dx + |y^n(g(y)) - y(g(y))| \\
\leq \ &M|g(y^n) - g(y)| + |y^n(g(y)) - y(g(y))| \\
\leq \ &ML_3|y^n - y| + |y^n(g(y)) - y(g(y))|,
\end{aligned}$$

$$\begin{aligned}
&|f(x, y^n, y^n(g(y^n))) - f(x, y, y(g(y)))| \\
\leq \ &L_1|y^n - y| + L_2|y^n(g(y^n)) - y(g(y))| \\
\leq \ &L_1|y^n - y| + ML_2L_3|y^n - y| + |y^n(g(y)) - y(g(y))|.
\end{aligned}$$

Consequently

$$f(x, y^n, y^n(g(y^n))) \to f(x, y, y(g(y))) \text{ as } n \to \infty.$$

Now,

$$\begin{aligned}
y(x) &= \lim_{n\to\infty} y^{n+1} \\
&= \lim_{n\to\infty} \left[y_0 + \int_{x_0}^{x} f(x, y^n, y^n(g(y^n)))dx \right] \\
&= y_0 + \int_{x_0}^{x} f(x, y, y(g(y)))dx.
\end{aligned}$$

(3) By the result we have obtained in (2),

$$|y(x) - y_0| \leq \int_{x_0}^{x} |f(x, y, y(g(y)))|dx \leq Ma.$$

□

We are now in a position to state and prove existence theorem.

Theorem 2.1. *Suppose f satisfies conditions* ***F-1,F-2*** *and* ***F-3****, and g satisfies conditions* ***G-1,G-2*** *and* ***G-3****. There is a continuously differentiable function $y = y(x) \in [y_0 - Ma, y_0 + Ma]$ for $x \in [x_0 - a, x_0 + a]$, which is a solution of* (1.1).

Proof. Take $y(x) = y^0(x) + [y^1(x) - y^0(x)] + [y^2(x) - y^1(x)] + [y^3(x) - y^2(x)] + \cdots$, where y^n is defined as in (2.1). Then since $y = \lim_{n\to\infty} y^n$(see Lemma(2.2)) and each y^n is continuous on $[x_0 - a, x_0 + a]$(see Lemma(2.1)), y is continuous on $[x_0 - a, x_0 + a]$. Since y satisfies the integral equation (see Lemma(2.2))

$$y_0 + \int_{x_0}^{x} f(x, y, y(g(y)))$$

and $|y(x) - y_0| \leq Ma$, and the function $f(x, u, v)$ is continuous on $\mathbb{R}^3$, $y \in C^1[x_0 - Ma, x_0 + Ma]$ is a solution of (1.1). $\square$

Remark 2.2. *The conditions on f and g are sufficient condition(not necessary conditions) for existence of solution to* (1.1).

The following example illustrates remark (2.2). The function $y(x) = x$ is a solution of the IVP

$$\begin{cases} y' &= \dfrac{yy(y)}{x^2}, \ x \in [1, 2], \\ y(\frac{3}{2}) &= \frac{3}{2}. \end{cases}$$

Here $f(x, y, z) = \dfrac{yz}{x^2}$ and $y \in [1, 2]$. Notice that $M = \max\{f(x, y, z) : (x, y, z) \in [1, 2]^3\} = 4 > 1$. This contradicts condition **F-1**.

3. Uniqueness of Solution

In this section, we prove the uniqueness of solution of problem(1.1).

Theorem 3.1. *Suppose f satisfies conditions **F-1,F-2** and **F-3**, and g satisfies conditions **G-1,G-2** and **G-3**. Then the solution of the initial value problem* (1.1) *is unique.*

Proof. Let $w \in C^1[x_0 - a, x_0 + a]$ be a solution of (1.1). Then w satisfies the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y, y(g(y)))dx, \ x \in [x_0 - a, x_0 + a].$$

Now

$$|y^0 - w| \leq \int_{x_0}^{x} |f(x, w, w(g(w)))|dx \leq M|x - x_0| = F_1|x - x_0|,$$

where $F_1 = M$.

$$\begin{aligned}
|y^1 - w| &\leq \int_{x_0}^{x} |f(x, y^0, y^0(g(y^0))) - f(x, w, w(g(w)))|dx \\
&\leq \int_{x_0}^{x} [L_1|y^0 - w| + L_2|y^0(g(y^0)) - w(g(w))|]dx \\
&\leq \int_{x_0}^{x} [L_1 \int_{x_0}^{x} |f(x, w, w(g(w)))|dx + \\
&\quad L_2 \int_{x_0}^{g(w)} |f(x, w, w(g(w)))|dx \\
&\leq \int_{x_0}^{x} [L_1 M|x - x_0| + L_2 M|g(w) - g(y_0)|]dx \\
&\leq \int_{x_0}^{x} [L_1 M|x - x_0| + L_2 L_3 M|w - y_0|]dx \\
&\leq \int_{x_0}^{x} [L_1 M|x - x_0| + L_2 L_3 M^2|x - x_0|]dx \\
&= [L_1 M + L_2 L_3 M^2] \frac{|x - x_0|^2}{2} \\
&= F_2 \frac{|x - x_0|^2}{2},
\end{aligned}$$

where $F_2 = L_1 M + L_2 L_3 M^2$.

$$\begin{aligned}
&|y^1(g(y^1)) - w(g(w))| \\
&= |y^1(g(y^1)) - y^1(g(w)) + y^1(g(w)) - w(g(w))| \\
&\leq |y^1(g(y^1)) - y^1(g(w))| + |y^1(g(w)) - w(g(w))| \\
&\leq \int_{g(w)}^{g(y^1)} |f(x, y, y(g(y)))|dx + |y^1(g(w)) - w(g(w))| \\
&\leq L_3 M|y^1 - w| + F_2 \frac{|g(w) - g(y^0)|^2}{2} \\
&\leq F_2 L_3 M \frac{|x - x_0|^2}{2} + F_2 L_3 \frac{|w - y^0|^2}{2} \\
&\leq F_2 \left(L_3 M + L_3^2 M^2\right) \frac{|x - x_0|^2}{2}
\end{aligned}$$

$$
\begin{aligned}
|y^2 - w| &\leq \int_{x_0}^{x} |f(x, y^1, y^1(g(y^1))) - f(x, w, w(g(w)))| dx \\
&\leq \int_{x_0}^{x} [L_1|y^1 - w| + L_2|y^1(g(y^1)) - w(g(w))|] dx \\
&\leq F_2 \left(L_1 + L_2L_3M + L_2L_3^2M^2\right) \int_{x_0}^{x} \frac{|x - x_0|^2}{2} dx \\
&\leq F_2 \left(L_1 + L_2L_3M + L_2L_3^2M^2\right) \frac{|x - x_0|^3}{6} \\
&= F_3 \frac{|x - x_0|^3}{6},
\end{aligned}
$$

where $F_3 = F_2(L_1 + L_2L_3M + L_2L_3^2M^2)$.

Now suppose that

$$|y^{n-1} - w| = F_n \frac{|x - x_0|^n}{n!},$$

where

$$F_n = F_{n-1} \left(L_1 + L_2L_3M + L_2L_3^{n-1}M^{n-1}\right).$$

Then

$$
\begin{aligned}
& |y^{n-1}(g(y^{n-1})) - w(g(w))| \\
= \; & |y^{n-1}(g(y^{n-1})) - y^{n-1}(g(w)) + y^{n-1}(g(w)) - w(g(w))| \\
\leq \; & |y^{n-1}(g(y^{n-1})) - y^{n-1}(g(w))| + |y^{n-1}(g(w)) - w(g(w))| \\
\leq \; & \int_{g(w)}^{g(y^{n-1})} |f(x, y, y(g(y)))| dx + |y^{n-1}(g(w)) - w(g(w))| \\
\leq \; & L_3M|y^{n-1} - w| + F_n \frac{|g(w) - g(y^0)|^n}{n!} \\
\leq \; & L_3MF_n \frac{|x - x_0|^n}{n!} + F_nL_3^n \frac{|w - y^0|^n}{n!} \\
\leq \; & L_3MF_n \frac{|x - x_0|^n}{n!} + L_3^nM^nF_n \frac{|x - x_0|^n}{n!}.
\end{aligned}
$$

Hence

$$\begin{aligned}
|y^n - w| &\leq \int_{x_0}^{x} |f(x, y^{n-1}, y^{n-1}(g(y^{n-1}))) - f(x, w, w(g(w)))|dx \\
&\leq \int_{x_0}^{x} [L_1|y^{n-1} - w| + L_2|y^{n-1}(g(y^{n-1})) - w(g(w))|]dx \\
&\leq \int_{x_0}^{x} \Bigg[L_1 F_n \frac{|x-x_0|^n}{n!} + L_2 \Bigg(L_3 M F_n \frac{|x-x_0|^n}{n!} + \\
&\quad L_3^n M^n F_n \frac{|x-x_0|^n}{n!}\Bigg)\Bigg] dx \\
&= \int_{x_0}^{x} [L_1 + L_2 L_3 M + L_2 L_3^n M^n] F_n \frac{|x-x_0|^n}{n!} dx \\
&= [L_1 + L_2 L_3 M + L_2 L_3^n M^n] F_n \frac{|x-x_0|^{n+1}}{(n+1)!} \\
&= F_{n+1} \frac{|x-x_0|^{n+1}}{(n+1)!},
\end{aligned}$$

where $F_{n+1} = F_n[L_1 + L_2 L_3 M + L_2 L_3^n M^n]$.

Since the series

$$\sum_{n=1}^{\infty} F_{n+1} \frac{|x-x_0|^{n+1}}{(n+1)!}$$

converges,

$$w = \lim_{n\to\infty} y_n.$$

Hence $w = y$. It follows that the solution is unique. □

4. Examples

In this section, two examples are presented to illustrate theorem (2.1) and (3.1).

Example 1. *Consider the IVP*

$$\begin{cases} y' &= \frac{1}{2}[y + y(y)], \ (x, y) \in S, \\ y(0) &= 0, \end{cases}$$

where $S = \{(x, y) : x \in [-1, 1]$ *and* $y \in [-1, 1]$.

Comparing this problem with (1.1) we have

$$f(x,y,z) = \frac{1}{2}(y+z), g(y) = y, x_0 = 0, y_0 = 0, a = 1 \text{ and } b = 1.$$

We see that

$$M = \max\{|f(x,y,y(g(y)))| : (x,y) \in S\} = 1, g(y_0) = x_0,$$

$Ma = 1 = b$,

$$\begin{aligned} & |f(x,u,u(g(u))) - f(x,v,v(g(v))))| \\ = \;& |u + u(g(u)) - v - v(g(v))| \le |u-v| + |u(g(u)) - v(g(v))| \end{aligned}$$

and

$$|g(u) - g(v)| = |u - v|.$$

Here we take $L_1 = L_2 = L_3 = 1$. We observe that conditions **F-1**,**F-2**,**F-3**,**G-1**,**G-2** and **G-3** have been fulfilled. Thus the given problem has unique solution in $[-1,1]$.

Example 2. *Consider the IVP*

$$\begin{cases} y' &= \frac{1}{16}x^2y^2[y(\frac{1}{2}y)]^2, \ (x,y) \in S, \\ y(\frac{1}{2}) &= 1, \end{cases}$$

where $S = \{(x,y) : x \in [0,1]$ *and* $y \in [0,2]$.

Comparing this problem with (1.1) we have

$$f(x,y,z) = \frac{1}{16}x^2y^2z^2, g(y) = \frac{1}{2}y, x_0 = \frac{1}{2}, y_0 = 1, a = \frac{1}{2} \text{ and } b = 1.$$

We see that

$$M = \max\{|f(x,y,y(g(y)))| : (x,y) \in S\} = 1, g(y_0) = x_0,$$

$Ma = \frac{1}{2} < 1 = b$,

$$\begin{aligned} & |f(x,u,u(g(u))) - f(x,v,v(g(v))))| \\ & \tfrac{1}{16}x^2|u^2[u(g(u))]^2 - v^2[v(g(v))]^2| \\ \le \;& |u-v| + |u(g(u)) - v(g(v))| \quad \text{by Lemma(1.1)} \end{aligned}$$

and

$$|g(u) - g(v)| = \frac{1}{2}|u - v|.$$

Here we take $L_1 = L_3 = 1$ and $L_2 = \frac{1}{2}$. We observe that conditions **F-1**,**F-2**,**F-3**,**G-1**,**G-2** and **G-3** have been fulfilled. Thus the given problem has unique solution in $[0, 1]$.

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