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OSCILLATIONS AND ASYMPTOTIC STABILITY OF ENTIRE SOLUTIONS OF LINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Think about the linear delay differential equation,

$$y'(q)+\sum_{n=1}^{m}P_n(q)y(q-\tau_n)=0,\quad q\geq q_0, \tag{1}$$

where $P_n\in C([q_0,\infty),R)$ and $\tau_n\geq 0$ for $n=1,2,\ldots,m$. By investigating the oscillatory solutions of the linear delay differential equations, we offer new adequate condition for the asymptotic stability of the solutions of (1). We also produce comparison result and stability of (1).

1. ESTABLISHMENT AND MAIN RESULTS

Here, we think about the Linear Delay Differential Equation

$$y'(q)+\sum_{n=1}^{m}P_n(q)y(q-\tau_n)=0,\ \ q\geq q_0, \tag{2}$$

where $P_n\in C([q_0,\infty),R)$ and $\tau_n\geq 0$ for $n=1,2,\ldots,m$.

We expect that the peruser knows about standard symbols and basic consequences of Nevanlinna Theory [2].

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For a meromorphic function $y(q)$, the order of q is defined by

$$\rho(y) = \limsup_{r \to \infty} \frac{\log\ T(r, y)}{\log r}.$$

and the hyper-order is characterized by,

$$\rho_2(y) = \limsup_{r \to \infty} \frac{\log \log T(r, y)}{\log r}.$$

Here $T(r, y)$ is the Nevanlinna characteristic of y for all r outside a set of finite logarithmic measure. Our point is to build up new adequate conditions for the oscillation of all entire solutions of equation (1). A continuous differentiable function characterized on $[\tau(T_0), \infty]$ for $T_0 \geq q_0$ also fulfilling equation (1). For $q \geq T_0$ is known as solution of equation (1), such an answer is called oscillatory in the event that it has discretionary huge zeros. Else it is called non-oscillatory.

We except for the analysis of asymptotic conduct of the function

$$h(q) = \frac{y(\tau(q))}{y(q)}$$

that equation (1) has a solution $y(q)$ which is positive for all enormous q.

Lemma 1.1 ([4]). *Assume that $m > 0$ and equation (1) has an gradually positive solution $y(q)$. Then $m \leq \frac{1}{2}$ and*

$$\lambda_1 \leq \lim_{q \to \infty} \inf h(q) \leq \lambda_2,$$

where λ_1 is minor and λ_2 is the major root of the equation $\lambda = e^{b\lambda}$.

Lemma 1.2 ([3]). *Let $g(z)$ be a non constant meromorphic function and $c \in C$. If $\tau_2(q) < 1$ and $\epsilon > 0$ then*

$$m\left(r, \frac{g(z+1)}{g(z)}\right) = O\left(\frac{T(r, g)}{r^{1-\sigma_2(g)-\epsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

2. Oscillatory Properties

In this part we will consider the oscillatory properties of equation (1).

Lemma 2.1. *Let $y(q)$ be an gradually positive entire solution of equation* (1) *and* $0 < m \leq \dfrac{1}{e}$. *Suppose that*

$$\sum_{n=1}^{m} \liminf_{q\to\infty} \int_{\tau_n(q)}^{q} P_n(s) \exp\Big(\lambda_1 \int_{\tau_n(s)}^{\tau_n(q)} P_n(v)\, dv\Big) ds > 1 - \frac{1}{\gamma}, \tag{3}$$

where $\gamma \in [\lambda_1, \lambda_2]$ and λ_1 is the minor and λ_2 is the major root of $\lambda = e^{b\lambda}$. Then

$$\liminf_{q\to\infty} T\Big(r, \frac{y(\tau(q))}{y(q)}\Big) > \gamma.$$

Proof. Let $q > q_0$ be sufficiently enormous so that $\tau(q) > q_0$. Integrating (1) from $\tau(q)$ to q, we obtain

$$y(\tau(q)) = y(q) + \sum_{m=1}^{n} \int_{\tau(q)}^{q} P_n\ (s)\ y(s)\ ds. \tag{4}$$

Let us take $0 < \lambda < \lambda_1$. Then the function

$$\varpi(q) = \sum_{n=1}^{m} y(q) \exp\ \Big(\lambda \int_{q_0}^{q} P_n(s)\ ds\Big), \qquad q \geq q_1, \tag{5}$$

is diminishing for suitable $q_1 \geq q_0$ (cf [[5], [6]]), Indeed by Lemma 1.1.

$$\frac{y(\tau(q_1))}{y(q)} > \lambda,$$

for $q \geq q_2$, where $q_2 \geq q_1$ is sufficiently large, and consequently,

$$0 = y'(q) + \sum_{n=1}^{m} P_n(q)\ y(q - \tau_n) >\ y'(q) + \sum_{n=1}^{m} \lambda P_n(q)\ y(q - \tau_n),$$

which suggests $\varpi'(q) < 0$ for $q \geq q_2$. Substituting (5) and (4), we derive for $q \geq q_2$ that

$$
\begin{aligned}
y(\tau(q)) &= y(q) + \sum_{n=1}^{m} \int_{\tau(q)}^{q} P_n(s)\, y(s) ds \\
&= y(q) + \sum_{n=1}^{m} \int_{\tau(q)}^{q} P_n \varphi(\tau(q)) \sum_{n=1}^{m} \exp(-\lambda) \int_{q_0}^{q} P_n(s) ds \\
y(\tau(q)) &= y(q) + \sum_{n=1}^{m} \varphi(\tau(q)) \int_{\tau(q)}^{q} P_n(s) \sum_{n=1}^{m} \exp\Big(-\lambda \int_{q_0}^{\tau_n(s)} P_n(v)\, dv \Big) ds \\
&\geq y(q) + \sum_{n=1}^{m} y(\tau(q)) \exp\Big(\lambda \int_{q_0}^{\tau_n(q)} P_n(v)\, dv \Big) \qquad (6) \\
&\quad \int_{\tau_n(q)}^{q} P_n(s) \sum_{n=1}^{m} \exp\Big(-\lambda \int_{q_0}^{\tau_n(q)} P_n(v) dv \Big)\, ds \\
&\geq y(q) + \sum_{n=1}^{m} y(\tau(q)) \int_{\tau_n(q)}^{q} P_n(s) \exp\Big(\lambda \int_{\tau_n(s)}^{\tau_n(q)} P_n(v)\, dv \Big)\, ds \\
0 &\geq y(q) + \sum_{n=1}^{m} y(\tau(q)) \Big[-1 + \int_{\tau_n(q)}^{q} P_n(s) \exp(\lambda \int_{\tau_n(s)}^{\tau_n(q)} P_n(u) du) ds \Big].
\end{aligned}
$$

From (3) it follows that there exists a constant c with the end goal that $c > 1 - \dfrac{1-\epsilon}{\gamma}$, where $0 < \epsilon < \gamma[c - (1 - \frac{1}{\gamma})] \leq 1$, and

$$
\sum_{n=1}^{m} \liminf_{q \to \infty} \int_{\tau_n(q)}^{q} P_n(s) \exp(\lambda_1 \int_{\tau_n(s)}^{\tau_n(q)} P_n(v) dv) ds \geq c \geq 1 - \frac{1-\epsilon}{\gamma}. \qquad (7)
$$

Then, for λ sufficiently close to λ_1, we get

$$
\sum_{n=1}^{m} \int_{\tau_n(q)}^{q} P_n(s) \exp(\lambda \int_{\tau_n(s)}^{\tau_n(q)} P_n(v) dv) ds > 1 - \frac{1-\epsilon}{\gamma}, \qquad q \geq q_3,
$$

where $q_3 \geq q_2$ is sufficiently large. In the event that it isn't accurate, at that point for all $0 < \lambda < \lambda_1$ we have

$$
\sum_{n=1}^{m} \liminf_{q \to \infty} \int_{\tau_n(q)}^{q} P_n(s) \exp(\lambda \int_{\tau_n(s)}^{\tau_n(q)} P_n(v) dv) ds \leq 1 - \frac{1-\epsilon}{\gamma} < C.
$$

By letting $\lambda \to \lambda_1$, the last inequality prompts

$$\sum_{n=1}^{m} \lim_{q\to\infty} \inf \int_{\tau_n(q)}^{q} P_n(s) \exp(\lambda_1 \int_{\tau_n(s)}^{\tau_n(q)} P_n(v)dv)ds < C.$$

We deduce from (1) and Lemma1.2 that

$$\begin{aligned} T(r, y(\tau(q))) &= T(r, y(q)) + \sum_{n=1}^{m} \int_{\tau_n(q)}^{q} P_n(s) T(r, y(s)) ds \\ &\leq m(r, y(\tau(q))) + m(r, \frac{y(q)}{y(\tau(q))}) + \sum_{n=1}^{m} \int_{\tau_n(q)}^{q} P_n(s) m(r, \frac{y(q)}{y(\tau(q)}) \\ &= m(r, y(\tau(q))) + S(r, w). \end{aligned}$$

Accordingly we acquire from (6),

$$0 > T(r, y(q)) - 1 - \frac{1-\epsilon}{\gamma} T(r, y(\tau(q))),$$

$$0 > \Big(1 - \frac{1-\epsilon}{\gamma}\Big) \frac{T(r, y(\tau(q)))}{T(r, y(q))},$$

$$T\Big(r, \frac{y(\tau(q))}{y(q)}\Big) \geq \frac{\gamma}{1-\epsilon} > \frac{(1-\epsilon)\gamma}{1-\epsilon} = \gamma \qquad q \geq q_3.$$

Thus, we have

$$\lim_{q\to\infty} \inf \; T\Big(r, \frac{y(\tau(q))}{y(q)}\Big) > \gamma.$$

This completes the proof. □

3. MAIN RESULTS

Theorem 3.1. *Let* $0 < m < \frac{1}{e}$. *Assume that*

$$\sum_{n=1}^{m} \lim_{q\to\infty} \inf \int_{\tau_n(q)}^{q} P_n(s) \exp(\lambda_1 \int_{\tau_n(s)}^{\tau_n(q)} P_n(v)dv)ds \leq 1 - \frac{1}{\lambda_2}.$$

Then all entire solutions of equations (1) *oscillate.*

Proof. Suppose that equation (1) ultimately has a positive solution $y(q)$. It follows from Lemma 2.1 that

$$\lim_{q\to\infty} \inf T\left(r, \frac{y(\tau(q_1))}{y(q)}\right) \lambda_2.$$

This repudiates the consequence of Lemma 2.1. □

Theorem 3.2. *Let* $0 < m < \frac{1}{e}$. *Suppose that there exists* $\beta \in (\lambda_1, \lambda_2)$ *such that*

(8) $$\sum_{n=1}^{m} \liminf_{q\to\infty} \int_{\tau_n(q)}^{q} P_n(s)\exp(\lambda_1 \int_{\tau_n(s)}^{\tau_n(q)} P_n(v)dv)ds \le 1 - \frac{1}{\beta},$$

(9) $$\sum_{n=1}^{m} \liminf_{q\to\infty} \int_{\tau_n(q)}^{q} P_n(s)\exp(\beta \int_{\tau_n(s)}^{\tau_n(q)} P_n(v)dv)ds \le 1 - \frac{1}{\lambda_2}.$$

Then all the solutions of equation (1) *oscillates.*

Proof. Suppose that equation (1) eventually has a positive solutions $y(q)$. By Lemma 2.1, condition (8) says that

$$\liminf_{q\to\infty} T(r, \frac{y(\tau(q_1))}{y(q)}) > \beta,$$

with regards to condition (9) also, rehashing the strategy as in verification of Lemma 2.1, we get

$$\liminf_{q\to\infty} T(r, \frac{y(\tau(q_1))}{y(q)}) > \lambda_2.$$

This negates the aftereffect of Lemma 1.1. The proof is complete. □

Comparison result and stability: The following corollary about solutions of (1) will be useful in this section.

Corollary 3.1. *Let* $u(q)$ *be a non oscillatory solution of* (1). *Set* $h(q) = \dfrac{y(q)}{u(q)}$, $q \ge T$, *where* $y(q)$ *is entire solution of* (1) *and* $T \ge q_0$ *is such that* $u(q) \ne 0$ *for* $q \ge T$. *Then*

$$h(q) = \sum_{n=1}^{m} P_n(q)\frac{z(q-\tau_n)}{u(q)}[h(q) - h(q-\tau_n)]q \ge T,$$

with

$$T(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}) \le S(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}).$$

Proof.

$$h(q) = \frac{1}{u^2(q)}[-u(q)\sum_{n=1}^{m} P_n(q)y(q-\tau_n) + y(q)\sum_{n-1}^{m} P_n(q)u(q-\tau_n)]$$

$$\frac{1}{u^2(q)}\sum_{n=1}^{m} P_n(q)[\frac{y(q)}{u(q)} - \frac{y(q-\tau_n)}{u(q-\tau_n)}]u(q)u(q-\tau_n)$$

$$\sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}[h(q)-h(q-\tau_n)]$$

Let $A_1 = h(q)$, $A_2 = h(q) - h(q-\tau_n)$. It is easy to see that A_1 and A_2 are of finite order. So, A_1 and A_2 are two small functions of $\sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}$, which means that

$$T(r, A_1) = T(r, A_2) = S(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)})$$

obviously,

$$T(r, g) = S(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}).$$

We rewrite this as

$$A_1 = \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)} A_2. \tag{10}$$

Next, we show that $A_2 \neq 0$. Suppose $A_2 = 0$ then

$$y(q) = -p(q)y(\tau(q)),$$

which suggests $2T(r, y(q)) \leq T(r, y(q)) + S(r, y)$ a contradiction. Then $A_2 \neq 0$. Suppose $A_1 = 0$ by using second fundamental theorem, we have

$$\begin{aligned}
&T\left(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}\right) \\
&\leq N\left(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}\right) + N\left(r, \frac{1}{\sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}}\right) \\
&+ N\left(r, \frac{1}{\sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)} - \frac{A_1}{A_2}}\right) + S\left(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}\right) \\
&\leq N\left(r, \frac{1}{A_2}\right) + S\left(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}\right) = S\left(r, \sum_{n=1}^{m} P_n(q)\frac{u(q-\tau_n)}{u(q)}\right),
\end{aligned}$$

which is a contradiction. So, $A_1 = 0$, which implies $h(q) = 0$, a contradiction. $\square$

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