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A NOTE ON CHAOS THEORY

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ABSTRACT. In this short paper, I gave a collective overview of Chaos in different dimensions and I believe that it would be of great help for projects related to this area of research.

1. INTRODUCTION

Chaos theory focuses on the study of dynamic systems with a random state of disorder. The mathematical theory's basic concepts will be presented through a simple analysis of exciting, dynamic systems in one-, two-, and three-dimensional space. In the beginning, a discussion will be conducted on the interval maps and observe that when maps are monotonic, their iterates behave in an orderly fashion. However, in the quadratic maps, the iterates exhibit archetypal characteristics of Chaos. Consequently, mapping in two dimensions presents a better variety of chaotic regimes than in the interval. Chaos's two main definitions: a state of nature devoid of order or intense confusion associated with unpredictability. The research paper will analyze one-, two- and three-dimensional Chaos using interval maps and quadratic maps.

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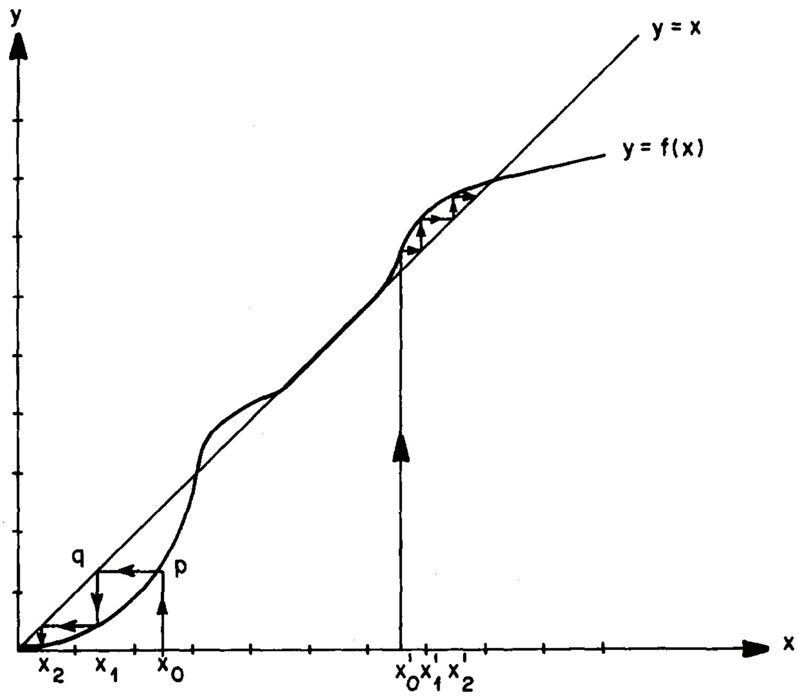

Fig. 1. Graph of function with $f' > 0$.

FIGURE 1.

2. ONE-DIMENSIONAL CHAOS (INTERVAL MAPS)

The iterates of the interval maps denoted as function $f : I \to I$, where $I = [0, 1]$ is the unit interval of the actual line providing a rich source of chaotic behavior. Therefore, for any given point x, of I the iterates are $f(x), f(f(x)) = f^2(x), f(f^2(x)), \ldots, f(f^{n-1}(x)) = f^n(x)$. Therefore, x being a fixed point, the iterates stay at x. and when it strays from the point x but return after a minimum of $n > 1$ is iteration, x is defined as a periodic point of period n. the set of iterates $x, f(x), \ldots, f^{n-1}(x)$ is called the periodic orbit of the period n or n-cycle. In order to analyze the theory in one dimension, we limit ourselves to smooth maps. Smooth maps are functions having continuous derivatives of all orders. In contrast to the Chaos, a consideration is done using monotonic interval maps. Monotonic interval maps are functions $f : I \to I$, and the derivative f' does not vanish.

Figure 1 above illustrates an interval map with a positive derivative. The points at which $y = f(x)$ intersect $y = x$ are the fixed points of f. To locate

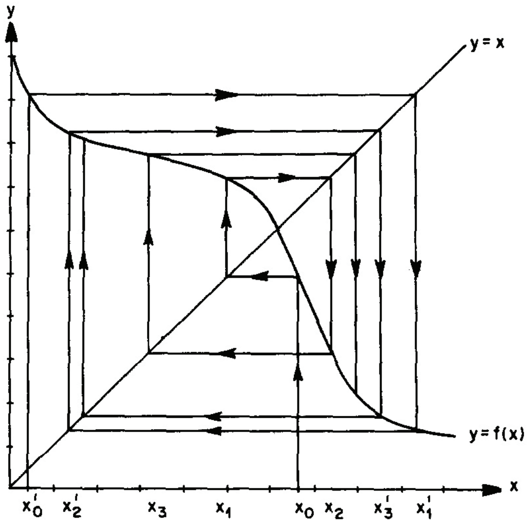

Fig. 2. Graph of function with $f' < 0$.

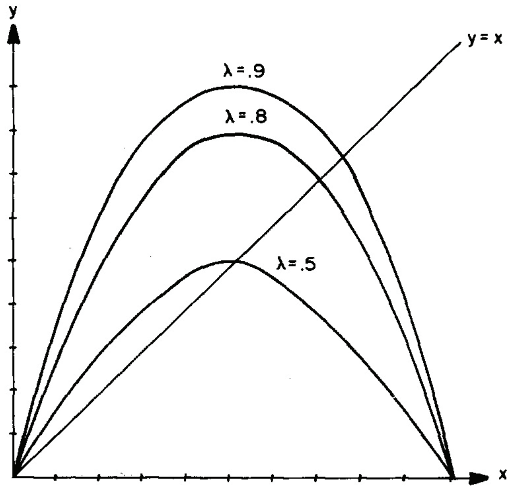

Fig. 3. f_λ for $\lambda = .5$, .8 and .9.

FIGURE 2.

$f(x_0)$ a vertical line is drawn through x_0 until it hits the graph at p followed by a horizontal line through p until it intersects $y = x$ at q. At that point, the x coordinate of g is $f(x_0)$. Repeat of the process leads to the tracing of orbit x_0, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$. It can be verified from Fig 1 that $x^{(1)}$ and $x^{(3)}$ are attractors while $x^{(2)}$ is a repeller. Alternatively, J attracts nearby points to the left and repels nearby points to the right. From research, it is evident that the only possible invariants sets are composed of fixed points. Therefore, the iterates behave in an orderly way; thus, no chaos exists. Likewise, decreasing the interval maps does not create Chaos either. Besides, attractors are characterized by $|f'| < 1$ while repellers $|f| > 1$, which is a general rule. Therefore, monotonic interval maps do not create Chaos. To create Chaos, a family of interval maps that are simple but create Chaos is derived from the formula below: $f_\lambda(x) = 4\lambda x(1 - x)$, under the condition $0 < \lambda < 1$. This equation has been researched exclusively by a host of mathematicians. The research presented two graphs for cases $\lambda = .5, \lambda = .8$, and $\lambda = .9$. as below;

From the graphs, the function f_λ achieves a maximum value at $x = .5$ shown when solving $f'_\lambda = 0$. The behavior of the three cases differs significantly.

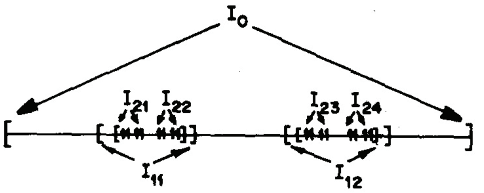

Fig. 5. Construction of a Cantor set.

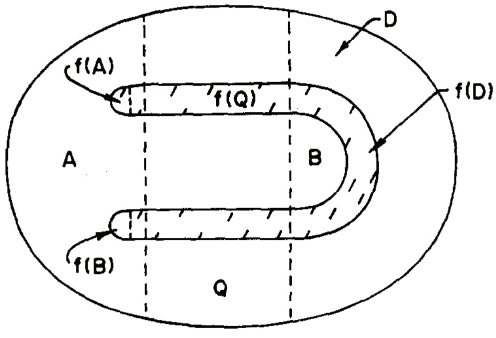

Fig. 6. The horseshoe.

FIGURE 3.

$$\frac{dx}{dt} = -10x + 10y,$$

$$\frac{dy}{dt} = -y + 200z - xz,$$

$$\frac{dz}{dt} = -\tfrac{8}{3}z + xy.$$

3. Two and three-dimensional Chaos

For two-dimensional Chaos, the focus is on a disk D, a subset of a plane consisting of all points inside a circle. In this case, we are looking at disk maps which gives a richer vein of chaotic behavior. The Focus area is the horseshoe described by merely specifying the image $f(D)$ under the map f. the disk is divide into pieces A, Q, and B, as shown in the figure below. The map has a strange invariant set with iterates jumping around wildly (see [1]). The peculiar invariant set is a product of cantos set that can denote f^{-1} as the inverse of f and f^{-n} be the n-fold composition of f^{-1} with itself. The set of points common to all the sets is an invariant set making it quite strange.

For three-dimensional Chaos, several differential equations with chaotic solutions can be obtained. One of the equations is the Lorenz equation which is in the first-order differential equation of the form below;

The equation was derived from Lorenz's simple metrological model and generated from observation of erratic behavior in computer solution, thus understanding the chaotic nature of solutions. Analysis of the Lorenz equations is difficult to undertake, but there are some interesting properties of the equation. The equation is non-linear, which points out that linear equations cannot exhibit Chaos. This can be verified by looking at differential equations. For Lorenz equation, three stationary point: $(x, y, z) = (0, 0, 0)$, $\approx (.015, .015, .000084)$, $\approx (199.97, 199.97, 14995.5)$. Therefore, each solution of the equation starts in the set, stays in the set for all time. Interestingly, the equation has two strange attractors that behave like two solar systems. The solution curves also act like trajectories of comets. In the system, if the comets are drawn too close to the system, they get sucked in. if not, they oscillate back and forth between the two systems in a chaotic way. The shape of the Lorenz attractor, when plotted, resembles a butterfly. Lorenz equation is a three-dimensional equation that allows for partial differentiation. However, when looking at chaos theory, the multidimensional approaches to differentiation to understand the different behaviors at the distant point are integral for understanding particle movement.

4. Conclusion

The chaos theory is derived from differential equations that do not have a linear outcome in the solution. It is the study of the deterministic difference equations displaying sensitivity that depends on the initial condition in a way that generates time paths that are random in look. Therefore, it is concerned with unpredictable courses of events. It may involve non-linear and complex linear equations. The Lorenz famous butterfly effect best illustrates it. Chaos can be deterministic and have a prediction form of a model. From the analysis done by various mathematicians, it is evident that chaos theory can be presented in non-linear equations, with long-term prediction being futile. The random nature of the outcomes makes predictions difficult. Chaos theory shows a challenge in determining position and effects as such predictions can be done at a point x and not at the different outcomes. Therefore, chaos theory has many approaches to problem-solving and understanding the paths and patterns in fluid and particle movements. It is a mathematical approach to understanding predictability in non-linear situations.

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