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FIXED POINT THEOREMS IN A GENERALIZED CONE *b*-METRIC SPACE

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ABSTRACT. In this paper, we introduce a generalization of a cone *b-metric space and to demonstrate the usefulness we prove some fixed point theorems of contraction type mappings in the generalized cone b-metric space.*

1. INTRODUCTION

In [1], Huang and Zhang introduced a cone metric spaces as a generalization of metric spaces by replacing the set of real numbers by an ordered Banach space. They proved some fixed point theorems for contractive mappings by using the normality of cone. The results by Huang and Zhang was then subsequently generalized by Rezapour and R. Hamlbarani, [4] omitting the assumption of the normality of the cone. Topological questions in cone metric spaces were investigated in [5], where it was proved that every cone metric space is a first-countable topological space. Hence, continuity is equivalent to sequential continuity and compactness is equivalent to sequential compactness.

In [2], Hussain and Shah, introduced a cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces with some topological properties and improved results pertaining to KKM mappings. Furthermore, they proved

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some fixed point existence results for multivalued mappings defined on cone b-metric spaces.

In 2013, Liu and Xu, [3], presented fixed point theorems of these mappings in cone metric spaces over Banach algebras by using tools of spectral radius and normal solid cones, and they gave an example to confirm that it is not equivalent to versions in the usual metric spaces.

Definition 1.1. *Let* $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ *be a real Banach space and* $\mathbb{P}$ *a subset of* $\mathbb{X}$. *The subset* $\mathbb{P}$ *is a cone, [1]* $\iff$

(i) $\mathbb{P}$ *is closed, non-empty and* $\mathbb{P} \neq \{\theta\}$*,where* θ *is the zero element of* $\mathbb{X}$*.*
(ii) *If* $x, y \in \mathbb{P}$ *and* a, b *are non-negative real numbers then* $ax + by \in \mathbb{P}$*.*
(iii) $\mathbb{P} \cap (-\mathbb{P}) = \{\theta\}$*.*

Define a partial ordering $\preceq$ with respect to $\mathbb{P}$ by $x \preceq y \iff y - x \in \mathbb{P}$ and if $x \neq y$ then we write $x \prec y$.

Definition 1.2. *Let* X *be a non-empty set. A function* $d : X \times X \to \mathbb{X}$ *is a cone* b*-metric, [2] on* X *if there exists a real number* $\alpha \geq 1$*, such that the following conditions hold for all* $x, y, z \in X$ *:*

(i) *if* $x, y \in X$ *then* $\theta \preceq d(x, y)$*, and* $d(x, y) = \theta \iff x = y$*;*
(ii) $d(x, y) = d(y, x)$*;*
(iii) $d(x, y) \preceq \alpha\,[d(x, z) + d(z, y)]$*.*

The pair (X, d) is a called a cone b-metric space.

2. Main result

Definition 2.1. *Let* X *be a non-empty set. A function* $\rho : X \times X \to \mathbb{X}$ *is an* α, β*-cone* b*-metric on* X *if there exists real numbers* $\alpha, \beta \geq 1$*, such that the following conditions hold for all* $x, y, z \in X$*:*

(i) *if* $x, y \in X$ *then* $\theta \preceq \rho(x, y)$ *and* $\rho(x, y) = \theta \iff x = y$*;*
(ii) $\rho(x, y) = \rho(y, x)$*;*
(iii) $\rho(x, y) \preceq \alpha\rho(x, z) + \beta\rho(z, y)$*.*

The pair (X, ρ) is a called an α, β-cone b-metric space. In the special case $\alpha = \beta$ we obtain a cone b-metric space.

Example 1. *Let* $\mathbb{X} = \mathbb{R}^2$*,* $\mathbb{P} = \{(x,y) \in \mathbb{X}; x, y \geq 0\}$ *and* $X = (1,3)$ *then define* $\rho : X \times X \to \mathbb{X}$*, such that*

$$\rho(x,y) = \begin{cases} \left(e^{|x-y|}, Ae^{|x-y|}\right), & \text{if } x \neq y, \\ \theta = (0,0), & \text{if } x = y, \end{cases}$$

where A is a constant greater than one. To show that ρ is an α, β-cone b-metric it suffices to verify property (iii) of definition 2.1.

For $x \neq y,\ z \in X$

$$\begin{aligned} e^{|x-y|} &\leq e^{|x-z|+|z-y|} \\ &= e^{\frac{1}{3}|x-z|+\frac{2}{3}|z-y|} e^{\frac{2}{3}|x-z|+\frac{1}{3}|z-y|} \\ &\leq \sup_{x,y,z \in X} e^{\frac{1}{3}|x-z|+\frac{2}{3}|z-y|} \left(\tfrac{2}{3}e^{|x-z|} + \tfrac{1}{3}e^{|z-y|}\right) \\ &\leq \tfrac{2}{3}e^2 e^{|x-z|} + \tfrac{1}{3}e^2 e^{|z-y|}. \end{aligned}$$

It follows that

$$\rho(x,y) \preceq \tfrac{2}{3}e^2 \left(e^{|x-z|}, Ae^{|x-z|}\right) + \tfrac{1}{3}e^2 \left(e^{|z-y|}, Ae^{|z-y|}\right),$$

with $\alpha = \frac{2}{3}e^2 \geq 1$ *and* $\beta = \frac{1}{3}e^2 \geq 1$*. Thus* (X,ρ) *is an* α,β*-cone* b*-metric space.*

Example 2. *Let* $\mathbb{X} = \mathbb{R}^n$*,* $\mathbb{P} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{X}; x_i \geq 0 \text{ for all } i\}$ *and* $X = (1,3)$ *then define* $\rho : X \times X \to \mathbb{X}$*, such that*

$$\rho(x,y) = \begin{cases} \left(A_1 e^{|x-y|}, A_2 e^{|x-y|}, \cdots, A_n e^{|x-y|}\right), & \text{if } x \neq y, \\ \theta, & \text{if } x = y, \end{cases}$$

where A_i are constants greater than one for all i. Then (X,ρ) is an α,β-cone b-metric space.

3. Preliminaries

Definition 3.1. *Let (X,ρ) be an α,β-cone b-metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then:*

- (i) *The sequence $\{x_n\}$ converges to $x \in X \iff$ for every $\theta \prec \epsilon \in \mathbb{X}$ there is a natural number N, such that $\rho(x_n, x) \prec \epsilon$ for all $n \geq N$.*
- (ii) *The sequence $\{x_n\}$ is a Cauchy in $(X,\rho) \iff$ for every $\theta \prec \epsilon \in \mathbb{X}$ there exist $N \in \mathbb{N}$, such that $\rho(x_n, x_m) \prec \epsilon$ for all $n, m \geq N$.*

(iii) *The space* (X, ρ) *is complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$.

Let $\mathbb{P}$ be a cone in an α, β-cone b-metric space, for a real Banach space $\mathbb{X}$. For $x, y, z \in \mathbb{X}$:

(i) if $x \preceq y$ and $y \prec z$ then $x \prec z$.
(ii) if $\theta \preceq x \prec y$ for each y in the interior point of $\mathbb{P}$ then $x = \theta$
(iii) if $x \preceq \lambda x$ for some $0 \leq \lambda < 1$ then $x = \theta$.

4. Fixed point theorem for a generalized cone b-metric spaces

Theorem 4.1. *Let* (X, ρ) *be a complete* α, β*-cone* b*-metric space and* $T : X \to X$ *be a mapping, such that*

$$\rho(Tx, Ty) \preceq \lambda \rho(x, y), \tag{4.1}$$

for all $x, y \in X$, *where* $0 \leq \lambda < \frac{1}{\beta}$. *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary and define an iterative sequence $\{x_n\}$ by

$$x_n = Tx_{n-1} = T^n x_0. \tag{4.2}$$

Then successively applying the inequality (4.1) we get

$$\begin{aligned} \rho(x_{n+1}, x_n) &= \rho(Tx_n, Tx_{n-1}) \\ &\preceq \lambda \rho(x_n, x_{n-1}) \\ &\vdots \\ &\preceq \lambda^n \rho(x_1, x_0). \end{aligned} \tag{4.3}$$

For $m, n \in \mathbb{N}$, and using inequality (4.3), we get

$$\begin{aligned} &\rho(x_n, x_{n+m}) \\ &\preceq \alpha \rho(x_n, x_{n+1}) + \beta \rho(x_{n+1}, x_{n+m}) \\ &\preceq \alpha \rho(x_n, x_{n+1}) + \beta [\alpha \rho(x_{n+1}, x_{n+2}) + \beta \rho(x_{n+2}, x_{n+m})] \end{aligned}$$

$$
\begin{aligned}
&\preceq \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) \\
&\quad + \alpha\beta^{m-1}\rho(x_{n+m-1}, x_{n+m}) \\
&\preceq \alpha\lambda^n\rho(x_0, x_1) + \alpha\beta\lambda^{n+1}\rho(x_0, x_1) + \cdots + \alpha\beta^{m-2}\lambda^{n+m-2}\rho(x_0, x_1) \\
&\quad + \alpha\beta^{m-1}\lambda^{n+m-1}\rho(x_0, x_1) \\
&\preceq \alpha\lambda^n\rho(x_0, x_1)\left(1 + \beta\lambda + \cdots + \beta^{m-2}\lambda^{m-2} + \beta^{m-1}\lambda^{m-1}\right) \\
&\preceq \alpha\lambda^n\left(\tfrac{1-\beta^m\lambda^m}{1-\beta\lambda}\right)\rho(x_1, x_0) \\
&\prec \alpha\lambda^n\left(\tfrac{1}{1-\beta\lambda}\right)\rho(x_1, x_0)
\end{aligned}
$$

Let $\theta \prec \epsilon$, then we can find $N \in \mathbb{N}$, such that $\rho(x_n, x_{n+m}) \prec \alpha\frac{\lambda^n}{1-\beta\lambda}\rho(x_1, x_0) \prec \epsilon$ for $n \geq N$ and any $m \in \mathbb{N}$. Taking the limit $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence.

By the completeness of (X, ρ) it follows that there exists $x^* \in X$ and $N \in \mathbb{N}$, such that $\rho(x_n, x^*) \prec \frac{\epsilon}{2\max\{\alpha,\beta\}}$ and $\rho(x_{n-1}, x^*) \prec \frac{\epsilon}{2\max\{\alpha,\beta\}}$ for all $n \geq N$. Furthermore, we get

$$
\begin{aligned}
\rho(Tx^*, x^*) &\preceq \alpha\rho(Tx^*, x_n) + \beta\rho(x_n, x^*) \\
&\preceq \alpha\lambda\rho(x^*, x_{n-1}) + \beta\rho(x_n, x^*) \\
&\prec \alpha\rho(x^*, x_{n-1}) + \beta\rho(x_n, x^*) \\
&\prec \alpha\tfrac{\epsilon}{2\max\{\alpha,\beta\}} + \beta\tfrac{\epsilon}{2\max\{\alpha,\beta\}} \\
&\prec \epsilon,
\end{aligned}
$$

for some $n \geq N$ and $m \in \mathbb{N}$. We obtain that $\rho(Tx^*, x^*) = 0$ thus $Tx^* = x^*$. Hence x^* is a fixed point of T.

To show that the fixed point is unique. Suppose that there exists another fixed point x^{**} then $\rho(x^*, x^{**}) = \rho(Tx^*, Tx^{**}) \preceq \lambda\rho(x^*, x^{**})$. Since $\lambda < \frac{1}{\beta} < 1$ we conclude that $x^* = x^{**}$. □

Theorem 4.2. *Let (X, ρ) be an α, β-cone b-metric space and $T : X \to X$ satisfies the contraction condition*

$$\rho(Tx, Ty) \preceq \lambda_1\rho(x, Tx) + \lambda_2\rho(y, Ty) + \lambda_3\rho(x, Ty) + \lambda_4\rho(y, Tx), \tag{4.4}$$

for $x, y \in X$, where $0 \leq \lambda_i < 1$, $i = 1, 2, 3, 4$, $\lambda_1 + \lambda_2 + \left(\frac{\alpha+\beta}{2}\right)\lambda_3 + \left(\frac{\alpha+\beta}{2}\right)\lambda_4 < 1$, $\lambda_1 + \lambda_2 + \frac{\beta(1+\alpha)}{1+\beta}\lambda_3 + \frac{\beta(1+\alpha)}{1+\beta}\lambda_4 < \frac{2}{1+\beta}$ and $\frac{\alpha}{2}\lambda_1 + \frac{\beta}{2}\lambda_2 + \frac{\beta^2}{2}\lambda_3 + \frac{\alpha\beta}{2}\lambda_4 < 1$. Then T has a fixed point in X.

Furthermore, if $\lambda_3 + \lambda_4 < \frac{1}{\alpha}$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}
\rho(x_{n+1}, x_n) &= \rho(Tx_n, Tx_{n-1}) \\
&\preceq \lambda_1\rho(x_n, Tx_n) + \lambda_2\rho(x_{n-1}, Tx_{n-1}) + \lambda_3\rho(x_n, Tx_{n-1}) + \lambda_4\rho(x_{n-1}, Tx_n) \\
&\preceq \lambda_1\rho(x_n, x_{n+1}) + \lambda_2\rho(x_{n-1}, x_n) + \lambda_4\rho(x_{n-1}, x_{n+1}) \\
&\preceq \lambda_1\rho(x_n, x_{n+1}) + \lambda_2\rho(x_{n-1}, x_n) + \lambda_4\left[\alpha\rho(x_{n-1}, x_n) + \beta\rho(x_n, x_{n+1})\right].
\end{aligned}$$

It follows that

$$(1 - \lambda_1 - \beta\lambda_4)\,\rho(x_{n+1}, x_n) \preceq (\lambda_2 + \alpha\lambda_4)\,\rho(x_{n-1}, x_n). \tag{4.5}$$

In a similar manner, we can show that

$$(1 - \lambda_2 - \beta\lambda_3)\,\rho(x_n, x_{n+1}) \preceq (\lambda_1 + \alpha\lambda_3)\,\rho(x_{n-1}, x_n). \tag{4.6}$$

Adding inequalities 4.5 and 4.6 we get

$$(2 - \lambda_1 - \lambda_2 - \beta\lambda_3 - \beta\lambda_4)\,\rho(x_n, x_{n+1}) \tag{4.7}$$

$$\preceq (\lambda_1 + \lambda_2 + \alpha\lambda_3 + \alpha\lambda_4)\,\rho(x_{n-1}, x_n). \tag{4.8}$$

If $\lambda = \frac{\lambda_1+\lambda_2+\alpha\lambda_3+\alpha\lambda_4}{2-\lambda_1-\lambda_2-\beta\lambda_3-\beta\lambda_4}$ then from assumption it can be shown that $0 \leq \lambda < 1$ and $\rho(x_n, x_{n+1}) \preceq \lambda\rho(x_{n-1}, x_n)$.

Thus, recussively applying the above inequality, we obtain that

$$\begin{aligned}
\rho(x_{n+1}, x_n) &\preceq \lambda\rho(x_n, x_{n-1}) \\
&\vdots \\
&\preceq \lambda^n\rho(x_1, x_0)
\end{aligned}$$

For $m, n \in \mathbb{N}$, we get

$$\begin{aligned}
&\rho(x_n, x_{n+m}) \\
&\preceq \alpha\rho(x_n, x_{n+1}) + \beta\rho(x_{n+1}, x_{n+m}) \\
&\preceq \alpha\rho(x_n, x_{n+1}) + \beta[\alpha\rho(x_{n+1}, x_{n+2}) + \beta\rho(x_{n+2}, x_{n+m})] \\
&\preceq \alpha\rho(x_n, x_{n+1}) + \alpha\beta\rho(x_{n+1}, x_{n+2}) + \cdots + \alpha\beta^{m-2}\rho(x_{n+m-2}, x_{n+m-1}) \\
&\quad + \alpha\beta^{m-1}\rho(x_{n+m-1}, x_{n+m}) \\
&\preceq \alpha\lambda^n\rho(x_0, x_1) + \alpha\beta\lambda^{n+1}\rho(x_0, x_1) + \cdots + \alpha\beta^{m-2}\lambda^{n+m-2}\rho(x_0, x_1) \\
&\quad + \alpha\beta^{m-1}\lambda^{n+m-1}\rho(x_0, x_1) \\
&\preceq \alpha\lambda^n\rho(x_0, x_1)\left(1 + \beta\lambda + \cdots + \beta^{m-2}\lambda^{m-2} + \beta^{m-1}\lambda^{m-1}\right) \\
&= \alpha\lambda^n\left(\tfrac{1-\beta^m\lambda^m}{1-\beta\lambda}\right)\rho(x_1, x_0) \\
&\prec \alpha\lambda^n\left(\tfrac{1}{1-\beta\lambda}\right)\rho(x_1, x_0)
\end{aligned}$$

Let $\theta \prec \epsilon$ then we can find $N \in \mathbb{N}$, such that $\rho(x_n, x_{n+m}) \prec \alpha\frac{\lambda^n}{1-\beta\lambda}\rho(x_1, x_0) \prec \epsilon$ for $n \geq N$ and any $m \in \mathbb{N}$. We conclude that $\{x_n\}$ is a Cauchy sequence.

Furthermore, we get

$$\begin{aligned}
&\rho(Tx^*, x^*) \\
&\preceq \alpha\rho(Tx^*, x_n) + \beta\rho(x_n, x^*) \\
&= \alpha\rho(Tx^*, Tx_{n-1}) + \beta\rho(x_n, x^*) \\
&\preceq \alpha\lambda_1\rho(x^*, Tx^*) + \alpha^2\lambda_2\rho(x_{n-1}, x^*) + \alpha\lambda_2\beta\rho(x^*, x_n) + \alpha\lambda_3\rho(x^*, x_n) \\
&\quad + \alpha^2\lambda_4\rho(x_{n-1}, x^*) + \alpha\lambda_4\beta\rho(x^*, Tx^*) + \beta\rho(x_n, x^*)
\end{aligned}$$

It follows that

$$\begin{aligned}
(1 - \alpha\lambda_1 - \alpha\lambda_4\beta)\,\rho(Tx^*, x^*) &\preceq \left(\alpha^2\lambda_2 + \alpha^2\lambda_4\right)\rho(x_{n-1}, x^*) \\
&\quad + (\alpha\lambda_2\beta + \alpha\lambda_3 + \beta)\,\rho(x^*, x_n)
\end{aligned} \tag{4.9}$$

In a similar manner, we get

$$\begin{aligned}
\rho(x^*, Tx^*) &\preceq \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*) \\
&= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*) \\
&\preceq \alpha\rho(x^*, x_n) + \beta\lambda_1\rho(x_{n-1}, x_n)
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_2\beta\rho(x^*, Tx^*) + \beta\lambda_3\rho(x_{n-1}, Tx^*) + \beta\lambda_4\rho(x^*, x_n) \\
\preceq\ & \alpha\rho(x^*, x_n) + \alpha\beta\lambda_1\rho(x_{n-1}, x^*) + \lambda_1\beta^2\rho(x^*, x_n) \\
&+ \lambda_2\beta\rho(x^*, Tx^*) + \alpha\beta\lambda_3\rho(x_{n-1}, x^*) \\
&+ \beta^2\lambda_3\rho(x^*, Tx^*) + \beta\lambda_4\rho(x^*, x_n)
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\left(1 - \beta\lambda_2 - \beta^2\lambda_3\right)\rho(x^*, Tx^*) &\preceq \left(\alpha + \lambda_1\beta^2 + \beta\lambda_4\right)\rho(x^*, x_n) \\
&\quad + \left(\alpha\lambda_1\beta + \alpha\beta\lambda_3\right)\rho(x_{n-1}, x^*)
\end{aligned}
\tag{4.10}
$$

Combining inequalities (4.9) and (4.10) we obtain

$$
\begin{aligned}
&\left(2 - \alpha\lambda_1 - \beta\lambda_2 - \beta^2\lambda_3 - \alpha\lambda_4\beta\right)\rho(Tx^*, x^*) \\
\preceq\ & \left(\alpha + \lambda_1\beta^2 + \beta\lambda_4 + \alpha\lambda_2\beta + \alpha\lambda_3\right. \\
&\left. +\beta\right)\rho(x^*, x_n) + \left(\alpha\lambda_1\beta + \alpha\beta\lambda_3 + \alpha^2\lambda_2 + \alpha^2\lambda_4\right)\rho(x_{n-1}, x^*).
\end{aligned}
\tag{4.11}
$$

By the completeness of (X, ρ) it follows that there exists $x^* \in X$ and $N \in \mathbb{N}$, such that $\rho(x_{n-1}, x^*) \prec \frac{\epsilon\left(2-\alpha\lambda_1-\beta\lambda_2-\beta^2\lambda_3-\alpha\lambda_4\beta\right)}{2(\alpha\lambda_1\beta+\alpha\beta\lambda_3+\alpha^2\lambda_2+\alpha^2\lambda_4)}$ and $\rho(x_n, x^*) \prec \frac{\epsilon\left(2-\alpha\lambda_1-\beta\lambda_2-\beta^2\lambda_3-\alpha\lambda_4\beta\right)}{2(\alpha+\lambda_1\beta^2+\beta\lambda_4+\alpha\lambda_2\beta+\alpha\lambda_3+\beta)}$ for all $n \geq N$. From inequality (4.11), we obtain that $\rho(Tx^*, x^*) = 0$ thus $Tx^* = x^*$. Hence x^* is a fixed point of T.

To show that the fixed point is unique. Suppose that there exists another fixed point x^{**} then

$$
\begin{aligned}
\rho(x^*, x^{**}) = \rho(Tx^*, Tx^{**}) &\preceq \lambda_3\rho(x^*, Tx^{**}) + \lambda_4\rho(x^{**}, Tx^*) \\
&\preceq \lambda_3\alpha\rho(x^*, x^{**}) + \lambda_4\alpha\rho(x^{**}, x^*).
\end{aligned}
$$

Since $\lambda_3 + \lambda_4 < \frac{1}{\alpha}$, we conclude that $x^* = x^{**}$. □

Theorem 4.3. *Let (X, ρ) be a complete α, β-cone b-metric space and $T : X \to X$ mapping satisfying*

$$
\rho(Tx, Ty) \preceq \lambda_1\rho(x, y) + \lambda_2\rho(x, Tx) + \lambda_3\rho(y, Ty), \tag{4.12}
$$

for all $x, y \in X$ with $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 < \frac{1}{\beta}$,$\alpha\lambda_2 < 1$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary in X then define the sequence $\{x_n\}$, such that $x_{n+1} = Tx_n$. For $n \in \mathbb{N}$, we get

$$\begin{aligned} \rho(x_{n+1}, x_n) &= \rho(Tx_n, Tx_{n-1}) \\ &\preceq \lambda_1 \rho(x_n, x_{n-1}) + \lambda_2 \rho(x_n, Tx_n) + \lambda_3 \rho(x_n, x_{n-1}). \end{aligned} \tag{4.13}$$

It follows from (4.13) that

$$(1 - \lambda_2)\, \rho(x_{n+1}, x_n) \preceq (\lambda_1 + \lambda_3)\, \rho(x_n, x_{n-1}). \tag{4.14}$$

In a similar manner, we can show that

$$(1 - \lambda_3)\, \rho(x_n, x_{n+1}) \preceq (\lambda_1 + \lambda_2)\, \rho(x_{n-1}, x_n). \tag{4.15}$$

In either case, if we define $\lambda = \frac{\lambda_1 + \lambda_3}{1 - \lambda_2}$ or $\lambda = \frac{\lambda_1 + \lambda_2}{1 - \lambda_3}$ then it can be show that $0 \leq \lambda < 1$ and $\rho(x_n, x_{n+1}) \preceq \lambda \rho(x_{n-1}, x_n)$. By repeating the process we obtain that

$$\rho(x_n, x_{n+1}) \preceq \lambda \rho(x_{n-1}, x_n) \preceq \cdots \preceq \lambda^n \rho(x_0, x_1). \tag{4.16}$$

For $m, n \in \mathbb{N}$, and inequality (4.16), we get

$$\begin{aligned} &\rho(x_n, x_{n+m}) \\ &\preceq \alpha \rho(x_n, x_{n+1}) + \beta \rho(x_{n+1}, x_{n+m}) \\ &\preceq \alpha \rho(x_n, x_{n+1}) + \beta [\alpha \rho(x_{n+1}, x_{n+2}) + \beta \rho(x_{n+2}, x_{n+m})] \\ &\preceq \alpha \rho(x_n, x_{n+1}) + \alpha\beta \rho(x_{n+1}, x_{n+2}) + \cdots + \alpha \beta^{m-2} \rho(x_{n+m-2}, x_{n+m-1}) \\ &\quad + \alpha \beta^{m-1} \rho(x_{n+m-1}, x_{n+m}) \\ &\preceq \alpha \lambda^n \rho(x_0, x_1) + \alpha \beta \lambda^{n+1} \rho(x_0, x_1) + \cdots + \alpha \beta^{m-2} \lambda^{n+m-2} \rho(x_0, x_1) \\ &\quad + \alpha \beta^{m-1} \lambda^{n+m-1} \rho(x_0, x_1) \\ &\preceq \alpha \lambda^n \rho(x_0, x_1) \left(1 + \beta\lambda + \cdots + \beta^{m-2} \lambda^{m-2} + \beta^{m-1} \lambda^{m-1}\right) \\ &= \alpha \lambda^n \left(\frac{1 - \beta^m \lambda^m}{1 - \beta\lambda}\right) \rho(x_1, x_0) \\ &\prec \alpha \lambda^n \left(\frac{1}{1 - \beta\lambda}\right) \rho(x_1, x_0). \end{aligned} \tag{4.17}$$

Let $\theta \prec \epsilon$ then we can find $N \in \mathbb{N}$, such that $\rho(x_n, x_{n+m}) \prec \alpha \frac{\lambda^n}{1 - \beta\lambda} \rho(x_1, x_0) \prec \epsilon$ for $n \geq N$ and any $m \in \mathbb{N}$. We conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, ρ) is complete there exists $x^* \in X$ and $N \in \mathbb{N}$ such that $\rho(x_n, x^*) \prec \epsilon$ for

$n \geq N$,

$$\begin{aligned}\rho(x^*, Tx^*) &\preceq \alpha\rho(x^*, x_n) + \beta\rho(x_n, Tx^*)\\ &= \alpha\rho(x^*, x_n) + \beta\rho(Tx_{n-1}, Tx^*)\\ &\preceq \alpha\rho(x^*, x_n) + \beta\left[\lambda_1\rho(x_{n-1}, x^*) + \lambda_2\rho(x_{n-1}, Tx_{n-1}) + \lambda_3\rho(x^*, Tx^*)\right].\end{aligned}$$

It follows that

$$(1 - \beta\lambda_3)\,\rho(x^*, Tx^*) \preceq \left(\alpha + \lambda_2\beta^2\right)\rho(x^*, x_n) + (\beta\lambda_1 + \beta\lambda_2\alpha)\,\rho(x_{n-1}, x^*). \tag{4.18}$$

In a similar manner, we get

$$\begin{aligned}\rho(Tx^*, x^*) &\preceq \alpha\rho(Tx^*, x_n) + \beta\rho(x_n, x^*)\\ &= \alpha\rho(Tx^*, Tx_{n-1}) + \beta\rho(x_n, x^*)\\ &\preceq \alpha\left[\lambda_1\rho(x^*, x_{n-1}) + \lambda_2\rho(x^*, Tx^*) + \lambda_3\rho(x_{n-1}, Tx_{n-1})\right] + \beta\rho(x_n, x^*).\end{aligned}$$

It follows that

$$(1 - \alpha\lambda_2)\,\rho(Tx^*, x^*) \preceq (\beta + \alpha\beta\lambda_3)\,\rho(x^*, x_n) + \left(\alpha\lambda_1 + \alpha^2\lambda_3\right)\rho(x_{n-1}, x^*). \tag{4.19}$$

Let $\theta \prec \epsilon$ then there exists $N_1 \in \mathbb{N}$, such that $\rho(x_n, x^*) \prec \frac{\epsilon(1-\beta\lambda_3)}{2(\alpha+\lambda_2\beta^2)}$ and $\rho(x_n, x^*) \prec \frac{\epsilon(1-\alpha\lambda_2)}{2(\beta+\alpha\beta\lambda_3)}$ for $n \geq N_1$.

Similarly, there exists $N_2 \in \mathbb{N}$, such that $\rho(x_{n-1}, x^*) \prec \frac{\epsilon(1-\beta\lambda_3)}{2(\beta\lambda_1+\beta\lambda_2\alpha)}$ and $\rho(x_{n-1}, x^*) \prec \frac{\epsilon(1-\alpha\lambda_2)}{2(\alpha\lambda_1+\alpha^2\lambda_3)}$ for all $n \geq N_2$.

It follows that $\rho(x^*, Tx^*) \prec \epsilon$ for $n \geq \max\{N_1, N_2\}$. Thus T has a fixed point.

To show that the fixed point is unique. Suppose that there exists another fixed point x^{**} then

$$\begin{aligned}\rho(x^*, x^{**}) = \rho(Tx^*, Tx^{**}) &\preceq \lambda_1\rho(x^*, x^{**}) + \lambda_2\rho(x^*, Tx^*) + \lambda_3\rho(x^{**}, Tx^{**})\\ &\preceq \lambda_1\rho(x^*, x^{**}).\end{aligned}$$

Since $\lambda_1 < 1$, we conclude that $x^* = x^{**}$. □

Theorem 4.4. *Let (X, ρ) be a complete α, β-cone b-metric space. If the mapping $T : X \to X$ satisfies the following condition*

$$\rho(Tx, Ty) \preceq a\left[\frac{\rho(x,Tx)\rho(y,Ty)}{\rho(x,y)}\right] + b\rho(x, y), \tag{4.20}$$

for all $x, y \in X$, $a, b \in [0, 1]$ with $a + b < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary then define $x_{n+1} = Tx_n$. Then

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho(Tx_{n-1}, Tx_n) \\ &\preceq a\left[\frac{\rho(x_{n-1}, Tx_{n-1})\rho(x_n, Tx_n)}{\rho(x_{n-1}, x_n)}\right] + b\rho(x_{n-1}, x_n) \\ &= a\rho(x_n, Tx_n) + b\rho(x_{n-1}, x_n). \end{aligned} \tag{4.21}$$

It follows that

$$\begin{aligned} (1-a)\rho(x_n, x_{n+1}) &\preceq b\rho(x_{n-1}, x_n) \\ \rho(x_n, x_{n+1}) &\preceq \frac{b}{(1-a)}\rho(x_{n-1}, x_n). \end{aligned} \tag{4.22}$$

If we define $\lambda = \frac{b}{1-a}$ then since $a + b < 1$ it follows that $0 < \lambda < 1$ thus we get

$$\rho(x_n, x_{n+1}) \preceq \lambda\rho(x_{n-1}, x_n). \tag{4.23}$$

The proof follows in a similar manner as theorem 4.1. □

5. Conclusion

In this paper, we have presented a generalization of a cone b-metric space which is suitable for applications in fixed point theory with contraction type mappings.

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