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FOURTH DIMENSION SUBGROUP FOR SOME 2-GROUPS

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ABSTRACT. The objective of this paper to discuss conditions on a finite 2-group G of class 3 for which fourth dimension subgroup is trivial.

1. INTRODUCTION

Let G be a finite group and $\mathbb{Z}G$ be its integral group ring. Let $\Delta(G)$ be the augmentation ideal of group ring $\mathbb{Z}G$. Dimension subgroup conjecture states that $D_n(G) = \gamma_n(G)$ for all $n \geq 1$ and for all groups G, where $D_n(G)$ is the dimension subgroup of G defined by $D_n(G) = G \cap \{1 + \Delta^n(G)\}$ and $\gamma_n(G)$ is the nth term of the lower central series of G. It has been proved that dimension subgroup conjecture holds in general for $n \leq 3$, (see [1], [2]). G. Higman [2] reduced the problem to p-groups by proving that if dimension subgroup conjecture is false, then there exist a p-group for which it is false. It has been proved in [2, 5] that the exponent of $D_4(G)/\gamma_4(G)$ is 2 and for a p-group G, p odd prime, $D_4(G) = \gamma_4(G)$. In [3], Rips gave a counterexample of a 2-group G for which $D_4(G) \neq \gamma_4(G)$. In [6], Tahara gave some conditions on a finite 2-group G for which $D_4(G) = \gamma_4(G)$.

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In this paper, we will find some more conditions on a finite 2-group G for which $D_4(G) = \gamma_4(G)$.

2. Preliminaries

Let G be a finite group of class 3, i.e., $\gamma_4(G) = \{e\}$. Since $G/\gamma_2(G)$ and $\gamma_2(G)/\gamma_3(G)$ are abelian. Write

$$G/\gamma_2(G) = C_{11} \oplus C_{12} \oplus \cdots \oplus C_{1s},$$

where C_{1i} is a cyclic group of order $d(i)$ generated by $\bar{x}_{1i}$, $1 \leq i \leq s$, and $d(1)|d(2)|\ldots|d(s)$ and

$$\gamma_2(G)/\gamma_3(G) = D_{21} \oplus D_{22} \oplus \cdots \oplus D_{2t},$$

where D_{2k} is a cyclic group of order $e(k)$ generated by $\bar{x}_{2k}$, $1 \leq k \leq t$, and $e(1)|e(2)|\ldots|e(t)$. Since $x_{1i}^{d(i)} \in \gamma_2(G)$ and $x_{2k}^{e(k)} \in \gamma_3(G)$, write

$$x_{1i}^{d(i)} = x_{21}^{b_{i1}} x_{22}^{b_{i2}} \ldots x_{2t}^{b_{it}} x_{3i}, \quad x_{3i} \in \gamma_3(G), \ 1 \leq i \leq s. \tag{2.1}$$

With the above notations, we now recall the structure of fourth dimension subgroup given by Tahara.

Theorem 2.1. ([5, Theorem 8]). *$D_4(G)$ is equal to the subgroup generated by the elements*

$$\prod_{1\leq i<j\leq s} [{x_{1i}}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}},$$

where u_{ij} are integers satisfying the following conditions:

$$u_{ij}\binom{d(j)}{2} \equiv 0 (\operatorname{mod} d(i)), \ 1 \leq i < j \leq s; \tag{2.2}$$

$$\sum_{1\leq h<i} u_{hi}\frac{d(i)}{d(h)} b_{hk} - \sum_{i<j\leq s} u_{ij} b_{jk} \equiv 0 (\operatorname{mod} (d(i), e(k))), \ 1 \leq i \leq s, \ 1 \leq k \leq t. \tag{2.3}$$

3. Results

It is well known that, if $G/\gamma_2(G)$ is direct sum of at most 3 cyclic groups, then $D_4(G) = \gamma_4(G)$. If $G/\gamma_2(G)$ is direct sum of n cyclic groups then following theorems give conditions on a finite group G, under which $D_4(G) = \gamma_4(G)$.

Theorem 3.1. [2, 4] *Let G be a finite 2-group and $G/\gamma_2(G)$ is direct sum of n cyclic groups each of order 2, then $D_4(G) = \gamma_4(G)$.*

Theorem 3.2. [2,4] *Let G be a finite 2-group and $G/\gamma_2(G) \cong C_{11} \oplus C_{12} \oplus \cdots \oplus C_{1n}$, where C_{1i}, $1 \leq i \leq n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-2) = d(n-1) = 2$, $d(n) \geq 4$. Then $D_4(G) = \gamma_4(G)$.*

Theorem 3.2 has been proved in [2, 4] using cohomology groups and polynomial maps. We will give the alternative proof of Theorem 3.2 using structure of $D_4(G)$.

Proof. It is enough to prove the result for a group G of class 3. It follows from Theorem 2.1 that any element g of $D_4(G)$ is of the form

$$\begin{aligned} g &= \prod_{1 \leq i < j \leq n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}}, \end{aligned} \tag{3.1}$$

where u_{ij} are integers satisfying conditions given in Theorem 2.1.

Since $d(i) = d(j) = 2$, $1 \leq i \leq n-1$, thus (2.2) gives that $u_{ij} \equiv \pmod{d(i)}$, i.e., $u_{ij} = d(j)m$ for some integer m. Consider

$$\begin{aligned} \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} [d(i) = d(j),\ 1 \leq i < j \leq n-1] \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}]^{d(j)m} \\ &= \prod_{1 \leq i < j \leq n-1} [x_{1i}^{d(i)}, x_{1j}^{d(j)}]^{m} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-\binom{d(j)}{2}m} \\ &= \{e\}, \end{aligned} \tag{3.2}$$

because $[x_{1i}^{d(i)}, x_{1j}^{d(j)}] \in [\gamma_2(G), \gamma_2(G)] \subseteq \gamma_4(G) = \{e\}$ and $[x_{1i}^{d(i)}, x_{1j}, x_{1j}] \in \gamma_4(G) = \{e\}$.

Finally consider,

$$\begin{aligned}
\prod_{i<n}[x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} &= \prod_{i<n}\prod_{1\le k\le t}[x_{2k}^{b_{ik}}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \qquad \text{[using (2.1)]}\\
&= \prod_{i<n}\prod_{1\le k\le t}[x_{2k}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}b_{ik}}\prod_{i<n}\prod_{1\le k\le t}[x_{2k}, x_{1n}, x_{2k}]^{u_{in}\frac{d(n)}{d(i)}\binom{b_{ik}}{2}}\\
&= \prod_{1\le k\le t}[x_{2k}, x_{1n}]^{\sum_{i<n}u_{in}\frac{d(n)}{d(i)}b_{ik}}.
\end{aligned}$$

For $i = n$ condition (2.3) becomes, $\sum_{i<n} u_{in}\frac{d(n)}{d(i)}b_{ik} = d(n)a + e(k)b$, for some integers a and b. Thus,

$$\begin{aligned}
\prod_{i<n}[x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} &= \prod_{1\le k\le t}[x_{2k}, x_{1n}]^{d(n)a+e(k)b}\\
&= \prod_{1\le k\le t}[x_{2k}, x_{1n}]^{d(n)a}\prod_{1\le k\le t}[x_{2k}, x_{1n}]^{e(k)b}\\
&= \prod_{1\le k\le t}[x_{2k}, x_{1n}^{d(n)}]^{a}\prod_{1\le k\le t}[x_{2k}^{e(k)}, x_{1n}]^{b}\\
&= \{e\}. \qquad (3.3)
\end{aligned}$$

From (3.1), (3.2) and (3.3). we get that $g = \{e\}$. Hence $D_4(G) = \gamma_4(G)$. □

In continuation of above results, we will prove following results.

Theorem 3.3. *Let G be a finite 2-group and $G/\gamma_2(G) \cong C_{11} \oplus C_{12} \oplus \cdots \oplus C_{1n}$, where $C_{1i} =< \bar{x}_{1i} >$, $1 \le i \le n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-2) = 2$. If $[x_{1n-1}, x_{1n}] = \{e\}$, then $D_4(G) = \gamma_4(G)$.*

Proof. It is enough to prove the result for a group G of class 3. It follows from Theorem 2.1 that any element g of $D_4(G)$ is of the form

$$\begin{aligned}
g &= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \\
&= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \cdot \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\
&= \prod_{1\le i<j\le n-2} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \cdot \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}}
\end{aligned}$$

(3.4)
$$= A.B.C. \text{ (say)},$$

where u_{ij} are integers satisfying conditions given in Theorem 2.1.

It can be easily seen that

$$A = \{e\}. \tag{3.5}$$

Now consider

$$\begin{aligned}
B &= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} = \prod_{i<n-1}\prod_{1\le k\le t} [x_{2k}^{b_{ik}}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \\
&= \prod_{i<n-1}\prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)} b_{ik}} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{\sum_{i<n-1} u_{in-1}\frac{d(n-1)}{d(i)} b_{ik}}.
\end{aligned}$$

For $i = n-1$ condition (2.3) becomes, $\sum_{i<n-1} u_{in-1}\frac{d(n-1)}{d(i)} b_{ik} - u_{n-1n}b_{nk} = d(n-1)a + e(k)b$, for some integers a and b. Thus

$$\begin{aligned}
B &= \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{u_{n-1n}b_{nk}+d(n-1)a+e(k)b} \\
&= [\prod_{1\le k\le t} x_{2k}^{b_{nk}}, x_{1n-1}]^{u_{n-1n}} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{d(n-1)a} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{e(k)b} \\
&= [x_{1n}^{d(n)}, x_{1n-1}]^{u_{n-1n}} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}^{d(n-1)}]^{a} [x_{2k}, x_{1n-1}, x_{1n-1}]^{-a\binom{d(n-1)}{2}} \\
&\quad \prod_{1\le k\le t} [x_{2k}^{e(k)}, x_{1n-1}]^{b} \qquad \text{[using (2.1)]}
\end{aligned}$$

$$= [x_{1n}^{d(n)}, x_{1n-1}]^{u_{n-1n}} = \{e\} \qquad \text{[using given assumption]} \tag{3.6}$$

Finally consider,

$$
\begin{aligned}
C &= \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} = \prod_{i<n}\prod_{1\le k\le t} [x_{2k}^{b_{ik}}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\
&= \prod_{i<n}\prod_{1\le k\le t} [x_{2k}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}b_{ik}} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}b_{ik}}
\end{aligned}
$$

For $i = n$ condition (2.3) becomes, $\sum_{i<n} u_{in}\frac{d(n)}{d(i)}b_{ik} = d(n)a + e(k)b$, for some integers a and b. Thus,

$$
\begin{aligned}
C &= \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{d(n)a+e(k)b} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{d(n)a} \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{e(k)b} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n}^{d(n)}]^{a} \prod_{1\le k\le t} [x_{2k}^{e(k)}, x_{1n}]^{b} \\
&= \{e\}. \qquad (3.7)
\end{aligned}
$$

From (3.5), (3.6) and (3.7), we get that, $D_4(G) = \gamma_4(G) = \{e\}$. □

Theorem 3.4. *Let G be a finite 2-group and $G/\gamma_2(G) \cong C_{11} \oplus C_{12} \oplus \cdots \oplus C_{1n}$, where $C_{1i} =< \bar{x}_{1i} >$, $1 \le i \le n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-3) = 2$. If $[x_{1i}, x_{1j}] = \{e\}$, $n-2 \le i < j \le n$, then $D_4(G) = \gamma_4(G)$.*

Proof. Let us assume that G is a finite group of class 3. Thus any element g of $D_4(G)$ is of the form,

$$
\begin{aligned}
g &= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{u_{in-2}\frac{d(n-2)}{d(i)}} \\
&\quad \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{u_{in-2}\frac{d(n-2)}{d(i)}} \\
&\quad \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\
&= A.B.C.D \text{ (say)},
\end{aligned}
$$

where u_{ij} are integers satisfying conditions given in Theorem 2.1.

It can be seen easily that that, $A = \{e\}$. For $i = n-2$ condition (2.3) becomes, $\sum_{i<n-2} u_{in-2}\frac{d(n-2)}{d(i)} b_{ik} - u_{n-2\ n-1} b_{n-1k} - u_{n-2n} b_{nk} = d(n-2)a + e(k)b$, for some integers a and b. Thus

$$
\begin{aligned}
B &= \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{u_{in-2}\frac{d(n-2)}{d(i)}} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n-2}]^{\sum_{i<n-2} u_{in-2}\frac{d(n-2)}{d(i)} b_{ik}} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n-2}]^{u_{n-2\ n-1} b_{n-1k} + u_{n-2n} b_{nk} + d(n-2)a + e(k)b} \\
&= \prod_{1\le k\le t} [x_{1n-1}^{d(n-1)}, x_{1n-2}]^{u_{n-2\ n-1}} \prod_{1\le k\le t} [x_{1n}^{d(n)}, x_{1n-2}]^{u_{n-2n}} \\
&\quad \prod_{1\le k\le t} [x_{2k}, x_{1n-2}^{d(n-2)}]^{a} \prod_{1\le k\le t} [x_{2k}^{e(k)}, x_{1n-2}^{d(n-2)}]^{b}.
\end{aligned}
$$

Thus by using given condition, i.e., $[x_{1i}, x_{1j}] = \{e\}$, $n-2 \le i < j \le n$, we get that

$$B = \{e\}.$$

For $i = n-1$ condition (2.3) becomes, $\sum_{i<n-1} u_{in-1}\frac{d(n-1)}{d(i)} b_{ik} - u_{n-1n} b_{nk} = d(n-1)a + e(k)b$, for some integers a and b. Thus

$$
\begin{aligned}
C &= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{\sum_{i<n-1} u_{in-1}\frac{d(n-1)}{d(i)} b_{ik}}
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{u_{n-1n}b_{nk}+d(n-1)a+e(k)b} \\
&= [\prod_{1\le k\le t} x_{2k}^{b_{nk}}, x_{1n-1}]^{u_{n-1n}} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{d(n-1)a} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{e(k)b} \\
&= [x_{1n}^{d(n)}, x_{1n-1}]^{u_{n-1n}} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}^{d(n-1)}]^{a} \prod_{1\le k\le t} [x_{2k}^{e(k)}, x_{1n-1}]^{b} \\
&= [x_{1n}^{d(n)}, x_{1n-1}]^{u_{n-1n}} = \{e\} \qquad \text{as } \; [x_{1n}, x_{1n-1}] = \{e\}.
\end{aligned}
$$

Similarly, it can be shown that, $D = \{e\}$. Hence $D_4(G) = \{e\}$. $\square$

In a similar way, the following theorem which is generalization of the Theorem 3.3 and 3.4 can be proved.

Theorem 3.5. *Let G be a finite 2-group and $G/\gamma_2(G) \cong C_{11} \oplus C_{12} \oplus \cdots \oplus C_{1n}$, where $C_{1i} =< \bar{x}_{1i} >$, $1 \le i \le n$, is a cyclic group of order $d(i)$, with $d(1) = d(2) = \cdots = d(n-k) = 2$, $1 \le k \le n$. If $[x_{1i}, x_{1j}] = \{e\}$, $n-k+1 \le i < j \le n$, then $D_4(G) = \gamma_4(G)$.*

Note: It can be seen easily that Theorem 3.1 and 3.2 are particular cases of Theorem 3.5.

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