ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **10** (2021), no.4, 2141–2147
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.28

SINGULAR FOLIATION GENERATED BY AN ORBIT OF FAMILY OF VECTOR FIELDS

Xurshid Sharipov[1], Bekmurod Boymatov, and Nematillo Abriyev

ABSTRACT. Geometry of orbit is a subject of many investigations because it has important role in many branches of mathematics such as dynamical systems, control theory. In this paper it is studied geometry of orbits of conformal vector fields. It is shown that orbits of conformal vector fields are integral submanifolds of completely integrable distributions. Also for Euclidean space it is proven that if all orbits have the same dimension they are closed subsets.

1. INTRODUCTION

In this paper, we study the geometry of the singular foliation generated by the orbits of smooth vector fields. If the dimensions of the fibers of a foliation with singularities are the same, then it is a regular foliation in the sense of the definition given in [5]. G. Sussman and P. Stefan showed that the decomposition of a manifold into orbits is a singular foliation [2, 7, 8]. Numerous studies have been devoted to the study of the geometry of a singular foliation [1, 2, 4–8].

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 54H15, 57R50, 53C12 57R30.

Key words and phrases. Killing vector field, Lie derivative, singular Riemannian foliation, orbit of vector fields.

Submitted: 31.03.2021; *Accepted:* 15.04.2021; *Published:* 20.04.2021.

2. Preliminaries

Let $M-$ be a smooth Riemannian manifold of dimension n with metric tensor g, D -family of smooth vector fields defined on the manifold M. The D family can contain a finite or an infinite number of smooth vector fields.

Definition 2.1. *The orbit $L(x)$ of the family D of vector fields passing through the point x from M is defined as the set of points y from M for which there are real numbers $t_1, t_2, \ldots, t_k$ and vector fields $X_1, X_2, \ldots, X_k$ from D (where k is an arbitrary natural number) such that*

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(\cdots(X_1^{t_1}(x))\cdots)).$$

It is clear that an orbit is a smooth curve (one-dimensional manifold) if D consists of one vector field.

Definition 2.2. *Let $G-$ be a transformation group acting on the manifold M. The subset $N \subset M$ is called $G-$ invariant, and the group G is called the symmetry group of the subset N if, for any $x \in N$ and $g \in G$, $g \cdot x \in N$.*

Definition 2.3. *Let $G-$ be a transformation group acting on the manifold M. A function $F : M \to R$ is called an invariant function of a transformation group if for all $x \in M$ and all $g \in G$, $F(g \cdot x) = F(x)$.*

The following theorem is known [3, 4].

Theorem 2.1. *Let the group $G-$ act on the manifold M and $F : M \to R-$ a smooth function. Then F is an invariant function if and only if each level set $\{x \in M : F(x) = c, c \in R\}$ will be a $G-$ invariant subset of the manifold M.*

Let $G-$ k -dimensional Lie group of transformations. Then it has k infinitesimal generators (vector fields).

The following theorem is known [3, 4].

Theorem 2.2. *The function $F(x)$ is an invariant of the group G if and only if it satisfies the equation $X(F) = 0$ for each infinitesimal generator X of the group G.*

3. MAIN RESULTS

Now consider the family of vector fields $D = \{X, Y\}$, where, $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$. Find the flow of the vector field $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

$$\left\{\begin{array}{c} \dot{x} = y \\ \dot{y} = x \\ \dot{z} = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} \dot{x} = y \\ \ddot{y} = \dot{x} \\ z = C_3 \end{array}\right. \Rightarrow \ddot{y} = y.$$

Let's make a replacement $y = e^{kt}, \dot{y} = ke^{kt}, \ddot{y} = k^2e^{kt}$, $k^2e^{kt} = e^{kt} \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$.

From here we find the solution, $y(t) = C_1e^t + C_2{}^{-t}$, $x(t) = C_1e^t - C_2{}^{-t}$, $z(t) = C_3$. From the initial conditions $y(0) = y$, $x(0) = x$, $z(0) = z$, find

$$\left\{\begin{array}{c} y(0) = C_1 + C_2 = y \\ x(0) = C_1 - C_2 = x \\ z(0) = C_3 = z \end{array}\right. \Rightarrow \left\{\begin{array}{c} C_1 = \frac{x+y}{2} \\ C_2 = \frac{y-x}{2} \\ C_3 = z \end{array}\right.$$

Means,

$$\left\{\begin{array}{c} x(t) = \frac{x+y}{2}e^t - \frac{y-x}{2}{}^{-t} = x\frac{e^t + {}^{-t}}{2} + y\frac{e^t - {}^{-t}}{2} \\ y(t) = \frac{x+y}{2}e^t + \frac{y-x}{2}{}^{-t} = x\frac{e^t - {}^{-t}}{2} + y\frac{e^t + {}^{-t}}{2} \\ z(t) = z \end{array}\right. \Rightarrow \left\{\begin{array}{c} x(t) = xcht + ysht \\ y(t) = xsht + ycht \\ z(t) = z \end{array}\right.$$

The flow of vector field $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, generates the following one-parameter transformation group

$$X^t(x, y, z) = (xcht + ysht, xsht + ycht, z).$$

Now we find the flow of the vector field $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$.

$$\left\{\begin{array}{c} \dot{x} = x \\ \dot{y} = y \\ \dot{z} = 2z \end{array}\right. \Rightarrow \left\{\begin{array}{c} \frac{dx}{ds} = x \\ \frac{dy}{ds} = y \\ \frac{dz}{ds} = 2z \end{array}\right. \Rightarrow \left\{\begin{array}{c} x = C_1e^s \\ y = C_2e^s \\ z = C_3e^{2s} \end{array}\right.$$

From here we find the solution, $x(s) = C_1e^s$, $y(s) = C_2e^s, z(s) = C_2e^{2s}$. From the initial conditions $y(0) = y$, $x(0) = x$, $z(0) = z$, find

$$\left\{\begin{array}{c} x(0) = C_1 = x \\ y(0) = C_2 = y \\ z(0) = C_3 = z \end{array}\right.$$

Means,

$$\begin{cases} x(s) = xe^s \\ y(s) = ye^s \\ z(t) = ze^{2s} \end{cases}.$$

The flow of vector field $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$, generates the following one-parameter transformation group

$$Y^s(x, y, z) = \left(xe^s,\ ye^s,\ ze^{2s}\right).$$

Now we find the invariant functions of the transformation group generated by the flows of these vector fields. By theorem-2.2, the function $I(x, y, z)$ is invariant under the group of transformations generated by the flow of the vector field $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, if and only if the equality $X(I) = y\frac{\partial I}{\partial x} + x\frac{\partial I}{\partial y} = 0$. From this equality we obtain

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{0} = \frac{dI}{0}.$$

From here we get

$$xdx = ydy, \quad 2xdx = 2ydy,$$

$$x^2 - y^2 = C,\ C = const.$$

Hence, the independent invariants of the transformation group generated by the flow of the vector field $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, are functions $I_1(x, y, z) = x^2 - y^2, I_2(x, y, z) = z$.

Now we write the vector field $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$ through the functions $I_1(x, y, z) = x^2 - y^2, I_2(x, y, z) = z$. To do this, find $\frac{dI_1(x,y,z)}{dt}, \frac{dI_2(x,y,z)}{dt}$, taking into account

$$\frac{dx}{dt} = x, \frac{dy}{dt} = y, \frac{dz}{dt} = 2z,$$

$$\frac{dI_1(x, y, z)}{dt} = \frac{d(x^2 - y^2)}{dt} = \frac{dx^2}{dt} - \frac{dy^2}{dt} = 2x\frac{dx}{dt} - 2y\frac{dy}{dt} = 2x^2 - 2y^2 = 2I_1,$$

$$\frac{dI_2(x, y, z)}{dt} = \frac{dz}{dt} = 2z = 2I_2.$$

So, the vector field Y takes the form $Y = 2I_1\frac{\partial}{\partial I_1} + 2I_2\frac{\partial}{\partial I_2}$. Find the invariant function of the transformation group by the generated flow of this vector field. By theorem-2.2, the function $I(x, y, z)$ is invariant under the transformation group generated by the flow of the vector field $Y = 2I_1\frac{\partial}{\partial I_1} + 2I_2\frac{\partial}{\partial I_2}$, if and only if the equality $Y(I) = 2I_1\frac{\partial I}{\partial I_1} + 2I_2\frac{\partial I}{\partial I_2} = 0$. From this equality we obtain

$$\frac{dI_1}{2I_1} = \frac{dI_2}{2I_2} = \frac{dI}{0}.$$

From here we get

$$\frac{dI_1}{2I_1} = \frac{dI_2}{2I_2}, \quad \ln I_1 = \ln I_2 + \ln C,$$

$$\frac{I_1}{I_2} = C, \; C = const.$$

This means that the invariant functions of the transformation group generated by streams of vector fields $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$, is a function

$$I(x, y, z) = \frac{I_1}{I_2} = \frac{x^2 - y^2}{z}.$$

Really,

$$\begin{aligned} I\left(X^t(x,y,z)\right) &= I\left(xcht + ysht,\; xsht + ycht,\; z\right) \\ &= \frac{(xcht + ysht)^2 - (xsht + ycht)^2}{z} \\ &= \frac{x^2ch^2t + 2xychtsht + y^2sh^2t - x^2sh^2t - 2xychtsht - y^2ch^2t}{z} \\ &= \frac{x^2 - y^2}{z} = I(x,y,z), \end{aligned}$$

$$I\left(Y^s(x,y,z)\right) = I\left(xe^s,\; ye^s,\; ze^{2s}\right) = \frac{(xe^s)^2 - (ye^s)^2}{ze^{2s}} = \frac{x^2 - y^2}{z} = I(x,y,z).$$

Theorem 3.1. *Layers of the foliation F generated by the family of vector fields $D = \{X, Y\}$, where, $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$, are the following sets:*

Zero dimension layer

$$L_0 = \{(0,0,0)\},$$

One-dimensional layers

$$L_1^+ = \{(x,y,z) : y = x = 0,\; z > 0\},$$

$$L_1^- = \{(x,y,z) : y = x = 0,\; z < 0\},$$

$$L_1^{++} = \{(x, y, z) : y = x,\ x > 0, y > 0, z = 0\},$$
$$L_1^{+-} = \{(x, y, z) : y = -x,\ x > 0, y < 0, z = 0\},$$
$$L_1^{-+} = \{(x, y, z) : y = -x,\ x < 0, y > 0, z = 0\},$$
$$L_1^{--} = \{(x, y, z) : y = x,\ x < 0, y < 0, z = 0\},$$

Two dimension layers

$$L_2^{+0} = \{(x, y, z) : |y| < x,\ x > 0,\ z = 0\},$$
$$L_2^{-0} = \{(x, y, z) : |y| < -x,\ x < 0,\ z = 0\},$$
$$L_2^{0+} = \{(x, y, z) : |x| < y,\ y > 0,\ z = 0\},$$
$$L_2^{0-} = \{(x, y, z) : |x| < -y,\ y < 0,\ z = 0\},$$
$$L_{c>0}^{++} = \left\{(x, y, z) : cz = x^2 - y^2,\ x > 0,\ z > 0,\ c > 0\right\},$$
$$L_{c>0}^{-+} = \left\{(x, y, z) : cz = x^2 - y^2,\ x < 0,\ z > 0,\ c > 0\right\},$$
$$L_{c>0}^{+-} = \left\{(x, y, z) : cz = x^2 - y^2,\ y > 0,\ z < 0,\ c > 0\right\},$$
$$L_{c>0}^{--} = \left\{(x, y, z) : cz = x^2 - y^2,\ y < 0,\ z < 0,\ c > 0\right\},$$
$$L_{c<0}^{++} = \left\{(x, y, z) : cz = x^2 - y^2,\ y > 0,\ z > 0,\ c > 0\right\},$$
$$L_{c<0}^{-+} = \left\{(x, y, z) : cz = x^2 - y^2,\ y < 0,\ z > 0,\ c < 0\right\},$$
$$L_{c<0}^{+-} = \left\{(x, y, z) : cz = x^2 - y^2,\ x > 0,\ z < 0,\ c > 0\right\},$$
$$L_{c<0}^{--} = \left\{(x, y, z) : cz = x^2 - y^2,\ x < 0,\ z < 0,\ c < 0\right\}.$$

Proof. The origin of coordinates is a fixed point of the families of vector fields $D = \{X, Y\}$. Therefore, the set $L_0 = \{(0, 0, 0)\}$ is a singular layer.

In the Oz axis, vector fields have the form $X = 0, Y = 2z\frac{\partial}{\partial z}$, therefore if from any point the set $L_1^+, \left(L_1^-,\right)$ using streams of vector fields X, Y, you can get to any other point of the same set, i.e. the set $L_1^+, \left(L_1^-,\right)$ is a layer of the foliation F.

In the coordinate plane Oxy vector fields have the form $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. In the sets $L_1^{++}, L_1^{+-}, L_1^{-+}, L_1^{-}$, vector fields will be collinear, therefore the sets $L_1^{++}, L_1^{+-}, L_1^{-+}, L_1^{-}$, are fibers of the foliation F. It can also be shown that the sets $L_2^{+0}, L_2^{-0}, L_2^{0+}, L_2^{0-}$ are fibers of the foliation F.

The sets $L_{c>0}^{++}, L_{c>0}^{-+}, L_{c>0}^{+-}, L_{c>0}^{-}, L_{c<0}^{++}, L_{c<0}^{-+}, L_{c<0}^{+-}, L_{c<0}^{-}$, are parts of the level surfaces of the invariant function $I(x, y, z) = \frac{x^2 - y^2}{z}$. Therefore, if consider one of these sets, then from any point of this set, using the streams of vector fields $X, Y,$, you can get to any other point of the same set, i.e. sets $L^{++}_{c>0}$, $L^{-+}_{c>0}$, $L^{+-}_{c>0}$, $L^{-}_{c>0}$, $L^{++}_{c<0}$, $L^{-+}_{c<0}$, $L^{+-}_{c<0}$, $L^{-}_{c<0}$, are fibers of the foliation F. The theorem is proved. □

ACKNOWLEDGMENT

The authors is very grateful for the helpful suggestions of the referee.

REFERENCES

[1] P. MOLINO: *Orbit-like foliations*, Geometric Study of Foliations (World Scientific, Singapore, (1994), 97–119.
[2] A.YA. NARMANOV, O. KASIMOV: *Geometry of singular Riemannian foliations*, Uzbek Mathematical Journal, **3** (2011), 129–135.
[3] G. ABDISHUKUROVA, A. NARMANOV, X. SHARIPOV: *Differential invariants of One Parametrical Group of Transformations*, Mathematics and Statistics, **8**(3) (2020), 347–352.
[4] P. OLVER: *Application of Lie Groups to Differential Equations*, Second edition, Springer, 1993.
[5] H. SUSSMAN: *Orbits of families of vector fields and integrability of distributions*, Transactions of the AMS. **180**(6) (1973), 171–188.
[6] H. SUSSMAN, N. LEVITT: *On controllability with two vector fields*, SIAM J. Control 13, **13**(6) (1973), 1271–1281.
[7] P. STEFAN: *Accessibility and foliations with singularities*, Bull. AMS, **80**(6) (1974), 1142–1145.
[8] I. TAMURA: *Topology of foliation: an introduction*, Translations of mathematical monographs. American Mathematical Soc., 2006.

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF UZBEKISTAN, TASHKENT, 100174, UZBEKISTAN.
Email address: sh_xurshid@yahoo.com

DEPARTMENT OF EXACT AND NATURAL SCIENCES, TASHKENT REGIONAL TERRITORIAL CENTER FOR RETRAINING AND ADVANCED TRAINING OF EMPLOYEES OF PUBLIC EDUCATION, TASHKENT, UZBEKISTAN.
Email address: bekshohruz2925@gmail.com

DEPARTMENT OF MATHEMATICS, SAMARKAND STATE UNIVERSITY, SAMARKAND, UZBEKISTAN.
Email address: nematilloabriyev@gmail.com