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SUPPLEMENTS IN A SEMIRING

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ABSTRACT. In this paper we define supplemented ideal of a semiring and study various characteristics of this ideal.

1. INTRODUCTION

The concept of supplemented module is introduced to study those modules which provides a supplement of each of its submodules. It is noted that supplement of a submodule of a given module, need not exist. Wisbauer [7] defined a module M to be supplemented if every submodule has a supplement in M. The detailed information about supplemented and related modules was given by Zoschinger. In dualizing the concept of Goldie Dimension Patrick Fleury [2] introduced the modules with finite spanning dimension. The notion of supplement plays an important role in study of dual Goldie dimension In recent times many concepts and types of supplemented modules are introduced. Here in this paper we try to generalize this concept of supplement modules in semirings and study various characteristics of it. H.S Vandiver [6] in 1934 first introduced the concept of Semiring. Semiring is a generalization of ring as well as distributive lattice. In recent times, many authors have generalized the concept of rings and modules to semiring.

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2. PRELIMINARIES

Throughout this paper S means a semiring with 0.

Definition 2.1. *Let S be a semiring and P and Q be two ideals of S with $P \subset Q$. Then P is said to be small in Q if for each ideal L $(\subseteq Q)$ of S, $P+L = Q \Rightarrow L = Q$. Equivalently, P is said to be small in Q if for each ideal L $(\subset Q)$ of S, $P + L \subset Q$ we denote it by $P \subset_s Q$.*

Example 1. *Let us consider a semiring $B(n, i) = \{0, 1, 2, 3, \ldots, n-1\}$ where $2 \leq n$ is an integer and $0 \leq i < n$. The operation $\oplus$ and $\odot$ on $B(n, i)$ are defined as follow.*

For $x, y \in B(n, i)$,

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y \leq n - 1 \\ l, & \text{where } l = (x + y) mod(n - i) \text{ and } i \leq l \leq (n - 1) \end{cases}$$

In a similar manner the operation $\odot$ is defined on $B(n, i)$. If we take $n = 6$ and $i = 2$ then $S = B(6, 2)$ is a semiring with the operation defined as above. The proper ideals of $S = B(6, 2)$ are $\{0, 4\}$ and $\{0, 2, 4\}$ and both the ideals are small in the semiring $B(6, 2)$.

Definition 2.2. *An ideal of a semiring S is small in S when S is considered an ideal of itself.*

Definition 2.3. *An ideal Q of a semiring S is said to be supplement of an ideal P of S if $P + Q = S$ but $P + Q' \neq S$ for any ideal $Q'(\subset Q)$ of S.*

Example 2. *Let $S = (S, +, ., 0)$ be a semiring defined as follows: $S = \{0, a, b, c, d, e, f, g\}$.*

+	0	a	b	c	d	e	f	g
0	0	a	b	c	d	e	f	g
a	a	b	c	0	e	f	g	d
b	b	c	0	a	f	g	d	e
c	c	0	a	b	g	d	e	f
d	d	e	f	g	d	e	f	g
e	e	f	g	d	e	f	g	d
f	f	g	d	e	f	g	d	e
g	g	d	e	f	g	d	e	f

Here, $A = \{0, a, b, c\}$ is supplement of $C = \{0, d\}$.

Definition 2.4. *If P is an ideal of semiring S, Q is another ideal of S, then S is said to be direct sum of P and Q denoted by $S = P \oplus Q$ if $S = P + Q$ and $P \cap Q = (0)$. In such case Q is called direct summand of S.*

Definition 2.5. *Let P be an ideal of semiring S. An ideal C of S is called complement of P if C is maximal among the ideals C_i of S such that $A \cap C_i = 0$.*

Definition 2.6. *The collection $L(S)$ of ideals of semiring S is said to be complemented if for each ideal Q of S, there is some ideal $Q^{'}$ of S such that $Q \cap Q^{'} = 0$ and $Q + Q^{'} = S$.*

Definition 2.7 (Modular law). *Let P, Q, R be ideal of semiring S such that $P \subseteq R$. Then $(P + Q) \cap R = P + (Q \cap R)$.*

Definition 2.8. *$J(S)$ is the sum of all small ideal of S.*

Definition 2.9. *If I is an ideal of semiring S then consedering I as semiring if if $J(I) = 0$ then I is called primitive ideal.*

Example 3. *Let $S = B(6, 2)$ be a semiring defined as in example 1 where $\{0, 4\}$ and $\{0, 2, 4\}$are its small ideal.In this example $J(S) = \{0, 2, 4\}$ and $J(0, 4) = 0$.*

Definition 2.10. *An ideal M of S is called supplemented if for any two ideals P and Q of S with $P \subseteq M$, $Q \subseteq M$, $P + Q = Q + P = M$, Q contains a supplement of P in M and P contains a supplement of Q in M.*

Example 4. *Let $S = \{0, a, b, c, d, e, f, g\}$ be a semiring defined as in Example 2, here S is a supplemented ideal.*

Note 2.11. S is said to be supplemented if it is supplemented when considered as an ideal of itself.

3. Main Results and Discussions

Lemma 3.1. *Let P, Q be ideal of S such that $P \subseteq Q$ and $Q \subset_s S$. Then $P \subset_s S$.*

Proof. Let C be an ideal of S such that $P + C = S$. Then,

$$\begin{aligned} & Q + C = S \\ & \Rightarrow C = S \qquad [\because Q \subset_s S] \end{aligned}$$

Hence, $P \subset_s S$. □

Lemma 3.2. *Let P and Q be ideals of semiring S and $P \subset_s Q$ then $P \subset_s S$.*

Proof. Let C be an ideal of S such that $P + C = S$. From (2.7) we have,

$$\begin{aligned} P + (C \cap Q) &= (P + C) \cap Q \\ &= Q \end{aligned}$$

Thus $C \cap Q$ is an ideal of S such that,

$$\begin{aligned} P + (C \cap Q) &= Q \\ \text{so, } (C \cap Q) &= Q && [\because P \subset_s Q] \\ \text{Thus, } P + C &= C \\ \Rightarrow C &= S \end{aligned}$$

Hence, $P \subset_s S$. □

Lemma 3.3. *If $P \subset_s S$, $Q \subset_s S$ then $P + Q \subset_s S$.*

Proof. Let C be an ideal of S such that,

$$\begin{aligned} (P + Q) + C &= S \\ P + (Q + C) &= S \\ Q + C &= S && [\because P \subset_s S] \\ C &= S && [\because Q \subset_s S] \\ \Rightarrow P + Q &\subset_s S. \end{aligned}$$

□

Lemma 3.4. *Let Q and Q^* be an ideal of semiring S such that $Q + Q^* = S$. Then Q^* is supplement of Q in S iff $Q \cap Q^* \subset_s Q^*$.*

Proof. Let, $Q \cap Q^* \not\subset_s Q^*$. Then for some ideal C of S with $C \subset Q^*$, $Q \cap Q^* + C = Q^*$. Given,

$$\begin{aligned} S &= Q + Q^* \\ &= Q + [Q \cap Q^* + C] \\ &= Q + C \end{aligned}$$

This implies Q^* is not supplement of Q in S.

Conversely, suppose

$$Q \cap Q^* \subset_s Q^*$$

Let $C(\subseteq Q^*)$ be an ideal of S such that $Q + C = S$.

$$\begin{aligned}(Q \cap Q^*) + C &= Q^* \cap (Q + C), by(2.7)\\ &= Q^*\\ \Rightarrow C &= Q^*\end{aligned}$$

Hence, Q^* is a supplement of Q in S. □

Lemma 3.5. *Let $L(S)$, the collection of ideals of S is complemented. Let Q be an ideal of S such that any ideal of Q is ideal of S. Then $L(Q)$ is complemented.*

Proof. Let C be an ideal of Q. Then C is an ideal of S. So, $C^{'}$ is complementary ideal of C in S, such that $C \cap C^{'} = 0$ and $C + C^{'} = S$.

Again, we consider $C^{''} = C^{'} \cap Q$, an ideal of Q. Then, $C \cap C^{''} = 0$ and $C + C^{''} = B$. Hence, $L(Q)$ is complemented. □

Lemma 3.6. *If P^* is a supplement (of P) in K and K is summand in S then P^* is a supplement in S.*

Proof. Given, $P + P^* = K$ and for some ideal $K^{'} of S\ K \oplus K^{'} = S$. Thus, $S = (P + P^*) \oplus K^{'} = (P \oplus K^{'}) + P^*$.

Let, Q and P^* be ideals of S with $Q \subseteq P^*$ such that

$$\begin{aligned}S &= (P \oplus K^{'}) + Q\\ &= (P + Q) \oplus K^{'}\end{aligned}$$

But $Q \subseteq P^*$ which implies $(P+Q) \subseteq K$. This gives, $P+Q = K$, a contradiction. Hence P^* is a supplement of S. □

Theorem 3.1. *If P^* is supplement of P in K^*, K^* is supplement of K in S, then P^* is supplement in S.*

Proof. Given,

$$\begin{aligned}&P^* + P = K^* \text{and} K^* + K = S\\ \Rightarrow &K + P^* + P = S\\ \Rightarrow &(K + P) + P^* = S\end{aligned}$$

Suppose Q and P^* are ideals of S with $Q \subseteq P^*$ such that,

$$(K+P)+Q=S$$
$$\Rightarrow K+(P+Q)=S$$

Since,

$$Q \subset P^*$$
$$\Rightarrow P+Q \subseteq K^*$$
$$\Rightarrow P+Q = K^*,$$
$$\text{as} K^* \text{is minimal w.r.t. } K+K^*=S.$$

which is a contradiction to minimality of P^* w.r.t $P+P^* = K^*$. Thus P^* is supplement in S. □

Theorem 3.2. *Let S be a supplemented and primitive ideal.Then any ideal of S is a direct summand of S.*

Proof. Since S is supplemented, for any ideal Q of S there is a supplement Q^* of Q in S. Thus,$Q+Q^*=S$. Hence, by using lemma (3.4), (3.2) and definition (2.9), we have,

$$Q \cap Q^* \subseteq_s Q^*$$
$$\Rightarrow Q \cap Q^* \subseteq_s S.$$

Hence, $Q \cap Q^* \subseteq J(S)=0$. Therefore, $Q \cap Q^*=0$. Thus Q is direct summand of S. □

Theorem 3.3. *Any summand of a supplemented ideal is supplemented.*

Proof. Let, S be supplemented ideal and A be summand of S. Let $S=A \oplus B$. To show A is supplemented. Let P and Q be ideal of A such that $P+Q=Q+P=A$. Now,

$$(P+Q) \oplus B = S$$
$$P+(Q \oplus B) = S.$$

Since S is supplemented, so there exist $X \oplus Y$, where $X \subset Q$, $Y \subset B$, such that $P+(X \oplus Y)=S$ and $X \oplus Y$ is minimal w.r.t this property. Now, $P \subseteq A$, $X \subseteq Q \subseteq A$. Therefore, $P+X \subseteq A$. Let, $P+X \neq A$,then $P+X \subset A$

$\Rightarrow (P+X)\oplus Y \subset A\oplus Y$

$\Rightarrow S \subset A\oplus Y$, which is not possible.

Thus, $P+X=A$ and so, $S=A\oplus Y$. Let, if possible Z is an ideal of X be such that $P+Z=A$. Then,

$$(P+Z)\oplus Y = S,$$
$$\Rightarrow P+(Z\oplus Y) = S, \text{where } Z\oplus Y \subseteq X\oplus Y,$$

a contradiction. Thus, X is minimal w.r.t $P+X=A$. Hence, A is supplemented. □

Conclusion

The notion of supplements in semiring will lead us to finite spanning dimension in semiring and hence will lead to the notion of dual Goldie dimension in semiring.

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