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FIXED POINT THEOREMS IN A WEIGHTED RECTANGULAR b-METRIC SPACE

Pravin Singh[1] and Virath Singh

ABSTRACT. In this paper, we provide a generalization of a rectangular b-metric space by relaxing the rectangular inequality to include unequal weights. We provide examples and restrictions for the Lipschitz constant enabling convergence of some sequences in the proof of some fixed point theorems.

1. INTRODUCTION

Branciari [3] introduced the concept of a rectangular metric space by replacing the triangle inequality in a metric space by an additional term. Bakhtin [2] introduced a b-metric space as a generalization of a metric space. George et al. [5] introduced a rectangular b-metric space as a generalization of a metric space, rectangular metric space and a b-metric space. This rectangular b-metric space has a coefficient $s > 1$ in the definition of the rectangular inequality. Here we consider a similar approach but allow for arbitrary constants or weights in the rectangular inequality which are all not necessarily greater than unity. We illustrate this with an example and prove an analogue of the Banach contraction principle and a Kannan's fixed point theorem in this weighted rectangular

[1]*corresponding author*

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b-metric space. Many authors have studied fixed point theorems in b-metric as well as rectangular metric spaces (see [4], [1]). Every metric space is a rectangular metric space, while every rectangular metric space is a weighted rectangular b-metric space and the latter is a rectangular b-metric space with coefficient s when all three weights are equal and greater than unity.

Definition 1.1. *Let X be a nonempty set and the mapping $\rho : X \times X \to [0, \infty)$ satisfy*

(i) $\rho(x, y) = 0$ *if and only if* $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$.

(iii) $\rho(x, y) \leq \omega_1 \rho(x, u) + \omega_2 \rho(u, v) + \omega_3 \rho(v, y)$ *for all* $x, y \in X$ *and all distinct points* $u, v \in X - \{x, y\}$, *for fixed postive real numbers* ω_1, ω_2 *and* ω_3.

Then ρ is called a weighted rectangular b-metric on X and (X, ρ) is called a weighted rectangular b-metric space.

If $\omega_1 = \omega_2 = \omega_3 = 1$, we have a rectangular b-metric space and if $\omega_1 = \omega_2 = \omega_3 = s > 1$, we get a rectangular b-metric space with coefficient s.

Definition 1.2. *Let (X, ρ) be a weighted rectangular b-metric space and $\{x_n\}$ be a sequence in X with $x \in X$. Then*

(i) *the sequence $\{x_n\}$ is convergent in (X, ρ) and converges to x, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(x_n, x) < \epsilon$ for all $n \geq N$ and is represented by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

(ii) *the sequence $\{x_n\}$ is Cauchy in (X, ρ) if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_{n+p}) < \epsilon$ for all $n > N$, $p > 0$ or equivalently* $\lim_{n\to\infty} \rho(x_n, x_{n+p}) = 0$ *for all* $p > 0$.

(iii) *(X, ρ) is said to be a complete weighted rectangular b-metric space if every Cauchy sequence in X is converges to some $x \in X$.*

2. Fixed Point Results

Theorem 2.1. *Let (X, ρ) be a complete weighted rectangular b-metric space with weights ω_1, ω_2 and ω_3 and $T : X \to X$ be a mapping such that*

$$\rho(Tx, Ty) \leq \lambda \rho(x, y), \tag{2.1}$$

for all $x, y \in X$ where $0 < \lambda < \min\{1, \frac{1}{\sqrt{\omega_3}}\}$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in x$ be arbitrary and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We first show that $\{x_n\}$ is a Cauchy sequence. If $x_{n+1} = x_n$, then x_n is a fixed point of T so we suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Let $\rho(x_n, x_{n+1}) = \rho_n$ then it follows from (2.1) that

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho(Tx_{n-1}, Tx_n) \leq \lambda\rho(x_{n-1}, x_n) \\ \rho_n &\leq \lambda\rho_{n-1}. \end{aligned} \tag{2.2}$$

Repeating the process we obtain

$$\rho_n \leq \lambda^n \rho_0. \tag{2.3}$$

We assume additionally that x_0 is not a periodic point of T, for if $x_0 = x_n$ then for $n \geq 2$

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho(x_n, Tx_n) = \rho(x_0, Tx_0) \\ \rho_n &= \rho_0. \end{aligned}$$

But $\rho_n < \rho_0$ from (2.3) is a contradiction. Hence we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Let $\rho_n^\star = \rho(x_n, x_{n+2})$, then using (2.1) we obtain

$$\begin{aligned} \rho(x_n, x_{n+2}) &= \rho(Tx_{n-1}, Tx_{n+1}) \leq \lambda\rho(x_{n-1}, x_{n+1}) \\ \rho_n^\star &\leq \lambda\rho_{n-1}^\star, \end{aligned} \tag{2.4}$$

and repeating the process we obtain

$$\rho_n^\star \leq \lambda^n \rho_0^\star. \tag{2.5}$$

Consider the sequence $\{x_n\}$ and $\rho(x_n, x_{n+k})$. If $k = 2m+1$, $m \geq 1$ then

$$\begin{aligned} \rho(x_n, x_{n+2m+1}) &\leq \omega_1\rho(x_n, x_{n+1}) + \omega_2\rho(x_{n+1}, x_{n+2}) + \omega_3\rho(x_{n+2}, x_{n+2m+1}) \\ &\leq \omega_1\rho_n + \omega_2\rho_{n+1} + \omega_3[\omega_1\rho(x_{n+2}, x_{n+3}) + \omega_2\rho(x_{n+3}, x_{n+4}) \\ &+ \omega_3\rho(x_{n+4}, x_{n+2m+1})] \\ &= \omega_1\rho_n + \omega_2\rho_{n+1} + \omega_3\omega_1\rho_{n+2} + \omega_3\omega_2\rho_{n+3} + \omega_3^2\rho(x_{n+4}, x_{n+2m+1}) \\ &\leq (\omega_1\rho_n + \omega_2\rho_{n+1}) + \omega_3(\omega_1\rho_{n+2} + \omega_2\rho_{n+3}) \\ &+ \omega_3^2(\omega_1\rho_{n+4} + \omega_2\rho_{n+5}) + \cdots + \omega_3^m\rho_{n+2m} \\ &= \sum_{q=0}^{m-1} \omega_3^q(\omega_1\rho_{n+2q} + \omega_2\rho_{n+2q+1}) + \omega_3^m\rho_{n+2m} \end{aligned}$$

$$
\begin{aligned}
&\leq && \sum_{q=0}^{m-1} \omega_3^q(\omega_1\lambda^{n+2q} + \omega_2\lambda^{n+2q+1})\rho_0 + \omega_3^m\lambda^{n+2m}\rho_0 \\
&= && \lambda^n\omega_1\rho_0 \sum_{q=0}^{m-1} \omega_3^q\lambda^{2q} + \lambda^{n+1}\omega_2\rho_0 \sum_{q=0}^{m-1} \omega_3^q\lambda^{2q} + \lambda^n\rho_0\omega_3^m\lambda^{2m} \\
&= && \lambda^n\omega_1\rho_0 \sum_{q=0}^{m-1} (\omega_3\lambda^2)^q + \lambda^{n+1}\omega_2\rho_0 \sum_{q=0}^{m-1} (\omega_3\lambda^2)^q + \lambda^n\rho_0(\omega_3\lambda^2)^m \\
&\leq && \lambda^n\omega_1\rho_0 \sum_{q=0}^{m-1} (\omega_3\lambda^2)^q + \lambda^{n+1}\omega_2\rho_0 \sum_{q=0}^{m-1} (\omega_3\lambda^2)^q \\
&+ && \lambda^n\rho_0 \ (\omega_3\lambda^2 < 1) \\
&< && \lambda^n\rho_0 \left[1 + \frac{\omega_1 + \omega_2\lambda}{1 - \omega_3\lambda^2}\right]. \qquad (2.6)
\end{aligned}
$$

If $k = 2m$, $m \geq 1$ then

$$
\begin{aligned}
\rho(x_n, x_{n+2m}) &\leq && \omega_1\rho(x_n, x_{n+1}) + \omega_2\rho(x_{n+1}, x_{n+2}) + \omega_3\rho(x_{n+2}, x_{n+2m}) \\
&\leq && \omega_1\rho_n + \omega_2\rho_{n+1} + \omega_3[\omega_1\rho(x_{n+2}, x_{n+3}) + \omega_2\rho(x_{n+3}, x_{n+4})] \\
&+ && \omega_3^2\rho(x_{n+4}, x_{n+2m}) \\
&= && \omega_1\rho_n + \omega_2\rho_{n+1} + \omega_3\omega_1\rho_{n+2} + \omega_3\omega_2\rho_{n+3} + \omega_3^2\rho(x_{n+4}, x_{n+2m}) \\
&\leq && (\omega_1\rho_n + \omega_2\rho_{n+1}) + \omega_3(\omega_1\rho_{n+2} + \omega_2\rho_{n+3}) + \omega_3^2(\omega_1\rho_{n+4} + \omega_2\rho_{n+5}) \\
&+ && \cdots + \omega_3^{m-2}(\omega_1\rho_{n+2m-4} + \omega_2\rho_{n+2m-3}) + \omega_3^{m-1}\rho(x_{n+2m-2}, x_{n+2m}) \\
&= && \sum_{q=0}^{m-2} \omega_3^q(\omega_1\rho_{n+2q} + \omega_2\rho_{n+2q+1}) + \omega_3^{m-1}\rho^\star_{n+2m-2} \\
&\leq && \sum_{q=0}^{m-2} \omega_3^q(\omega_1\lambda^{n+2q} + \omega_2\lambda^{n+2q+1})\rho_0 + \omega_3^{m-1}\lambda^{n+2m-2}\rho_0^\star \\
&= && \lambda^n\omega_1\rho_0 \sum_{q=0}^{m-2} \omega_3^q\lambda^{2q} + \lambda^{n+1}\omega_2\rho_0 \sum_{q=0}^{m-2} \omega_3^q\lambda^{2q} + \lambda^n\rho_0^\star\omega_3^{m-1}\lambda^{2m-2} \\
&= && \lambda^n\omega_1\rho_0 \sum_{q=0}^{m-2} (\omega_3\lambda^2)^q + \lambda^{n+1}\omega_2\rho_0 \sum_{q=0}^{m-2} (\omega_3\lambda^2)^q + \lambda^n\rho_0^\star(\omega_3\lambda^2)^{m-1} \\
&\leq && \lambda^n\omega_1\rho_0 \sum_{q=0}^{m-2} (\omega_3\lambda^2)^q + \lambda^{n+1}\omega_2\rho_0 \sum_{q=0}^{m-2} (\omega_3\lambda^2)^q + \lambda^n\rho_0^\star
\end{aligned}
$$

$$< \quad \lambda^n \left[\rho_0^\star + \frac{(\omega_1 + \omega_2\lambda)\rho_0}{1 - \omega_3\lambda^2}\right]. \tag{2.7}$$

It follows from (2.6) and (2.7) that $\lim_{n\to\infty} \rho(x_n, x_{n+k}) = 0$ for all $k > 0$, therefore $\{x_n\}$ is a Cauchy sequence in X and by the completeness of (X, d) there exists $x^\star \in X$ such that $\lim_{n\to\infty} x_n = x^\star$. Now $x^\star$ is a fixed point of T as

$$\begin{aligned} \rho(x^\star, Tx^\star) &\leq \omega_1\rho(x^\star, x_n) + \omega_2\rho(x_n, x_{n+1}) + \omega_3\rho(x_{n+1}, x^\star) \\ &= \omega_1\rho(x^\star, x_n) + \omega_2\rho_n + \omega_3\rho(x_{n+1}, x^\star). \end{aligned} \tag{2.8}$$

Taking the limit as $n \to \infty$ in (2.8) we have $\rho(x^\star, Tx^\star) = 0$. If $x^{\star\star}$ is another fixed point of T then

$$\rho(x^\star, x^{\star\star}) = \rho(Tx^\star, Tx^{\star\star}) \leq \lambda\rho(x^\star, x^{\star\star}) < \rho(x^\star, x^{\star\star}),$$

a contradiction. Hence $\rho(x^\star, x^{\star\star}) = 0$ or $x^\star = x^{\star\star}$ proving uniqueness.

Theorem 2.2. *Let (X, ρ) be a complete weighted rectangular b-metric space with weights ω_1, ω_2 and ω_3 and $T : X \to X$ be a mapping such that*

$$\rho(Tx, Ty) \leq \lambda[\rho(x, Tx) + \rho(y, Ty)], \tag{2.9}$$

for all $x, y \in X$ where $\lambda < \frac{1}{2}$ if $\omega_3 \leq 1$ and $\lambda < \frac{1}{1+\sqrt{\omega_3}}$ if $\omega_3 > 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and define $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is a Cauchy sequence. If $x_{n+1} = x_n$, then x_n is a fixed point of T so we suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. As before define $\rho(x_n, x_{n+1}) = \rho_n$ then

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho(Tx_{n-1}, Tx_n) \leq \lambda[\rho(x_{n-1}, Tx_{n-1}) + \rho(x_n, Tx_n)] \\ &= \lambda[\rho(x_{n-1}, x_n) + \rho(x_n, x_{n+1})] \\ \rho_n &\leq \lambda[\rho_{n-1} + \rho_n] \\ \rho_n &\leq \frac{\lambda}{1 - \lambda}\rho_{n-1} \\ &= \beta\rho_{n-1}, \end{aligned} \tag{2.10}$$

where $\beta = \frac{\lambda}{1-\lambda} < 1$ and repeating the process in (2.10) we obtain

$$\rho_n \leq \beta^n \rho_0. \tag{2.11}$$

□

We also assume that x_0 is not a periodic point of T otherwise $x_0 = x_n$ for $n \geq 2$ implies that

$$\begin{aligned} \rho(x_0, Tx_0) &= \rho(x_n, Tx_n) \\ \rho(x_0, x_1) &= \rho(x_n, x_{n+1}) \\ \rho_0 &= \rho_n \leq \beta^n \rho_0, \end{aligned} \tag{2.12}$$

which is a contradiction as $\rho_0 \neq 0$ or $x_1 \neq x_0$. Thus we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Now

$$\begin{aligned} \rho(x_n, x_{n+2}) &= \rho(Tx_{n-1}, Tx_{n+1}) \leq \lambda[\rho(x_{n-1}, Tx_{n-1}) + \rho(x_{n+1}, Tx_{n+1})] \\ &= \lambda[\rho(x_{n-1}, x_n) + \rho(x_{n+1}, x_{n+2})] \\ &= \lambda[\rho_{n-1} + \rho_{n+1}] \\ &\leq \lambda[\beta^{n-1}\rho_0 + \beta^{n+1}\rho_0] \\ &= \lambda\beta^{n-1}[1+\beta^2]\rho_0 = \gamma\beta^{n-1}\rho_0 \text{ (where } \gamma = \lambda(1+\beta^2)) \\ \rho_n^\star &\leq \gamma\beta^{n-1}\rho_0. \end{aligned} \tag{2.13}$$

For $\{x_n\}$ we consider $\rho(x_n, x_{n+k})$. For $k = 2m+1$, $m \geq 1$ as in the proof of Theorem 2.1 we can show that

$$\begin{aligned} \rho(x_n, x_{n+2m+1}) &\leq \sum_{q=0}^{m-1} \omega_3^q(\omega_1\rho_{n+2q} + \omega_2\rho_{n+2q+1}) + \omega_3^m\rho_{n+2m} \\ &\leq \sum_{q=0}^{m-1} \omega_3^q(\omega_1\beta^{n+2q} + \omega_2\beta^{n+2q+1})\rho_0 + \omega_3^m\beta^{n+2m}\rho_0 \\ &= \beta^n\omega_1\rho_0 \sum_{q=0}^{m-1}(\omega_3\beta^2)^q + \beta^{n+1}\omega_2\rho_0\sum_{q=0}^{m-1}(\omega_3\beta^2)^q + \beta^n\rho_0(\omega_3\beta^2)^m \\ &\leq \beta^n\omega_1\rho_0 \sum_{q=0}^{m-1}(\omega_3\beta^2)^q + \beta^{n+1}\omega_2\rho_0\sum_{q=0}^{m-1}(\omega_3\beta^2)^q \\ &+ \beta^n\rho_0 \text{ (note } \omega_3\beta^2 < 1) \\ &< \beta^n\rho_0\left[1 + \frac{\omega_1+\omega_2\beta}{1-\omega_3\beta^2}\right]. \end{aligned} \tag{2.14}$$

Similarly for $k = 2m$, $m \geq 1$ we can show that

$$\begin{aligned}
\rho(x_n, x_{n+2m}) &\leq \sum_{q=0}^{m-2} \omega_3^q(\omega_1\rho_{n+2q} + \omega_2\rho_{n+2q+1}) + \omega_3^{m-1}\rho^\star_{n+2m-2} \\
&\leq \sum_{q=0}^{m-2} \omega_3^q(\omega_1\beta^{n+2q} + \omega_2\beta^{n+2q+1})\rho_0 + \omega_3^{m-1}\gamma\beta^{n+2m-2}\rho_0^\star \\
&= \beta^n\omega_1\rho_0\sum_{q=0}^{m-2}(\omega_3\beta^2)^q + \beta^{n+1}\omega_2\rho_0\sum_{q=0}^{m-2}(\omega_3\beta^2)^q + \gamma\beta^{n-1}\rho_0^\star(\omega_3\beta^2)^{m-1} \\
&\leq \beta^n\omega_1\rho_0\sum_{q=0}^{m-2}(\omega_3\beta^2)^q + \beta^{n+1}\omega_2\rho_0\sum_{q=0}^{m-2}(\omega_3\beta^2)^q + \gamma\beta^{n-1}\rho_0^\star \\
&< \beta^{n-1}\left[\gamma\rho_0^\star + \frac{(\omega_1\beta + \omega_2\beta^2)\rho_0}{1-\omega_3\beta^2}\right]. \qquad (2.15)
\end{aligned}$$

Hence the $\lim_{n\to 0}\rho(x_n, x_{n+k}) = 0$ for all $k > 0$ and $\{x_n\}$ is Cauchy in (X, ρ). By the completeness of (X, ρ) there exists $x^\star \in X$ such that $x_n \to x^\star$. We show that $x^\star$ is a fixed point of T:

$$\begin{aligned}
\rho(x^\star, Tx^\star) &\leq \omega_1\rho(x^\star, x_n) + \omega_2\rho(x_n, x_{n+1}) + \omega_3\rho(x_{n+1}, Tx^\star) \\
&= \omega_1\rho(x^\star, x_n) + \omega_2\rho_n + \omega_3\rho(Tx_n, Tx^\star) \\
&\leq \omega_1\rho(x^\star, x_n) + \omega_2\rho_n + \omega_3\lambda[\rho(x_n, Tx_n) + \rho(x^\star, Tx^\star)],
\end{aligned}$$

$$(1-\omega_3\lambda)\rho(x^\star, Tx^\star) \leq \omega_1\rho(x^\star, x_n) + (\omega_2 + \omega_3\lambda)\rho_n. \qquad (2.16)$$

Taking the limit as $n \to \infty$ in (2.16) we have $\rho(x^\star, Tx^\star) = 0$. If $x^{\star\star}$ is another fixed point of T then

$$\begin{aligned}
\rho(x^\star, x^{\star\star}) = \rho(Tx^\star, Tx^{\star\star}) &\leq \lambda[\rho(x^\star, Tx^\star + \rho(x^{\star\star}, Tx^{\star\star})] \\
&= \lambda[\rho(x^\star, x^\star) + \rho(x^{\star\star}, x^{\star\star})] = 0.
\end{aligned}$$

Hence $\rho(x^\star, x^{\star\star}) = 0$ or $x^\star = x^{\star\star}$ proving uniqueness. □

Example 1. *Let $X = (2, 3)$ and define $\rho(x, y)$ by*

$$\rho(x, y) = \begin{cases} e^{|x-y|} & x \neq y, \\ 0, & x = y. \end{cases}$$

Then $\rho(x,y)$ is a weighted rectangular b-metric. We only need to prove the rectangular inequality:

$$\begin{aligned}
\rho(x,y) &= e^{|x-u+u-v+v-y|}\\
&\leq e^{|x-u|+|u-v|+|v-y|}\\
&= e^{\frac{1}{2}|x-u|+\frac{1}{10}|u-v|+\frac{2}{5}|v-y|}e^{\frac{1}{2}|x-u|+\frac{9}{10}|u-v|+\frac{3}{5}|v-y|}\\
&\leq \left(\frac{1}{2}e^{|x-u|}+\frac{1}{10}e^{|u-v|}+\frac{2}{5}e^{|v-y|}\right)\sup_X e^{\frac{1}{2}|x-u|+\frac{9}{10}|u-v|+\frac{3}{5}|v-y|} \qquad (2.17)\\
&= \left(\frac{1}{2}e^{|x-u|}+\frac{1}{10}e^{|u-v|}+\frac{2}{5}e^{|v-y|}\right)e^2\\
&= \frac{e^2}{2}e^{|x-u|}+\frac{e^2}{10}e^{|u-v|}+\frac{2e^2}{5}e^{|v-y|}\\
&= \omega_1\rho(x,u)+\omega_2\rho(u,v)+\omega_3\rho(v,y). \qquad (2.18)
\end{aligned}$$

Here in (2.17) *we have used Jenkin's inequality [6]. Now let $T: X \to X$ be defined by $Tx = \frac{1}{x}+2$ then*

$$\begin{aligned}
\rho(Tx,Ty) &= e^{\left|\frac{1}{x}-\frac{1}{y}\right|} = e^{\frac{|x-y|}{|xy|}} \leq e^{\frac{1}{4}|x-y|}\\
&< \frac{1}{4}e^{|x-y|} = \frac{1}{4}\rho(x,y). \qquad (2.19)
\end{aligned}$$

Hence $\lambda = \frac{1}{4} < \frac{\sqrt{2.5}}{e}$. Then by Theorem (2.1) *T has a unique fixed point $1+\sqrt{2}$ in X.*

3. Conclusion

We have succeeded in bridging a gap that existed between a rectangular metric space and a rectangular b-metric space by introducing a weighted rectangular b-metric space involving weights in the rectangular inequality, which under certain circumstances fits into the realm of a rectangular b-metric space. The results in the paper also demonstrate that common fixed point results can be extended to this space.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KWAZULU-NATAL
PRIVATE BAG X54001, DURBAN,
SOUTH AFRICA.
Email address: singhp@ukzn.ac.za

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KWAZULU-NATAL
PRIVATE BAG X54001, DURBAN,
SOUTH AFRICA.
Email address: singhv@ukzn.ac.za