Advances in Mathematics: Scientific Journal **10** (2021), no.4, 2167–2179
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.31

LH LABELING OF GRAPHS

Farisa M.[1] and Parvathy K.S.

ABSTRACT. A graph G with n vertices is said to have an LH labeling if there exists a bijective function $f : V(G)$ to $\{1, 2, 3, \ldots, n\}$ such that the induced map $f^* : E(G) \to N$, the set of natural numbers defined by $f^*(uv) = \frac{LCM(f(u),f(v))}{HCF(f(u),f(v))}$ is injective (LCM and HCF denotes the least common multiple and highest common factor respectively). A graph that admits an LH labeling is called an LH graph. This article explores the results of LH Labeling of some standard graphs.

1. INTRODUCTION

Graph labeling is one of the potential areas of research in graph theory which assigns numerical values or functions to the vertices or edges or both, subject to certain conditions. Graph labeling were first introduced in 1960's. Labeled graphs have application ranging from computer science to social science and on engineering to mention a few. Latest survey of all the graph labeling techniques can be found in Gallian survey [2]. Every graph can be labeled in infinitely many ways. It motivated us to define a new vertex labeling called LH labeling of graphs. Here, we intend to apply the elementary class concepts LCM(Least common Multiple) and HCF(Highest Common Factor) in labeling.

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C78.

Key words and phrases. Graph labeling, Strong Multiplicative labeling, LCM, HCF.

Submitted: 13.03.2021; *Accepted:* 02.04.2021; *Published:* 26.04.2021.

We consider simple, finite, connected and undirected graph $G = G(V, E)$. We provide a brief summary of definitions and other information which serve as prerequisites for the present investigation.

The triangular snake T_n [8] is obtained from the path P_n by replacing each edge of the path by a triangle C_3. Similarly, A quadrilateral snake G_n is obtained from a path $u_1, u_2, \ldots, u_{n+1}$ by replacing every edge of a path$u_1, u_2, \ldots, u_{n+1}$ by replacing every edge of a path by a cycle C_4, and is denoted by G_n [4].

The corona graph $P_n \odot K_1$ is called a Comb.

The Helm $H_n, n \geq 3$ is the graph obtained from a wheel W_n by attaching a pendant edge at each rim vertex [6].

Bistar is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ and is denoted by $B_{m,n}$. Let S_m be a star with central vertex v_0 and pendant vertices $v_1, v_2, \ldots, v_m$ and let $[P_n; S_m]$ be the graph obtained from n copies of S_m with vertices $v_{0j}, v_{1j}, v_{2j}, \ldots, v_{mj} (1 \leq j \leq n)$ and joining v_{0j} and v_{0j-1} by means of an edge, $1 \leq j \leq n-1$ [4].

A Twig TW(n) $n \geq 3$, is a tree obtained from a path by attaching exactly two pendent edges to each internal vertex of the path.

A graph with p vertices is said to be strongly multiplicative if the vertices can be labeled $1, 2, \ldots, p$ so that the values on the edges, obtained as the products if the labels of their end verices, are all distinct. [3]

2. LH LABELING

In this section we define a new type of vertex labeling called *LH Labeling* and discuss the LH labeling of some class of graphs. Also we prove that the Petersen graph and hypercube are LH graphs.

Definition 2.1. *A graph G with n vertices is said to have an LH labeling if there exists a bijective function$f : V(G)$ to $\{1, 2, 3, \ldots, n\}$ such that the induced map $f^* : E(G) \rightarrow N$, the set of natural numbers defined by $f^*(uv) = \frac{LCM(f(u), f(v))}{HCF(f(u), f(v))}$ (LCM denotes Least common Multiple & HCF denotes Highest Common Factor) is injective. A graph that admits an LH labeling is called an LH graph.*

By numbering the vertices $1, 2, \ldots, n$ serially, we observe that the path P_n, Odd Cycles, and the star $K_{1,n}$ are LH graphs. The complete graph K_2 and K_3 are LH graphs.

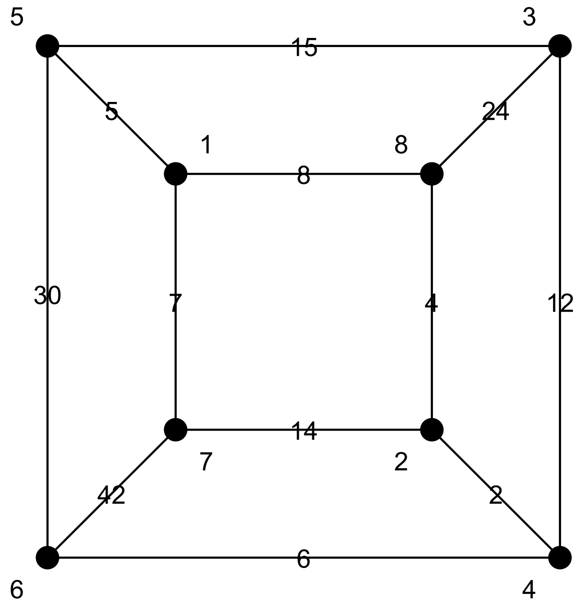

FIGURE 1. LH labelling of Q_3

Observation

Since the edge labels in any LH labeling are large numbers, We have, if G be an LH graph with n vertices, then $2 \leq l(e) \leq n^2 - n$, where $l(e)$ denotes the label of the edge e.

L.W. Beineke and S.M. Hegde have introduced strong multiplicative graphs in [3], which coincides with this concept.

Remark 2.1. *If the labels of each pair of adjacent vertices of a given graph G are relatively prime, then LH graphs and strong multiplicative graphs are the same.*

Remark 2.2. *The complete graph K_4 is not an LH graph. By labeling the vertices of K_4 using the numbers 1,2,3,4 produces the edge label 2 two times, so K_4 is not an LH graph. Hence we conclude that K_n is not an LH gaph for $n \geq 4$.*

Theorem 2.1. *K_n is not an LH gaph for $n \geq 4$.*

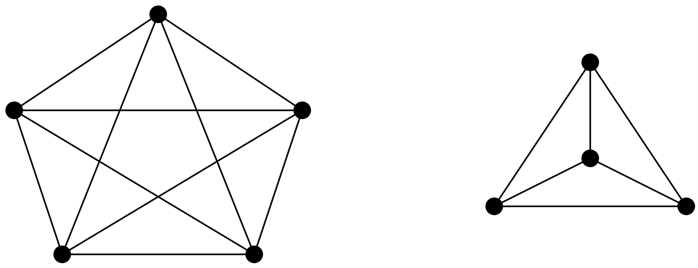

FIGURE 2. Some Non LH graphs

Observed that $K_4 - e$ is an LH graph.

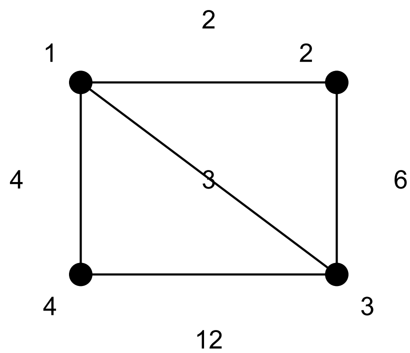

FIGURE 3. LH labeling of $K_4 - e$

In our next remark, we make an observation about the subgraphs of an LH graph.

Remark 2.3. *Every spanning subgrah of an LH graph is LH, but every subgraph need not be LH. K_4 is a subgraph of the LH graph given in Fig. 4, which is not LH.*

The Petersen graph is an LH graph. The labeling pattern is given in Fig. 5.

3. LH GRAPHS

In this section we show that some well-known graphs are LH.

Theorem 3.1. *The comb graph $P_n \odot K_1$ is an LH graph.*

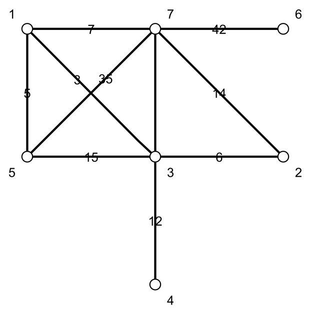

FIGURE 4.

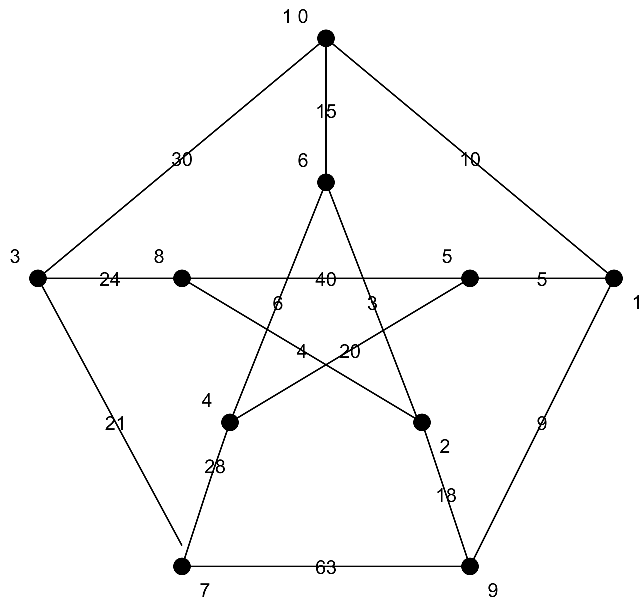

FIGURE 5. LH labelling of Petersen graph

Proof. Consider a comb graph G with vertex set $\{v_t, v'_t, 1 \leq t \leq n\}$ where $v'_t, t = 1$ to n are the pendant vertices. Then, $|V(G)| = 2n$ and $E(G) = \{v_t v_{t+1}, v_t v'_t, 1 \leq t \leq n\}$.

We define the vertex labeling f from $V(G)$ to $\{1, 2, 3, \ldots, 2n\}$ by

$$f(v_t) = 2t - 1,\ t = 1, 2, 3, \ldots, n.$$
$$f(v'_t) = 2t,\ \ i = 1, 2, 3, \ldots, n.$$

One can observe that the labeling defined above satisfies the conditions of LH labeling and the comb graph is an LH graph. □

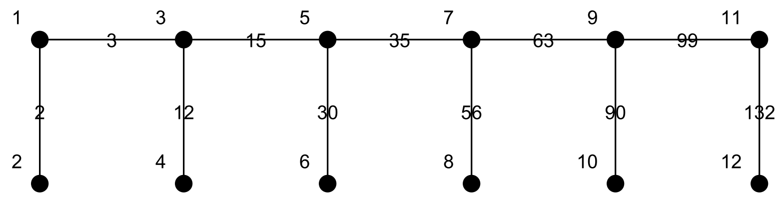

FIGURE 6. LH labeling of $P_6 \odot K_1$

Theorem 3.2. *The helm $H_n, n \geq 3$ is an LH graph.*

Proof. Let G be the helm graph H_n with central vertex u_0. Let $u_t, t = 1$ to n be the rim vertices and $u_1', u_2', \ldots, u_n'$ be the pendant vertices of H_n. Then $V(G) = \{u_0, u_t, u_t', 1 \leq t \leq n\}$, and $|V(G)| = 2n+1$, $E(G) = \{u_0u_t, u_tu_t', 1 \leq t \leq n\} \cup \{u_1u_n, u_tu_{t+1}, 1 \leq t \leq n-1\}$.

We define the vertex labeling $f : V(G) \to \{1, 2, 3, \ldots, 2n+1\}$ as

$$f(u_0) = 1$$
$$f(u_1) = 2$$
$$f(u_1') = 3$$
$$f(u_t) = 2t+1, t = 2, 3, \ldots, n.$$
$$f(u_t') = 2t, t = 2, 3, \ldots, n.$$

One can observe that the edge labels induced by the labeling function defined above satisfies the conditions of LH labeling and the graph H_n is an LH graph. □

Theorem 3.3. *The triangular snake T_n is an LH graph.*

Proof. Let G denote the triangular snake T_n which can be obtained from a path $u_1, u_2, \ldots, u_n, u_{n+1}$, by joining u_i, u_{i+1} to a new vertex $v_j, j = 1$ to n. Then,

$V(G) = \{u_t, 1 \leq t \leq n+1, v_t, 1 \leq t \leq n\}$,

$|V(G)| = 2n+1$,

$E(G) = \{u_tu_{t+1}, u_tv_t, u_{t+1}v_t, 1 \leq t \leq n\}$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 2n+1\}$ by

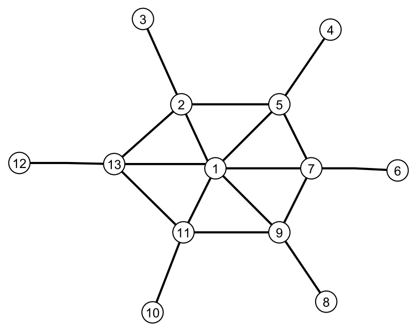

FIGURE 7. LH labeling of H_6

$$f(u_t) = 2t - 1, \quad t = 1, 2, 3 \dots, n + 1.$$
$$f(v_t) = 2t, \qquad t = 1, 2, 3 \dots, n.$$

One can observe that the labeling defined above satisfies the conditions of LH labeling and the graph under consideration is an LH graph. □

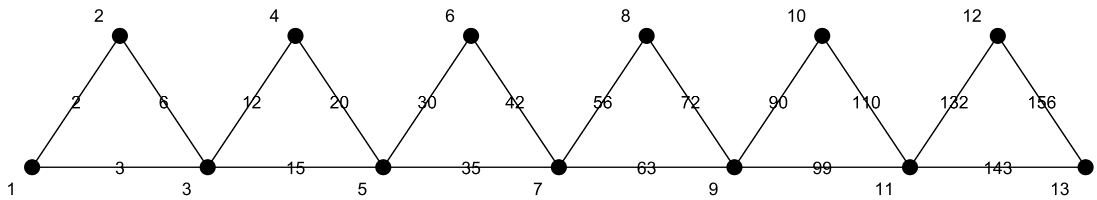

FIGURE 8. LH labeling of T_6

Theorem 3.4. *A quadrilateral snake G_n is an LH graph.*

Proof. Consider a quadrilateral snake G_n with vertex set $\{v_i, i = 1$ to $n+1, u_j$ and $w_j, j = 1$ to $n\}$ which is obtained from the path $v_1, v_2, \dots, v_{n+1}$ by joining v_i, v_{i+1} to new vertices u_i and w_i respectively and joining u_i and $w_i, 1 \leq i \leq n$. Then, $|V(G)| = 3n + 1$, $E(G) = \{v_t v_{t+1}, u_t w_t, t = 1$ to $n\} \cup \{u_i v_i, v_{i+1} w_i, i = 1$ to $n\}$.

Define $f : V(G_n) \to \{1, 2, 3, \dots, 3n + 1\}$ by

$$f(v_m) = 3m - 2, m = 1, 2, 3, \dots, n + 1.$$
$$f(u_i) = 3m - 1, m = 1, 2, 3, \dots, n.$$
$$f(w_m) = 3m, \quad m = 1, 2, 3, \dots, n.$$

One can observe that the labeling defined above satisfies the conditions of LH labeling and the graph under consideration is an LH graph. □

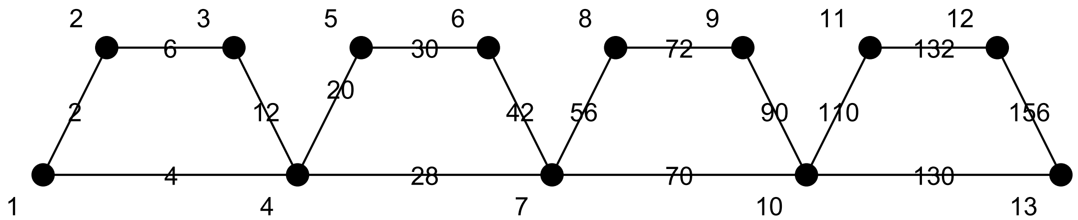

FIGURE 9. LH labeling of G_4

Theorem 3.5. *A Twig graph $TW(n), n \geq 3$ is an LH graph.*

Proof. Consider a twig graph G, which is obtained from a path $u_1, u_2, \ldots, u_{n-1}, u_n$ by attaching exactly two pendent edges v_i and w_i to each internal vertex of the path. Then, $V(G) = \{u_i, i = 1 \leq i \leq n$ and $v_i, w_i, i = 1, 2, 3, \ldots, n-2\}$, $|V(G)| = 3n - 4$, and $E(G) = \{u_i u_{i+1}, i = 1, 2, 3, \ldots, n.\} \cup \{u_{j+1} v_j, u_{j+1} w_j, j = 1, 2, 3, \ldots, n-2\}$. □

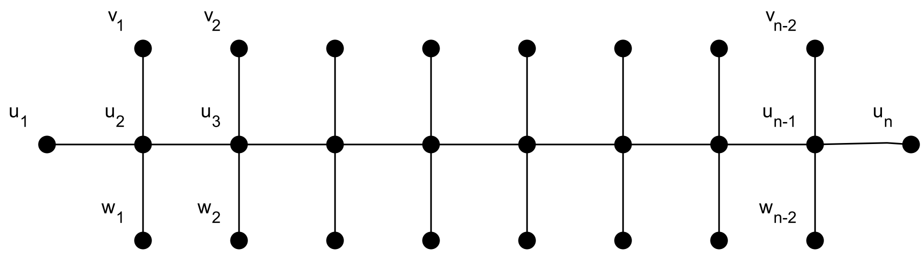

FIGURE 10. Ordinary labeling of $TW(n)$

We define the function $f : V(G) \to \{1, 2, 3, 4, \ldots, 3n-4\}$ by

$$f(u_m) = 3m - 2 \text{ for } i = 1, 2, 3, \ldots, n-1.$$
$$f(v_i) = 3m - 1 \text{ for } i = 1, 2, 3, \ldots, n-2.$$
$$f(w_i) = 3m \text{ for } i = 1, 2, 3 \ldots, n-2.$$
$$f(u_n) = 3n - 4.$$

One can observe that the labeling defined above satisfies the conditions of LH labeling and the graph under consideration is an LH graph.

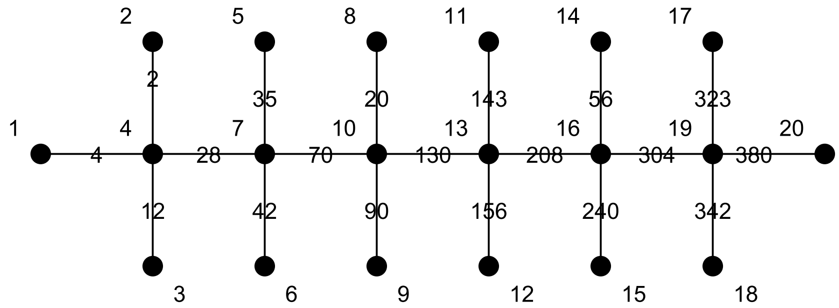

FIGURE 11. LH labeling of $TW(8)$

Theorem 3.6. *The bistar graph $B_{m,n}$ is an LH graph.*

Proof. Let G be the graph $B_{m,n}$. First number the m pendant vertices by u_1, u_2, $\ldots, u_m$ and n pendant vertices by v_1, v_2, $v_3 \ldots, v_n$. The vertex adjacent to u_i is u and vertex adjacent to v_i is v. Then, $V(G) = \{u, v\} \cup \{u_i, 1 \le i \le m\} \cup \{v_j, 1 \le j \le n\}$, $|V(G)| = m + n + 2$, and $E(G) = \{uu_i, i = 1, 2, \ldots, m\} \cup \{vv_j, j = 1, 2, 3, \ldots, n\}$. Vertex labeling from $V(G)$ to the set $\{1, 2, 3, \ldots, |V(G)|\}$ is defined as follows:

Case 1 : When $m = n$, we have $|V(G)| = 2n + 2$, and

$$\begin{aligned}
&f(u) = 2n + 1\\
&f(u_i) = 2i + 2, i = 1, 2, 3, \ldots, n\\
&f(v) = 2n - 1\\
&f(v_i) = 2i - 1, i = 1, 2, \ldots, n\\
&f(v_n) = 2.
\end{aligned}$$

Case 2 : When $m = 1$, we have $V(G) = \{u, u_1, v, v_i, i = 1, 2, 3 \ldots, n\}$, $|V(G)| = n + 3$, and

$$\begin{aligned}
&f(u) = n + 2\\
&f(u_1) = n + 3\\
&f(v) = 1\\
&f(v_i) = i + 1, i = 1, 2, 3, \ldots, n.
\end{aligned}$$

Case 3 : When $n = 1$, we have $V(G) = \{u, u_i, i = 1, 2, 3, \ldots, m\}$, $|V(G)| = m+3$, and

$$\begin{aligned}
&f(u) = 1\\
&f(u_i) = i + 1, i = 1, 2, 3, \ldots, m\\
&f(v) = m + 2\\
&f(v_1) = m + 3.
\end{aligned}$$

Case 4 : When $m < n$, we have

i) n odd and m even or n even and m odd

$$\begin{aligned}
&f(u) = m + n + 2\\
&f(u_1) = m + n + 1\\
&f(u_i) = f(u_{i-1}) - 2, i = 2, 3, 4, \ldots, m.\\
&f(v) = m + n.
\end{aligned}$$

The remaining $v_i, i = 1, 2, 3, \ldots, n$ can be numbered using the digits 1, 3, 5, 7, $\ldots$, $m + n - 2$, 2, 4, 6, $\ldots$, $f(u_m) - 2$.

ii) n even and m even or n odd and m odd

$$\begin{aligned}
&f(u) = m + n + 1\\
&f(u_1) = m + n + 2\\
&f(u_i) = f(u_{i-1}) - 2, \text{ for } i = 2, 3, 4, \ldots, m.\\
&f(v) = m + n - 1
\end{aligned}$$

The remaining $v_i, i = 1, 2, \ldots, n$ can be numbered using the digits 1, 3, 5, 7, $\ldots$, $m + n - 3$, 2, 4, 6, $\ldots$, $f(u_m) - 2$.

Case 5 : When $m > n$

i) m odd and n even or m even and n odd

$$\begin{aligned}
&f(v) = m + n + 2\\
&f(v_1) = m + n + 1\\
&f(v_i) = f(v_{i-1}) - 2, \text{ for } i = 2, 3, 4, \ldots, n.\\
&f(u) = m + n
\end{aligned}$$

The remaining $u_i, i = 1, 2, 3, \ldots, m$ can be numbered using the digits 1, 3, 5, 7, $\ldots$, $m + n - 2$, 2, 4, 6, $\ldots$, $f(v_m) - 2$.

ii) n even and m even or n odd and m odd

$f(v) = m + n + 1.$
$f(v_1) = m + n + 2.$
$f(v_i) = f(v_{i-1}) - 2$, for $i = 2, 3, 4, \ldots, n.$
$f(u) = m + n - 1$

The remaining $u_i, i = 1$ to m can be numbered using the digits 1, 3, 5, 7, $\ldots$, $m + n - 3$, 2, 4, 6, $\ldots$, $f(v_n) - 2$.

One can observe that the induced edge labels are distinct in all cases. Hence $B_{m,n}$ is an LH graph. $\square$

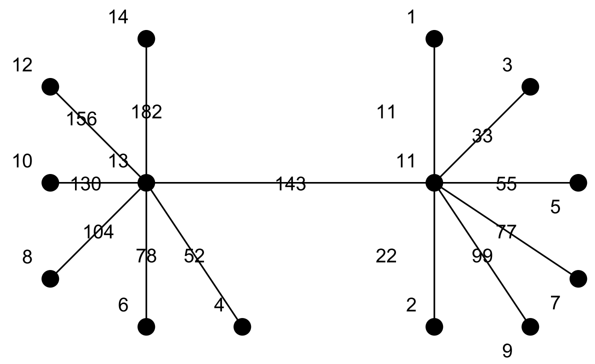

FIGURE 12. LH labeling of $B_{6,6}$

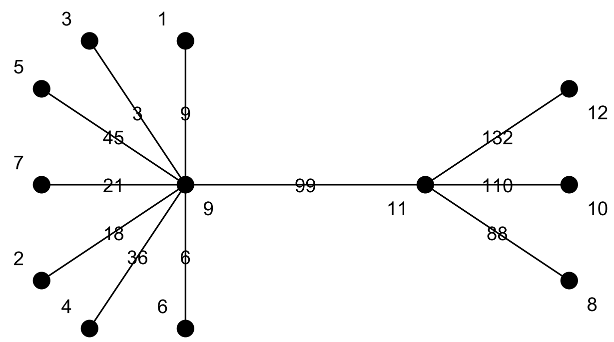

FIGURE 13. LH labeling of $B_{7,3}$,

Theorem 3.7. *The graph* $[P_n : S_2]$ *is an LH graph.*

Proof. Consider a $[P_n : S_2]$ graph G. Let the vertex set of the path P_n is given by $v_i, 1 \leq i \leq n$ and $u_i, w_i, 1 \leq i \leq n$ be the vertices which are made adjacent with v_i. Then, $V(G) = \{u_i, v_i, w_i, 1 \leq i \leq n\}$, $|V(G)| = 3n$, and $E(G) = \{v_i v_{i+1}, 1 \leq i \leq n-1\} \cup \{v_i w_i, v_i u_i, 1 \leq i \leq n\}$.

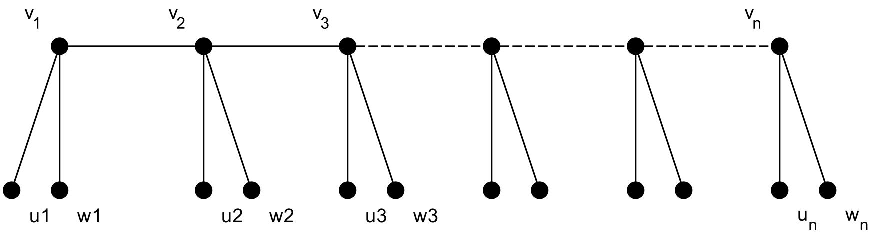

FIGURE 14. Ordinary labeling of $[P_n : S_2]$

We define $f : V(G) \to \{1, 2, 3, \ldots, 3n\}$ by

$$f(v_m) = 3m - 2, 1 \leq m \leq n.$$
$$f(u_m) = 3m - 1, 1 \leq m \leq n.$$
$$f(w_m) = 3m, \quad 1 \leq m \leq n$$

We can observe that the graph G with the labeling function defined above satisfies the conditions of an LH graph. $\square$

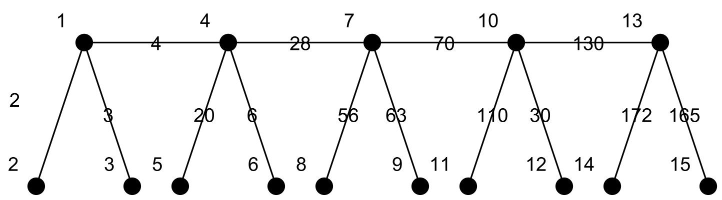

FIGURE 15. LH labeling of $[P_5 : S_2]$

REFERENCES

[1] D.B. WEST: *Introduction to Graph Theory*, Second Edition, PHI Learning Private Limited, 2011.

[2] J.A. GALLIAN: *A dynamic survey of graph labeling*,The electronic journal of combinatorics, 2016.

[3] L. W. BEINEKE, S. M. HEGDE: *Strongly Multiplicative Graphs*, Discussions Mathematicae,Graph Theory **21** (2001), 63-75.

[4] R. VASUKI, A. NAGARAJAN, S. AROCKIARAJ: *Even vertex odd mean labeling of graphs*, SUT journal of Mathematics, **49**(2) (2013), 79-92.

[5] S.M. HEGDE, S. SHETTY: *Combinatorial labelings of Graphs*, Applied Mathematics E-Notes, **6**(2006), 251-258.

[6] K. K. KANANI, T. M. CHHAYA: *Strongly multiplicative labeling of some standard graphs*, International Journal of Mathematics and Soft Computing, **7**(0.1) (2017), 13-21.

[7] A. SUGUMARAN, P. VISHNU PRAKASH: *Prime Cordial Labeling for Theta Graph*, Annals of Pure and Applied Mathematics, **14**(3) (2017), 379-386.

[8] K. ESHGHI: *introduction to graceful graphs*, Discussions Mathematicae, Sharif University of Technology, 2002.

[9] K. VAIDYA, N.H. SHAH: *Graceful and Odd graceful labeling of Some graphs*, International Journal of Mathematics and Soft Computing, **3**(1) (2013), 61-68.

[10] L. BIJUKUMAR: *Odd Sequential Labeling of Some New Families of Graphs*, International J.Math. Combin, **3** (2014), 89-96.

RESEARCH SCHOLAR DEPARTMENT OF MATHEMATICS
ST.MARY'S COLLEGE,
THRISSUR, KERALA,
INDIA.
Email address: dharu8011@yahoo.com

ASSOCIATE PROFESSOR, POST GRADUATE AND RESEARCH DEPARTMENT OF MATHEMATICS
ST.MARY'S COLLEGE
THRISSUR, KERALA,
INDIA.
Email address: parvathy.math@gmail.com