ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.4, 2181–2185
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.32

COMPACT COMPLEMENT TOPOLOGIES IN BITOPOLOGICAL SPACE

Janat A. Emwas[1] and Hasan Z. Hdeib

ABSTRACT. Let (X, τ_0, τ_1) be a Hausdorff space, where X is an infinite set. The compact complement topologies in bitopological space (X, τ_0, τ_1) are τ_0^* and τ_1^* where $\tau_0^* = \{\emptyset\} \cup \{X \setminus M_0 : M_0$ is τ_0-compact$\}$ and $\tau_1^* = \{\emptyset\} \cup \{X \setminus M_1 : M_1$ is τ_1-compact$\}$. Throughout this paper, some properties of the space (X, τ_0^*, τ_1^*) are studied in ZF and we prove some conditions hold in ZF.

1. INTODUCTION

A bitopological space is a space of the form (X, τ_0, τ_1), where τ_0 and τ_1 are two topologies on X, this concept initiated by Kelly [2].

Throughout this paper, we suppose that τ_0 and τ_1 are topologies on an infinite set X such that (X, τ_0, τ_1) is a Hausdorff bitological space and in this paper, we study elementary properties of compact complement topologies in bitopological space.

In particular, X stands for a set and "P" will be used for "Pairwise", i.e., P-Hausdorff will mean pairwise Hausdorf, and so on.

---

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 22xx, 22Cxx, 22Axx.

Key words and phrases. Compact complement bitopology, Pairwise irreducible (Pairwise hyper connected).

Submitted: 05.04.2021; *Accepted:* 21.04.2021; *Published:* 28.04.2021.

2. Main properties of compact complement Topologies in Bitopological space.

Definition 2.1. *A bitopological space (X, τ_0, τ_1) is called Hausdorff (compact) space if (X, τ_0) and (X, τ_1) are Hausdorff (compact).*

Because it is hold in ZF that a compact subspace of a Hausdourff space is closed, so it is easy to show that τ_0^* and τ_1^* are topologies on X. Clearly, (X, τ_0^*, τ_1^*) is a bitopological space and we name it compact complement bitoplogy.

For a subset Y of X and the topologies τ_0 and τ_1 on X, let $\tau_0 \mid Y = \{v_0 \cap Y : v_0 \in \tau_0\}$ and $\tau_1 \mid Y = \{v_1 \cap Y : v_1 \in \tau_1\}$ are called relative topologies on Y then $(Y, \tau_0 \mid Y, \tau_1 \mid Y)$ is called a bitopological subspace of (X, τ_0, τ_1).

Theorem 2.1. *Let (X, τ_0, τ_1) be a bitopological space and $Y \subseteq X$. Then $\tau_i^* \mid Y \subseteq \tau_i \mid Y$, i.e. , $\tau_i^* \mid Y$ is coarser than $\tau_i \mid Y$; $\forall i = 0, 1$.*

Proof. We want to prove it when $i = 0$, it is enough to show that $\tau_0^* \subseteq \tau_0$. Let $U_0 \in \tau_0^*$ and $U_0 \neq \emptyset$. Then $U_0 = X \setminus M_0$ for τ_0– compact subspace M_0, since a compact subspace of a Haudorff space is closed, M_0 is a τ_0 – closed set. Thus $U_0 \in \tau_0$ also, in consequence, $\tau_0^* \subseteq \tau_0$. In a similar way we can prove that $\tau_1^* \subseteq \tau_1$. □

Theorem 2.2. *Let (X, τ_0, τ_1) be a bitopological space and $Y \subseteq X$. The following conditions satisfy in ZF:*

(a) *If Y is compact in (X, τ_0, τ_1), then $\tau_i^* \mid Y = \tau_i \mid Y$, $\forall\, i = 0, 1$.*
(b) *If $\tau_i^* \mid Y = \tau_i \mid Y$ then there exists a τ_i - compact set C_i such that $Y \subseteq C_i$, $\forall\, i = 0, 1$.*

Proof.

(a) Assume that Y is compact set, then Y is τ_0 – compact and τ_1 – compact. Let V_0 be τ_0 – open set. Since (X, τ_0) is Hausdorff, the set Y is τ_0 – closed, thus $A = Y \cap (X \setminus V_0)$ is a τ_0 – closed subset of τ_0– compact set Y. So A is a τ_0 – compact. Notice that $V_0 \cap Y = Y \cap (X \setminus A)$. This proves that $V_0 \cap Y \in \tau_0^* \mid Y$ and $\tau_0 \mid Y \subseteq \tau_0^* \mid Y$, also by previous theorem $\tau_0^* \mid Y \subseteq \tau_0 \mid Y$, so the equality hold when $i = 0$. Similarly, $\tau_1^* \mid Y = \tau_1 \mid Y$.
(b) Assume that Y is a subset of X where $\tau_0^* \mid Y = \tau_0 \mid Y$. Let $V_0 \in \tau_0$ and $\emptyset \neq V_0 \cap Y \neq Y$. Because $V_0 \cap Y \in \tau_0 \mid Y$ then $V_0 \cap Y \in \tau_0^* \mid Y$, there is

a τ_0 – compact set C_0 such that $V_0 \cap Y = Y \setminus C_0$. Fix a point $x_0 \in V_0 \cap Y$. Thus $x_0 \notin C_0$, since C_0 is τ_0 – compact, there exists a pair U_0, W_0 of disjoint τ_0– open sets such that $x_0 \in U_0$ and $C_0 \subseteq W_0$. So $W_0 \cap Y \neq \emptyset$. Since $V_0 \cap Y \neq Y$. Because $U_0 \cap Y$ and $W_0 \cap Y$ are in $\tau_0^* \mid Y$, there is τ_0 – compact sets M_0, N_0 such that $U_0 \cap Y = Y \setminus M_0$ and $W_0 \cap Y = Y \setminus N_0$. Now let $K_0 = M_0 \cup N_0$. Thus K_0 is τ_0 – compact and $Y = Y \setminus (U_0 \cap W_0) = (Y \setminus U_0) \cup (Y \setminus W_0) \subseteq M_0 \cup N_0 = K_0$. Similarly, when we take i =1.

□

Corollary 2.1. *Let (X, τ_0, τ_1) be a biotopological space. Then (X, τ_0, τ_1) is compact if and only if $\tau_0 = \tau_0^*$ and $\tau_1 = \tau_1^*$.*

Remark 2.1. *In general, $\tau_i^* \mid Y \neq (\tau_i \mid Y)^*$ where i=0, 1. For example, let Y be the open interval $(0, 1)$ of $\mathbb{R}$, where τ_u is the usual topology of $\mathbb{R}$, then $\tau_u^* \mid Y \neq (\tau_u \mid Y)^*$ as we can see in [3].*

We have the following obvious results because $\tau_i^* \subseteq \tau_i \quad i = 0, 1$.

Proposition 2.1.

(a) *If (X, τ_0, τ_1) is separable, then (X, τ_0^*, τ_1^*) is separable.*
(b) *If (X, τ_0, τ_1) is hereditarily separable, so is (X, τ_0^*, τ_1^*).*
(c) *If (X, τ_i) is lindelöf, then (X, τ_i^*) is lindelöf $i = 0, 1$.*
(d) *If (X, τ_i) is hereditarily lindelöf so is (X, τ_i^*) $i = 0, 1$.*

Definition 2.2. *A bitopological space (X, τ_0, τ_1) will be called P - Hausdorff if for any two points $x_0 \neq x_1$ there exists τ_0 – open set U_0 of x_0 and a τ_1 – open set U_1 of x_1 which are disjoint.*

Proposition 2.2. *If (X, τ_0, τ_1) is a P – Hausdorff space. Let U_0 and U_1 be τ_0 – open and τ_1 – open subsets of X respectively where $U_0 \cap U_1 = \emptyset$, then $U_0^{\tau_1} \cap U_1 = \emptyset$ and $U_1^{\tau_0} \cap U_0 = \emptyset$.*

Proof. Assume that $U_0^{\tau_1} \cap U_1 \neq \emptyset$ and Let $x \in U_0^{\tau_1} \cap U_1$, so $x \in U_0^{\tau_1}$ that means every τ_1 – open neighborhood of x intersects U_0. But U_1 is τ_1 – open set containing x which does not intersect U_0. This contradiction proves that $U_0^{\tau_1} \cap U_1 = \emptyset$. In a similar way we can show that $U_1^{\tau_0} \cap U_0 = \emptyset$. □

Fora and Hdeib [1] show the following proposition:

Proposition 2.3. *If* (X, τ_0, τ_1) *is a* P *- Hausdorff, then both* τ_0 *and* τ_1 *are* T_1 *– topologies.*

Proposition 2.4. *If* (X, τ_0, τ_1) *is a* P *- Hausdorff, then both* τ_0^* *and* τ_1^* *are* T_1 *– topologies*

Proof. Let $x \in X$ then $\{x\}$ is τ_i – compact $\forall i = 0, 1$, because finite sets are compact, we get that $X \setminus \{x\}$ is τ_i^* - open set. Thus, $\{x\}$ is τ_i^* - closed set, that means τ_0^* and τ_1^* are T_1 – topologies. □

Proposition 2.5. (X, τ_0, τ_1) *is not compact if and only if* (X, τ_0^*, τ_1^*) *is not Hausdorff. Also, if* (X, τ_0, τ_1) *is not compact, then any two non – empty* τ_0^* *- open sets have a non-empty intersection or any two non-empty* τ_1^* *- open sets have a non-empty intersection.*

Proof. Let (X, τ_0, τ_1) be not compact, thus (X, τ_0) is not compact space or (X, τ_1) is not compact. Suppose that (X, τ_0) is not compact space and U_0, V_0 be any two non-empty τ_0^* -open sets, then $X \setminus U_0$ and $X \setminus V_0$ are τ_0 – compact, thus $(X \setminus U_0)$ $\cup$ $(X \setminus V_0)$ is τ_0 – compact. Hence, $X \neq (X \setminus U_0) \cup (X \setminus V_0) = X \setminus (U_0 \cap V_0)$. Thus $(U_0 \cap V_0) \neq \emptyset$; so (X, τ_0^*, τ_1^*) is not Hausdorff. Similarly, when we suppose that (X, τ_1) is not compact we get $(X, \tau_0^*, \tau_1^*))$ is not Hausdorff. Conversely, suppose that (X, τ_0^*, τ_1^*) is not Hausdorff, so we get $\tau_0 \neq \tau_0^*$ or $\tau_1 \neq \tau_1^*$ because (X, τ_0, τ_1) is Hausdorff, thus by Corollary 2.1 we get (X, τ_0, τ_1) is not compact. □

Corollary 2.2. *If* (X, τ_0, τ_1) *is not compact, then we have the following results:*

(a) *Every* τ_i^* *- open set is connected in* (X, τ_i^*) *where* $i = 0, 1$.

(b) (X, τ_i^*) *is connected and Locally connected where* $i = 0, 1$.

Proof. Let (X, τ_0, τ_1) be not compact, so (X, τ_0) is not compact or (X, τ_1) is not compact. Suppose that (X, τ_0) is not compact and $\emptyset \neq V_0 \in \tau_0^*$. If V_0 were disconnected in (X, τ_0^*), there is a pair U_0, W_0 of non-empty disjoint τ_0^* - open sets which is contradiction by Proposition 2.5. Thus, V_0 is connected in (X, τ_0^*). Similarly, when we take (X, τ_1) is not compact. Hence, (a) holds and clearly, (b) follows from (a). □

Definition 2.3. *A space* (X, τ_0, τ_1) *is said to be pairwise irreducible or pairwise hyper connected if all* τ_i *– pen sets are connected in* (X, τ_i) *where* $i = 0, 1$.

Remark 2.2. *By Corollary 2.2, if* (X, τ_0, τ_1) *is not compact, then the space* (X, τ_0^*, τ_1^*) *is pairwise hyper connected.*

Definition 2.4. *A subset G of X, where (X, τ_0, τ_1) is a bitopological space, will be called pairwise G_δ – set if $G = \bigcap \{G_n : 1 \leq n \leq \infty\}$, where G_n is pairwise open set.*

Definition 2.5. *A subset A of X, where (X, τ_0, τ_1) is a bitopological space, will be called pairwise σ^* - compact if $A = \bigcup \{C_n : 1 \leq n \leq \infty\}$, where C_n is τ_0 - open set and $A = \bigcup \{K_n : 1 \leq n \leq \infty\}$, where K_n is τ_1 - open set.*

Although some other results can be said about compact complement bitopology, we want to finish our paper with the following theorem:

Theorem 2.3. *Let $x_0 \in X$. If $\{x_0\}$ is pairwise G_δ – set in (X, τ_0^*, τ_1^*), then $X \setminus \{x_0\}$ is pairwise σ^* - compact set of (X, τ_0, τ_1).*

Proof. Assume that $\{x_0\} = \bigcap_{n\in\omega} V_n$ such that $\{V_n : n \in \omega\}$ is a family of pairwise open sets in (X, τ_0^*, τ_1^*). Then V_n are both τ_0^* - open and τ_1^* - open in X. Hence, $V_n = X \setminus M_n$ where M_n are τ_0 – compact sets and $V_n = X \setminus N_n$ where N_n are τ_1 – compact sets. Thus, the sets $M_n = X \setminus V_n$ are all τ_0 – compact sets and $N_n = X \setminus V_n$ are all τ_1 – compact sets, so $X \setminus \{x_0\} = \bigcap_{n\in\omega} M_n$ and $X \setminus \{x_0\} = \bigcap_{n\in\omega} N_n$. Hence, $X \setminus \{x_0\}$ is pairwise σ^* – compact. □

We can find more applications of this study in problems of [4] and [5].

References

[1] A. A. Fora, H. Z. Hdeib: *On pairwise lindelöf spaces*, Revista Colombiana de Matemáticas, **17**(1-2) (1983), 37–57.

[2] J. Kelly: *Bitopological spaces*, Proceedings of the London Mathematical Society, **3**(1) (1963), 71–89.

[3] K. Keremedis, C. Özel, A. Piekosz, M. A. Shumrani, E. Wajch: *Compact complement topologies and k-spaces*, arXiv preprint arXiv:1806.10177, 2018.

[4] D. H. Pahk, B. D. Choi: *Notes on pairwise compactness*, Kyungpook Mathematical Journal, **11**(1) (1971), 45–52.

[5] C. Patty: *Bitopological spaces*, Duke Mathematical Journal, **34**(3) (1967), 387–391.

Department of Mathematics, University of Jordan, Amman, Jordan.
Email address: janatmanasrah@yahoo.com

Department of Mathematics, University of Jordan, Amman, Jordan.
Email address: Zahedib@ju.edu.jo