Advances in Mathematics: Scientific Journal **10** (2021), no.4, 2187–2197
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.33

SOME FIXED POINT RESULTS IN A GENERALIZED G_b-METRIC SPACE

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ABSTRACT. In this paper, we provide a generalization of the G_b-metric and prove some fixed point results of contractive type mappings in this space.

1. INTRODUCTION

Mustafa et al., introduced a new structure of generalized metric spaces which they called *G*-metric spaces as a generalization of metric spaces, to develop and introduce a new fixed point theory for various mappings in these new structure, [1]. Various authors have proved some fixed point theorems in these spaces, [2,3,5].

Recently, Sedghi et al., [4] have introduced D^*-metric spaces which is a modification of the definition of D-metric spaces introduced by Dhage, [6] and proved some basic properties in D^*-metric spaces, [7]. Furthmore, they introduced the concept of S-metric spaces and presented some properties for common fixed point theorem for a self-mapping on complete S-metric spaces, [8].

Using the concepts of G-metric and b-metric, Ahhajani et al, define a new type of metric which they called a G_b-metric. They studied some basic properties of such a metric and prove common fixed point theorem for mappings satisfying

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2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Fixed point, G_b-metric, rectangle inequality.
Submitted: 31.03.2021; *Accepted:* 15.04.2021; *Published:* 28.04.2021.

weakly compatible condition in complete partially ordered G_b-metric spaces and presented a nontrivial example to verify their effectiveness and applicability, [9].

Definition 1.1. *Let X be a nonempty set. A generalization of a metric or G-metric is a function $G : X \times X \times X \to [0, \infty)$ satisfying the following properties, [1]:*

(i) *for all* $x, y, z \in X$ $G(x, y, z) = 0 \iff x = y = z$;

(ii) *for all* $x, y \in X$, $x \neq y$, $0 < G(x, x, y)$;

(iii) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$, $z \neq y$;

(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$ *symmetry in all variables;*

(v) *for all* $x, y, z, w \in X$, $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$.

The pair (X, G) is a G-metric space.

Definition 1.2. *Let X be a nonempty set and $s \geq 1$ be a real number. A generalization of a G-metric is a function $G_b : X \times X \times X \to [0, \infty)$ satisfying the following properties, [9]:*

(i) *for all* $x, y, z \in X$ $G_b(x, y, z) = 0 \iff x = y = z$;

(ii) *for all* $x, y \in X$, $x \neq y$, $0 < G_b(x, x, y)$;

(iii) *for all* $x, y, z \in X$, $z \neq y$ $G_b(x, x, y) \leq G_b(x, y, z)$;

(iv) $G_b(x, y, z) = G_b(x, z, y) = G_b(y, x, z) = \cdots$ *symmetry in all variables;*

(v) *for all* $x, y, z, w \in X$, $G_b(x, y, z) \leq s\,[G_b(x, w, w) + G_b(w, y, z)]$.

The pair (X, G_b) is a generalized b-metric space or G_b-metric space.

Example 1. *Let $X = \mathbb{R}$ then define a mapping $G_b : X \times X \times X \to [0, \infty)$ by $G_b(x, y, z) = \frac{1}{9}(|x - y| + |y - z| + |x - z|)^2$. Then (X, G_b) is a G_b-metric space, [9].*

Definition 1.3. *Let X be a nonempty set and $\alpha, \beta \geq 1$ are real numbers. A generalization of a G_b-metric is a function $G_b^{\alpha\beta} : X \times X \times X \to [0, \infty)$ satisfying the following properties:*

(i) *for all* $x, y, z \in X$ $G_b^{\alpha\beta}(x, y, z) = 0 \iff x = y = z$;

(ii) *for all* $x, y \in X$, $x \neq y$, $0 < G_b^{\alpha\beta}(x, x, y)$;

(iii) *for all* $x, y, z \in X$, $z \neq y$, $G_b^{\alpha\beta}(x, x, y) \leq G_b^{\alpha\beta}(x, y, z)$;

(iv) $G_b^{\alpha\beta}(x, y, z) = G_b^{\alpha\beta}(x, z, y) = G_b^{\alpha\beta}(y, x, z) = \cdots$ *symmetry in all variables;*

(v) *for all* $x, y, z, w \in X$, $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(x, w, w) + \beta G_b^{\alpha\beta}(w, y, z)$.

The pair $(X, G_b^{\alpha\beta})$ is an α, β generalized b-metric space or $G_b^{\alpha\beta}$-metric space. If $\alpha = \beta = 1$ then $G^{\alpha\beta} = G$. If $\alpha = \beta = s$ then $G_b^{\alpha\beta} = G_b$. For every generalized

b-metric G_b it is not always possible to find an $\alpha, \beta \geq 1$ such that $1 \leq \alpha, \beta < s$ satisfying the property (v) of definition 1.3.

Example 2. *Let $X = (1, 3)$ then define $G_b^{\alpha\beta} : X \times X \times X \to [0, \infty)$ by*

$$G_b^{\alpha\beta}(x, y, z) = \begin{cases} e^{|x-y|+|y-z|+|z-x|}, & x \neq y \neq z, \\ 0, & x = y = z, \end{cases}$$

To show that $G_b^{\alpha\beta}(x, y, z)$ is a $G_b^{\alpha\beta}$-metric we verify properties (i)-(v) of definition 1.3. Properties (i)-(iv) are easily verified. We only verify property (v) of definition 1.3. Let $x, y, z \in X$ such that $x \neq y \neq z$ then

$$\begin{aligned} & G_b^{\alpha\beta}(x, y, z) \\ & \leq e^{|x-w|+|y-w|+|y-z|+|z-w|+|x-w|} \\ & \leq e^{2|x-w|+|y-w|+|y-z|+|z-w|} \\ & \leq \sup_{x,y,z \in X} e^{\frac{2}{3}|2(x-w)|+\frac{1}{3}[|y-w|+|y-z|+|z-w|]} e^{\frac{1}{3}|2(x-w)|+\frac{2}{3}[|y-w|+|y-z|+|z-w|]} \\ & \leq \frac{e^{\frac{14}{3}}}{3} e^{|2(x-w)|} + \frac{2e^{\frac{14}{3}}}{3} e^{[|y-w|+|y-z|+|z-w|]} \\ & = \frac{e^{\frac{14}{3}}}{3} G_b^{\alpha\beta}(x, w, w) + \frac{2e^{\frac{14}{3}}}{3} G_b^{\alpha\beta}(w, y, z), \end{aligned}$$

with $\alpha = \frac{e^{\frac{14}{3}}}{3} \geq 1$ and $\beta = \frac{2e^{\frac{14}{3}}}{3} \geq 1$, $\alpha \neq \beta$ and $\alpha < \beta$.

2. PROPERTIES

The following properties can be deduced from the definition 1.3.

Proposition 2.1. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space. For all $x, y, z \in X$:*

(i) $G_b^{\alpha\beta}(x, y, z) \leq \alpha G_b^{\alpha\beta}(y, x, x) + \beta G_b^{\alpha\beta}(x, x, z)$;
(ii) $G_b^{\alpha\beta}(x, y, y) \leq (\alpha + \beta) G_b^{\alpha\beta}(y, x, x)$.

Definition 2.1. *A $G_b^{\alpha\beta}$-metric $G_b^{\alpha\beta}$ is symmetric if $G_b^{\alpha\beta}(x, x, y) = G_b^{\alpha\beta}(x, y, y)$ for all $x, y \in X$.*

Definition 2.2. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space then for $x_0 \in X$, $\epsilon > 0$, a $G_b^{\alpha\beta}$ ball with center x_0 and radius ϵ is*

$$B_{G_b^{\alpha\beta}}(x_0, \epsilon) = \left\{ y \in X; G_b^{\alpha\beta}(x_0, y, y) < \epsilon \right\}$$

.

Proposition 2.2. *Let $x_0 \in X$, $\epsilon > 0$ and if $y \in B_{G_b^{\alpha\beta}}(x_0, \epsilon)$ then there exists $\delta > 0$ and $c \geq 1$ such that $B_{G_b^{\alpha\beta}}(y, \delta) \subset B_{G_b^{\alpha\beta}}(x_0, c\epsilon)$.*

Proof. Let $y \in B_{G_b^{\alpha\beta}}(x_0, \epsilon)$ then $G_b^{\alpha\beta}(x_0, y, y) < \epsilon$ and taking $\delta = \epsilon - G_b^{\alpha\beta}(x_0, y, y) > 0$. Now, let $w \in B_{G_b^{\alpha\beta}}(y, \delta)$ then $G_b^{\alpha\beta}(y, w, w) < \delta$. Then it follows that

$$\begin{aligned} G_b^{\alpha\beta}(x_0, w, w) &\leq \alpha G_b^{\alpha\beta}(x_0, y, y) + \beta G_b^{\alpha\beta}(y, w, w) \\ &\leq \alpha G_b^{\alpha\beta}(x_0, y, y) + \beta(\epsilon - G_b^{\alpha\beta}(x_0, y, y)) \\ &\leq \alpha G_b^{\alpha\beta}(x_0, y, y) + \beta\epsilon \\ &\leq (\alpha + \beta)\epsilon. \end{aligned}$$

Thus $w \in B_{G_b^{\alpha\beta}}(x_0, (\alpha + \beta)\epsilon)$. Since $\alpha, \beta \geq 1$ and taking $c = \alpha + \beta \geq 2$ we conclude. □

Definition 2.3. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-space and $\{x_n\}$ a sequence in X:*

- (i) *The sequence $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_k) < \epsilon$ for all $n, m, k \geq N$.*
- (ii) *The sequence $\{x_n\}$ is a $G_b^{\alpha\beta}$-convergent sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ and $x \in X$ such that $G_b^{\alpha\beta}(x_n, x_m, x) < \epsilon$ for all $n, m \geq N$.*

Proposition 2.3. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space. The sequence $\{x_n\}$ is $G_b^{\alpha\beta}$-Cauchy $\iff$ for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.*

Proof.

$\Longrightarrow$: From definition 2.3 it follows easily if we take $k = m$.

$\Longleftarrow$: If $\epsilon_{\alpha\beta} = \frac{\epsilon}{2\max\{\alpha,\beta\}}$ then $\epsilon_{\alpha\beta} > 0$ for any $\epsilon > 0$ then there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_m, x_m) < \epsilon_{\alpha\beta}$ for all $n, m \geq N$. From definition 1.3, property (iv) and (v), we get

$$\begin{aligned} G_b^{\alpha\beta}(x_n, x_m, x_k) &\leq \alpha G_b^{\alpha\beta}(x_n, x_m, x_m) + \beta G_b^{\alpha\beta}(x_m, x_m, x_k) \\ &< \alpha \left(\frac{\epsilon}{2\max\{\alpha, \beta\}} \right) + \beta \left(\frac{\epsilon}{2\max\{\alpha, \beta\}} \right) \\ &= \epsilon \end{aligned}$$

for all $n, m, k \geq N$. □

Proposition 2.4. *Let $(X, G_b^{\alpha\beta})$ be a $G_b^{\alpha\beta}$-metric space. The following statements are equivalent:*

(a) *If* $\{x_n\}$ *is a* $G_b^{\alpha\beta}$*-convergent sequence.*
(b) *for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $G_b^{\alpha\beta}(x_n, x_n, x) < \epsilon$ *for all* $n \geq N$.
(c) *for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $G_b^{\alpha\beta}(x_n, x, x) < \epsilon$ *for all* $n \geq N$.

Proposition 2.5. *Let* $(X, G_b^{\alpha\beta})$ *be a* $G_b^{\alpha\beta}$*-metric space. Then the following statements are equivalent:*

(i) $\{x_n\}$ *is* $G_b^{\alpha\beta}$*-convergent to* $x \in X$.
(ii) $G_b^{\alpha\beta}(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
(iii) $G_b^{\alpha\beta}(x_n, x, x) \to 0$ *as* $n \to \infty$.

Definition 2.4. *A* $G^{\alpha\beta}$*-metric space* $(X, G_b^{\alpha\beta})$ *is* $G_b^{\alpha\beta}$ *complete if every* $G_b^{\alpha\beta}$*- Cauchy sequence is* $G^{\alpha\beta}$*-convergent.*

3. Fixed Point Theorems

Theorem 3.1. *Let* $(X, G_b^{\alpha\beta})$ *be a* $G_b^{\alpha\beta}$*-complete metric space. If a mapping* $T : X \to X$ *satisfies the following*

$$G_b^{\alpha\beta}(Tx, Ty, Tz) \leq \lambda G_b^{\alpha\beta}(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$*, where* $0 \leq \lambda < \frac{1}{\beta}$*. Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. Then it follows from inequality 3.1, for the sequence $\{x_n\}$ we get

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) = G_b^{\alpha\beta}(Tx_{n-1}, Tx_n, Tx_n) \leq \lambda G_b^{\alpha\beta}(x_{n-1}, x_n, x_n). \tag{3.2}$$

Repeatedly applying inequality 3.2, we get

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G^{\alpha\beta}(x_0, x_1, x_1). \tag{3.3}$$

For $n, m \in \mathbb{N}$ and proposition 2.3, we get

$$\begin{aligned} &G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) \\ &\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \end{aligned}$$

$$
\begin{aligned}
&+ \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \alpha\beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\
&\leq \alpha \left(\lambda^n + \lambda^{n+1}\beta + \cdots + \lambda^{n+m-1}\beta^{m-1}\right) G_b^{\alpha\beta}(x_0, x_1, x_1) \\
&\leq \alpha\lambda^n \frac{1}{1-\beta\lambda} G_b^{\alpha\beta}(x_0, x_1, x_1).
\end{aligned}
$$

Since $\lambda < \frac{1}{\beta}$ we conclude that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) < \epsilon$ for $n \geq N$, thus $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence in $(X, G_b^{\alpha\beta})$. Since $(X, G_b^{\alpha\beta})$ is a complete-$G_b^{\alpha\beta}$ metric space there exists x^* and $N_1 \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_n, x^*) < \epsilon$ for $n \geq N_1$. To show that x^* is a fixed point of T. Using the contraction condition we get

$$
\begin{aligned}
G_b^{\alpha\beta}(x_{n+1}, Tx^*, Tx^*) &= G_b^{\alpha\beta}(Tx_n, Tx^*, Tx^*) \\
&\leq \lambda G_b^{\alpha\beta}(x_n, x^*, x^*).
\end{aligned}
$$

Taking the limit as $n \to \infty$, we get $G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) = 0$. Thus $Tx^* = x^*$.

To prove uniqueness we assume that T has fixed points x^* and x^{**}. Then it follows that

$$
G_b^{\alpha\beta}(x^{**}, x^*, x^*) = G_b^{\alpha\beta}(Tx^{**}, Tx^*, Tx^*) \leq \lambda G_b^{\alpha\beta}(x^{**}, x^*, x^*).
$$

Since $0 \leq \lambda < 1$, we get $x^{**} = x^*$. □

Theorem 3.2. *Let $X, G_b^{\alpha\beta}$ be a $G_b^{\alpha\beta}$-complete metric space and a mapping $T : X \to X$ such that*

$$
\begin{aligned}
&G_b^{\alpha\beta}(Tx, Ty, Tz) \\
&\leq \lambda \max\left\{G^{\alpha\beta}(x, y, z), G^{\alpha\beta}(x, Tx, Tx), G^{\alpha\beta}(y, Ty, Ty),\right. \\
&\quad \left. G^{\alpha\beta}(z, Tz, Tz), G^{\alpha\beta}(x, Ty, Ty), G^{\alpha\beta}(y, Tz, Tz), G^{\alpha\beta}(z, Tx, Tx)\right\},
\end{aligned} \tag{3.4}
$$

for all $x, y, z \in X$ with $0 \leq \lambda < \frac{1}{\alpha+\beta}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ define a sequence $x_{n+1} = Tx_n$ then for sequence $\{x_n\}$, we get from inequality 3.4,

$$\begin{aligned} &G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\ &\leq \lambda \max\Big\{G_b^{\alpha\beta}(x_{n-1}, x_n, x_n), G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}), G_b^{\alpha\beta}(x_{n-1}, x_{n+1}, x_{n+1}), \\ &\qquad G_b^{\alpha\beta}(x_n, x_n, x_n)\Big\} \\ &\leq \lambda \max\Big\{G_b^{\alpha\beta}(x_{n-1}, x_n, x_n), G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}), \alpha G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) \\ &\qquad +\beta G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1})\Big\} \end{aligned}$$

$$= \lambda\alpha G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda\beta G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}). \tag{3.5}$$

From inequality 3.5, and that $\lambda < \frac{1}{\alpha+\beta}$, we get

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \leq \frac{\lambda\alpha}{1-\lambda\beta} G^{\alpha\beta}(x_{n-1}, x_n, x_n). \tag{3.6}$$

For $n, m \in \mathbb{N}$, recussively applying inequality 3.6, we get

$$\begin{aligned} &G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) \\ &\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\ &\leq \alpha G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \alpha\beta G_b^{\alpha\beta}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \alpha\beta^2 G_b^{\alpha\beta}(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \alpha\beta^{m-1} G_b^{\alpha\beta}(x_{n+m-1}, x_{n+m}, x_{n+m}) \\ &\leq \alpha \left(\left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^n + \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^{n+1} \beta + \cdots + \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^{n+m-1} \beta^{m-1} \right) \\ &\quad G_b^{\alpha\beta}(x_0, x_1, x_1) \\ &\leq \alpha \left(\frac{\lambda\alpha}{1-\lambda\beta}\right)^n \frac{1-\beta\lambda}{1-\beta\lambda(1+\alpha)} G_b^{\alpha\beta}(x_0, x_1, x_1). \end{aligned}$$

Since $\lambda < \frac{1}{\alpha+\beta}$ we conclude that for every $\epsilon > 0$ there exsits $N \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_{n+m}, x_{n+m}) < \epsilon$ for $n \geq N$, thus $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence in $(X, G_b^{\alpha\beta})$. Since $(X, G_b^{\alpha\beta})$ is a complete-$G_b^{\alpha\beta}$ metric space there exists x^* and $N_1 \in \mathbb{N}$ such that $G_b^{\alpha\beta}(x_n, x_n, x^*) < \epsilon$ for $n \geq N_1$.

To show that x^* is a fixed point of T. For the $G_b^{\alpha\beta}$-convergent sequence $\{x_n\}$, we get

$$\begin{aligned} &G_b^{\alpha\beta}(x_n, Tx^*, Tx^*) \\ &\le \lambda \max\Big\{G_b^{\alpha\beta}(x_{n-1}, x^*, x^*), G_b^{\alpha\beta}(x_{n-1}, Tx^*, Tx^*), \\ &\quad G_b^{\alpha\beta}(x^*, Tx^*, Tx^*), G_b^{\alpha\beta}(x_{n-1}, x_n, x_n), G_b^{\alpha\beta}(x^*, x_n, x_n)\Big\}. \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in inequality 3.7, we get $G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) \le \lambda G_b^{\alpha\beta}(x^*, Tx^*, Tx^*)$. Since $\lambda < \frac{1}{\alpha+\beta} < \frac{1}{2}$ the inequality is only valid if $G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) = 0$ which implies $Tx^* = x^*$. For the uniqueness of the fixed point we assume that $x^{**} \in X$ is a fixed point of T. Then from inequality 3.4, we get

$$\begin{aligned} &G_b^{\alpha\beta}(Tx^*, Tx^*, Tx^{**}) \\ &\le \lambda \max\big\{G^{\alpha\beta}(x^*, x^*, x^{**}), G^{\alpha\beta}(x^*, Tx^*, Tx^*), G^{\alpha\beta}(x^{**}, Tx^{**}, Tx^{**}), \\ &\quad G^{\alpha\beta}(x^{**}, Tx^{**}, Tx^{**}), G^{\alpha\beta}(x^*, Tx^{**}, Tx^{**}), G^{\alpha\beta}(x^*, Tx^{**}, Tx^{**}), \\ &\quad G^{\alpha\beta}(x^{**}, Tx^*, Tx^*)\big\}. \end{aligned} \tag{3.8}$$

Thus, we obtain from proposition 2.1,

$$\begin{aligned} G_b^{\alpha\beta}(x^*, x^*, x^{**}) &\le \lambda \max\Big\{G_b^{\alpha\beta}(x^*, x^*, x^{**}), G_b^{\alpha\beta}(x^*, x^{**}, x^{**})\Big\} \\ &\le \lambda(\alpha+\beta) G^{\alpha\beta}(x^*, x^*, x^{**}). \end{aligned}$$

It follows that

$$[1 - \lambda(\alpha+\beta)]\, G_b^{\alpha\beta}(x^*, x^*, x^{**}) \le 0.$$

Since $1 - \lambda(\alpha+\beta) > 0$, we conclude that $G_b^{\alpha\beta}(x^*, x^*, x^{**}) = 0$ thus $x^* = x^{**}$. □

Theorem 3.3. *Let $X, G_b^{\alpha\beta}$ be a $G_b^{\alpha\beta}$-complete metric space and a mapping $T : X \to X$ such that*

$$\begin{aligned} G_b^{\alpha\beta}(Tx, Ty, Tz) \le{}& \lambda_1 G^{\alpha\beta}(x, y, z) + \lambda_2 G^{\alpha\beta}(x, Tx, Tx) + \lambda_3 G^{\alpha\beta}(y, Ty, Ty) \\ &+ \lambda_4 G^{\alpha\beta}(z, Tz, Tz) + \lambda_5 G^{\alpha\beta}(x, Ty, Ty) + \lambda_6 G^{\alpha\beta}(y, Tz, Tz) \\ &+ \lambda_7 G^{\alpha\beta}(z, Tx, Tx), \end{aligned} \tag{3.9}$$

for all $x, y, z \in X$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + (\alpha+\beta)\lambda_5 + \lambda_6 + \lambda_7 < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$, where $x_{n+1} = Tx_n$. For the sequence $\{x_n\}$ and from inequality 3.9, we get

$$\begin{aligned}
&G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\leq \lambda_1 G^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda_2 G^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda_3 G^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \lambda_4 G^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) + \lambda_5 G^{\alpha\beta}(x_{n-1}, x_{n+1}, x_{n+1}) + \lambda_6 G^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \lambda_7 G^{\alpha\beta}(x_n, x_n, x_n) \\
&\leq (\lambda_1 + \lambda_2) G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + (\lambda_3 + \lambda_4 + \lambda_6) G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \lambda_5 \left(\alpha G_b^{\alpha\beta}(x_{n-1}, x_n, x_n) + \beta G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1} \right). \qquad (3.10)
\end{aligned}$$

From inequality 3.10, we get

$$\begin{aligned}
&[1 - (\lambda_3 + \lambda_4 + \lambda_6) - \lambda_5 \beta]\, G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \\
&\leq (\lambda_1 + \lambda_2 + \lambda_5 \alpha) G_b^{\alpha\beta}(x_{n-1}, x_n, x_n).
\end{aligned}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + (\alpha + \beta)\lambda_5 + \lambda_6 < 1$, we get

$$G_b^{\alpha\beta}(x_n, x_{n+1}, x_{n+1}) \leq \left(\frac{\lambda_1 + \lambda_2 + \lambda_5 \alpha}{1 - (\lambda_3 + \lambda_4 + \lambda_6) - \lambda_5 \beta} \right) G_b^{\alpha\beta}(x_{n-1}, x_n, x_n).$$

Following an argument as in theorem 3.1, we can conclude that the sequence $\{x_n\}$ is a $G_b^{\alpha\beta}$-Cauchy sequence in X. Since $(X, G^{\alpha\beta})$ is complete it follows that the sequence $\{x_n\}$ is $G^{\alpha\beta}$-convergent to $x^* \in X$. To show that x^* is a fixed point for T. From inequality 3.9, we have

$$\begin{aligned}
&G_b^{\alpha\beta}(x_n, Tx^*, Tx^*) \\
&\leq \lambda_1 G^{\alpha\beta}(x_{n-1}, x^*, x^*) + \lambda_2 G^{\alpha\beta}(x_{n-1}, x_n, x_n) + \lambda_3 G^{\alpha\beta}(x^*, Tx^*, Tx^*) \\
&\quad + \lambda_4 G^{\alpha\beta}(x^*, Tx^*, Tx^*) + \lambda_5 G^{\alpha\beta}(x_{n-1}, Tx^*, Tx^*) + \lambda_6 G^{\alpha\beta}(x^*, Tx^*, Tx^*) \\
&\quad + \lambda_7 G^{\alpha\beta}(x^*, x_n, x_n)
\end{aligned}$$

Taking the limit $n \to \infty$ in the above inequality, we get

$$(1 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6) G_b^{\alpha\beta}(x^*, Tx^*, Tx^*) \leq 0.$$

It follows that $Tx^* = x^*$. To prove uniqueness we assume that x^{**} is a fixed point for T. Then from inequality 3.9, we get

$$\begin{aligned} & G_b^{\alpha\beta}(x^*, x^{**}, x^*) \\ & \leq \lambda_1 G^{\alpha\beta}(x^*, x^{**}, x^*) + \lambda_2 G^{\alpha\beta}(x^*, Tx^*, Tx^*) + \lambda_3 G^{\alpha\beta}(x^{**}, Tx^{**}, Tx^{**}) \\ & \quad + \lambda_4 G^{\alpha\beta}(x^*, Tx^*, Tx^*) + \lambda_5 G^{\alpha\beta}(x^*, Tx^{**}, Tx^{**}) + \lambda_6 G^{\alpha\beta}(x^{**}, Tx^*, Tx^*) \\ & \quad + \lambda_7 G^{\alpha\beta}(x^*, Tx^*, Tx^*) \end{aligned}$$

It follows that

$$\begin{aligned} G_b^{\alpha\beta}(x^*, x^{**}, x^*) \leq \lambda_1 G_b^{\alpha\beta}(x^*, x^{**}, x^*) + \lambda_5 G_b^{\alpha\beta}(x^*, x^{**}, x^{**}) \\ + \lambda_6 G_b^{\alpha\beta}(x^{**}, x^*, x^*). \end{aligned}$$

Using proposition 2.1, we get

$$(1 - \lambda_1 - \lambda_6 - \lambda_5(\alpha + \beta)) G_b^{\alpha\beta}(x^*, x^{**}, x^*) \leq 0. \tag{3.11}$$

Thus, we get $G_b^{\alpha\beta}(x^*, x^{**}, x^*) = 0$ which implies $x^{**} = x^*$. □

4. Conclusion

In this investigation, we demonstrate that we can relax the rectangle-inequality to obtain a generalization G_b-metric in which the weights $\alpha \geq 1$ and $\beta \geq 1$ are not equal and that there is an improvement in the bounds for the parameters considered in the fixed point theorems with contraction type mappings.

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