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NEW GENERALIZATIONS OF HESITANT AND INTERVAL-VALUED FUZZY IDEALS OF SEMIGROUPS

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ABSTRACT. In this paper, we introduce the notion of sup-hesitant fuzzy ideals of semigroups, which is the general notion of hesitant fuzzy ideals and interval-valued fuzzy ideals. In addition, sup-hesitant fuzzy ideals are characterized in terms of sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. Finally, sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions of sup-hesitant fuzzy ideals of semigoups are discussed, and their relations are investigated.

1. INTRODUCTION

Zadeh [1] introduced the concept of a fuzzy set. Given a set S, a fuzzy subset of S (or a fuzzy set in S) is described as an arbitrary mapping $f\colon S \to [0,1]$ where $[0,1]$ is the usual interval of real numbers. Fuzzy set theory has been widely and successfully applied in many different areas to handle uncertainties. But it presents limitations to deal with imprecise and vague information when different sources of vagueness appear simultaneously. In order to overcome such limitations, different extensions of fuzzy sets have been introduced in the

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literature such as intuitionistic fuzzy sets, type-2 fuzzy sets, rough sets, soft sets, interval-valued fuzzy sets, fuzzy multisets, hesitant fuzzy sets, neuromorphic sets and so on.

Zadeh [2] made an extension of the concept of a fuzzy set by an interval-valued fuzzy set. Biswas [3] defined the interval-valued fuzzy subgroups of Rosenfeld's nature, and investigated some elementary properties. After that, the theory of interval-valued fuzzy sets was studied in semigroups. Narayan and Manikantan [4] introduced and discussed interval-valued fuzzy ideals of semigroups generated by interval-valued fuzzy sets. Thillaigovindan and Chinnadurai [5] studied interval-valued fuzzy interior (quasi-, bi-, two-sided) ideals of semigroups and characterized regular semigroups in terms of interval-valued fuzzy sets. Singaram and Kandasamy [6] characterized intra-regular semigroups by means of interval-valued fuzzy left ideals, right ideals, bi-ideals, interior ideals and ideals. Kar et.al [7] studied interval-valued fuzzy ideals and interval-valued prime fuzzy ideals of semigroups. Interval-valued fuzzy set theory is also applied to other algebraic structures seen in [8–14].

Hesitant fuzzy sets are introduced by Torra and Narukawa [15, 16]as another extension of fuzzy sets. Several researches have applied to algebraic structures, for example, Jun et al. [17] studied hesitant fuzzy left (right, generalized bi-, bi-, two-sided) ideals of semigroups. Talee et al. [18] introduced hesitant fuzzy interior ideals in a semigroup and investigated some properties of hesitant fuzzy ideals, hesitant fuzzy interior ideals and hesitant fuzzy bi-ideals. Phakawat et al. [19] introduced the concepts of $\sup$-hesitant fuzzy UP-subalgebras, $\sup$-hesitant fuzzy UP-filters, $\sup$-hesitant fuzzy UP-ideals, and $\sup$-hesitant fuzzy strongly UP-ideals and discussed their properties. Muhiuddin and Jun [20] introduced $\sup$-hesitant fuzzy subalgebra in BCK/BCI-algebrasis, $\sup$-hesitant fuzzy translation and $\sup$-hesitant fuzzy extension of $\sup$-hesitant fuzzy subalgebras and investigated their related properties. Muhiuddin et al. [21] introduced Sup-hesitant fuzzy ideals in BCK/BCI-algebras, characterized Sup-hesitant fuzzy ideals and investigated relations between Sup-hesitant fuzzy subalgebras and Sup-hesitant fuzzy ideals. Harizavi and Jun [22] introduced the notion of Sup-hesitant fuzzy quasi-associative ideal in BCI-algebras and investigated relations between Sup-hesitant fuzzy ideals and Sup-hesitant fuzzy quasi-associative ideals.

In this paper we introduce the notion of $\sup$-hesitant fuzzy ideals of semigroups, which is the general notion of hesitant fuzzy ideals and interval-valued fuzzy ideals. In addition, $\sup$-hesitant fuzzy ideals are characterized in terms of sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. Finally, $\sup$-hesitant fuzzy translation and $\sup$-hesitant fuzzy extension of $\sup$-hesitant fuzzy ideals of semigoups are discussed, and their relations are investigated.

2. PRELIMINARIES

In this section we first give some basic definitions and results which will be used in this paper.

Let S be a semigroup. By a **left (right) ideal** of S we mean a nonempty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). A nonempty subset of S is called an **ideal** of S if it is both a left and a right ideal of S.

A **fuzzy subset** f [1] in a nonempty set X (or a fuzzy subset of X) is an arbitrary function from the set X into $[0, 1]$ where $[0, 1]$ is the unit segment of the real line. A fuzzy set f of a semigroup S is called a **fuzzy ideal** of S if it satisfies: $\max\{f(x), f(y)\} \leq f(xy)$ for all $x, y \in S$.

Let $\mathcal{D}[0, 1]$ denote the family of all closed subintervals of $[0, 1]$, that is

$$\mathcal{D}[0,1] = \{[a^-, a^+] | a^-, a^+ \in [0,1] \text{ and } a^- \leq a^+\}.$$

We consider two elements $\tilde{a} = [a^-, a^+]$ and $\tilde{b} = [b^-, b^+]$ in $\mathcal{D}[0, 1]$. We denote $[1, 1]$ by $\tilde{1}$ and $[0, 0]$ by $\tilde{0}$. Define the operations $\preceq$, $=$, $\prec$ and **Max** in case of two elements in $\mathcal{D}[0, 1]$ as follow:

(1) $\tilde{a} \preceq \tilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(2) $\tilde{a} = \tilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,
(3) $\tilde{a} \prec \tilde{b}$ if and only if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \neq \tilde{b}$,
(4) **Max**$\{\tilde{a}, \tilde{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Let X be a nonempty set. A mapping $\tilde{\mu} : X \to \mathcal{D}[0, 1]$ is called an **interval-valued fuzzy subset** [2] on X, where $\tilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in X$, μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$ for all $x \in X$. An interval-valued fuzzy set $\tilde{\mu}$ on a semigroup S is called an **interval-valued fuzzy ideal** [4, 5] of S if it satisfies: **Max**$\{\tilde{\mu}(x), \tilde{\mu}(y)\} \preceq \tilde{\mu}(xy)$ for all $x, y \in S$.

Torra [15, 16] defined hesitant fuzzy subset on a nonempty set X in terms of a function h that when applied to X returns a subset of $[0, 1]$, that is, $h : X \to \mathcal{P}[0, 1]$ where $\mathcal{P}[0, 1]$ denote the set of all subset of $[0, 1]$. A hesitant fuzzy set h on a semigroup S is called a **hesitant fuzzy ideal** [17, 18] of S if it satisfies: $h(x) \cup$

$h(y) \subseteq h(xy)$ for all $x, y \in S$. It is well known that every interval-valued fuzzy set on an arbitrary semigroup S is a hesitant fuzzy set on S and the concepts of a hesitant fuzzy ideal and an interval-valued fuzzy ideal of S do not coincide, seen in the following example.

Example 1. *Let $S = \{a, b, c, d\}$ and define a binary operation " $\cdot$ " on S as follows:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then S is a semigroup under the binary operation " $\cdot$ " .

(1) *Define a hesitant fuzzy set h on S by $h(a) = [0, 1]$, $h(b) = [0, 0.7]$, $h(c) = \emptyset$ and $h(d) = \{0, 0.2, 0.3\}$. Then h is a hesitant fuzzy ideal of S but not an interval-valued fuzzy ideal of S because h is not an interval-valued fuzzy set on S.*

(2) *Define an interval-valued fuzzy set $\tilde{\mu}$ on S by $\tilde{\mu}(a) = \tilde{1}$, $\tilde{\mu}(b) = [0.3, 0.7]$ and $\tilde{\mu}(c) = \tilde{\mu}(d) = \tilde{0}$. Then $\tilde{\mu}$ is an interval-valued fuzzy ideal of S and a hesitant fuzzy set on S. However, $\tilde{\mu}$ is not a hesitant fuzzy ideal of S because $\tilde{\mu}(d) \cup \tilde{\mu}(c) = \tilde{0} = \{0\} \not\subseteq [0.3, 0.7] = \tilde{\mu}(b) = \tilde{\mu}(dc)$.*

In this paper, we introduce a new concep of a hesitant fuzzy set on a semigroup is a generalization of a hesitant and an interval-valued fuzzy ideal, and investigate its related properties.

3. sup-HESITANT FUZZY IDEALS OF SEMIGROUPS

In this section, we introduce the concept of a sup-hesitant fuzzy ideal, which is the general concept of a hesitant and an interval-valued fuzzy ideal. In addition, characterizations of sup-hesitant fuzzy ideals are investigated by sets, fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. We also talk about relationship between ideals and the general idea of âĂŃâĂŃ the characteristic hesitant and interval-valued fuzzy sets. Finally, sup-hesitant fuzzy translations and sup-hesitant fuzzy extensions of sup-hesitant fuzzy ideals of semigroups are discussed.

In what follows, the power set of $[0,1]$ is denoted by $\mathcal{P}[0,1]$. For any element Δ of $\mathcal{P}[0,1]$, define SupΔ by

$$\text{Sup}\Delta = \begin{cases} \sup\Delta, & \text{if } \Delta \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Then Sup$\tilde{\mu}(x) = \sup\tilde{\mu}(x) = \mu^+(x)$ for every interval-valued fuzzy set $\tilde{\mu}$ on a nonempty set X and for all $x \in X$. For any hesitant fuzzy set h on X and any element Δ of $\mathcal{P}[0,1]$, consider the set

$$\mathcal{S}[h;\Delta] = \{x \in X \mid \text{Sup}h(x) \geq \text{Sup}\Delta\}.$$

Definition 3.1. *A hesitant fuzzy set h on a semigroup S is called a* sup***-hesitant fuzzy ideal of S related to*** *Δ (briefly, Δ-*sup*-hesitant fuzzy ideal of S) if the set $\mathcal{S}[h;\Delta]$ is an ideal of S. We say that h is a* sup***-hesitant fuzzy ideal*** *of S if h is a Δ-*sup*-hesitant fuzzy ideal of S for all $\Delta \in \mathcal{P}[0,1]$ when $\mathcal{S}[h;\Delta] \neq \emptyset$.*

Lemma 3.1. *Every interval-valued fuzzy ideal of a semigroup is a* sup*-hesitant fuzzy ideal.*

Proof. Assume that $\tilde{\mu}$ is an interval-valued fuzzy ideal of a semigroup S and $\Delta \in \mathcal{P}[0,1]$ such that $\mathcal{S}[\tilde{\mu};\Delta] \neq \emptyset$. Let $x \in S$ and $y \in \mathcal{S}[\tilde{\mu};\Delta]$. Then $\sup\tilde{\mu}(y) \geq$ SupΔ. Since $\tilde{\mu}$ is an interval-valued fuzzy ideal of S, we have $\tilde{\mu}(y) \preceq \tilde{\mu}(xy)$ and $\tilde{\mu}(y) \preceq \tilde{\mu}(yx)$. Thus Sup$\Delta \leq \sup\tilde{\mu}(y) = \mu^+(y) \leq \mu^+(xy) = \sup\tilde{\mu}(xy)$ and Sup$\Delta \leq \sup\tilde{\mu}(y) = \mu^+(y) \leq \mu^+(yx) = \sup\tilde{\mu}(yx)$. Thus $xy, yx \in \mathcal{S}[\tilde{\mu};\Delta]$. Hence $\mathcal{S}[\tilde{\mu};\Delta]$ is an ideal of S and so $\tilde{\mu}$ is a Δ-sup-hesitant fuzzy ideal of S. Therefore $\tilde{\mu}$ is a sup-hesitant fuzzy ideal of S. □

Lemma 3.2. *Every hesitant fuzzy ideal of a semigroup is a* sup*-hesitant fuzzy ideal.*

Proof. Assume that h is a hesitant fuzzy ideal of a semigroup S and $\Delta \in \mathcal{P}[0,1]$ such that $\mathcal{S}[h;\Delta] \neq \emptyset$. Let $x \in S$ and $y \in \mathcal{S}[h;\Delta]$. Then Sup$h(y) \geq$ SupΔ. Since h is a hesitant fuzzy ideal of S, we have $h(y) \subseteq h(xy)$ and $h(y) \subseteq h(yx)$. Thus Sup$h(y) \leq$ Sup$h(xy)$ and Sup$h(y) \leq$ Sup$h(yx)$, which implies that $xy, yx \in \mathcal{S}[h;\Delta]$. Hence $\mathcal{S}[h;\Delta]$ is an ideal of S which implies that h is a Δ-sup-hesitant fuzzy ideal of S. Therefore h is a sup-hesitant fuzzy ideal of S. □

The following example shows that the converses of Lemma 3.1 and Lemma 3.2 do not hold in general.

Example 2. *Let $S = \{a, b, c, d\}$ and define a binary operation " $\cdot$ " on S as follows:*

$\cdot$	a	b	c	d
a	b	b	d	d
b	b	b	d	d
c	d	d	c	d
d	d	d	d	d

Then S is a semigroup under the binary operation " $\cdot$ " .

(1) *Define a hesitant fuzzy set h on S by $h(b) = h(d) = \emptyset$ and $h(a) = h(c) = \tilde{0}$. Then h is a* sup*-hesitant fuzzy ideal of S but not a hesitant fuzzy ideal of S because $h(a) \cup h(c) = \tilde{0} = \{0\} \not\subseteq \emptyset = h(d) = h(ac)$.*

(2) *Define an interval-valued fuzzy set $\tilde{\mu}$ on S by $\tilde{\mu}(a) = \tilde{\mu}(b) = \tilde{\mu}(c) = \tilde{1}$ and $\tilde{\mu}(d) = [0, 1]$. Then $\tilde{\mu}$ is a* sup*-hesitant fuzzy ideal of S but not an interval-valued fuzzy ideal of S because $\tilde{\mu}(bc) = \tilde{\mu}(d) = [0, 1] \prec \tilde{1} =$ **Max**$\{\tilde{\mu}(b), \tilde{\mu}(c)\}$.*

By Lemma 3.1, Lemma 3.2, and Example 2, we have that a sup-hesitant fuzzy ideal of an arbitrary semigroup is a generalization of a hesitant and an interval-valued fuzzy ideal.

Let h be a hesitant fuzzy set on a semigroup S and define the fuzzy set $\mathcal{F}_h$ of S by $\mathcal{F}_h : S \to [0, 1], x \mapsto \text{Sup}h(x)$. The following lemma, we characterize a sup-hesitant fuzzy ideal h of a semigroup S by the fuzzy set $\mathcal{F}_h$.

Lemma 3.3. *A hesitant fuzzy set h on a semigroup S is a* sup*-hesitant fuzzy ideal of S if and only if $\mathcal{F}_h$ is a fuzzy ideal of S.*

Proof.

($\Rightarrow$) Assume that h is a sup-hesitant fuzzy ideal of S. Let $x, y \in S$ and $\Delta = h(y)$. Then $y \in \mathcal{S}[h; \Delta]$. Thus h is a Δ-sup-hesitant fuzzy ideal of S which imples that $\mathcal{S}[h; \Delta]$ is an ideal of S. Hence $xy, yx \in \mathcal{S}[h; \Delta]$ and so

$$\mathcal{F}_h(yx) = \text{Sup}h(yx) \geq \text{Sup}\Delta = \text{Sup}h(y) = \mathcal{F}_h(y).$$

Hence $\mathcal{F}_h(yx) \geq \mathcal{F}_h(y)$ and similarly, we have $\mathcal{F}_h(xy) \geq \mathcal{F}_h(y)$. Therefore $\mathcal{F}_h$ is a fuzzy ideal of S.

($\Leftarrow$) Assume that $\mathcal{F}_h$ is a fuzzy ideal of S and $\Delta \in \mathcal{P}[0, 1]$ such that $\mathcal{S}[h; \Delta] \neq \emptyset$. Let $x \in S$ and $y \in \mathcal{S}[h; \Delta]$. Then

$$\text{Sup}h(xy) = \mathcal{F}_h(xy) \geq \mathcal{F}_h(y) = \text{Sup}h(y) \geq \text{Sup}\Delta,$$

which implies that $xy \in \mathcal{S}[h;\Delta]$. Similarly, we have $yx \in \mathcal{S}[h;\Delta]$. Hence $\mathcal{S}[h;\Delta]$ is an ideal of S, that is h is a Δ-$\sup$-hesitant fuzzy ideal of S. Therefore h is a $\sup$-hesitant fuzzy ideal of S. □

Let h be a hesitant fuzzy set on a semigroup S and Δ an element of $\mathcal{P}[0,1]$, we define the hesitant fuzzy set $\mathcal{H}(h;\Delta)$ on S by

$$\mathcal{H}(h;\Delta) : S \to \mathcal{P}[0,1], x \mapsto \{t \in \Delta \,|\, \text{Sup}h(x) \geq t\}.$$

We denote $\mathcal{H}(h; \bigcup_{x\in S} h(x))$ by $\mathcal{H}_h$ and denote $\mathcal{H}(h;[0,1])$ by $\mathcal{I}_h$. Then $\mathcal{I}_h$ is an interval-valued fuzzy set on S.

Remark 3.1. *If h is a hesitant fuzzy set on a semigroup S, then $h(x) \subseteq \mathcal{H}_h(x) \subseteq \mathcal{I}_h(x)$ and Sup$h(x) =$ Sup$\mathcal{H}_h(x) = \sup \mathcal{I}_h(x)$ for all $x \in S$.*

Now, we study a $\sup$-hesitant fuzzy ideal h of a semigroup S by the hesitant fuzzy set $\mathcal{H}(h;\Delta)$.

Lemma 3.4. *A hesitant fuzzy set h on a semigroup S is a $\sup$-hesitant fuzzy ideal of S if and only if $\mathcal{H}(h;\Delta)$ is a hesitant fuzzy ideal of S for all $\Delta \in \mathcal{P}[0,1]$.*

Proof. ($\Rightarrow$) Let $\Delta \in \mathcal{P}[0,1]$ and $x, y \in S$. If $\mathcal{H}(h;\Delta)(x) \cup \mathcal{H}(h;\Delta)(y)$ is empty, then $\mathcal{H}(h;\Delta)(x) \cup \mathcal{H}(h;\Delta)(y) \subseteq \mathcal{H}(h;\Delta)(xy)$. On the other hand, let $t \in \mathcal{H}(h;\Delta)(x) \cup \mathcal{H}(h;\Delta)(y)$. Then

$$\text{Sup}\{h(x) \cup h(y)\} \geq \max\{\text{Sup}h(x), \text{Sup}h(y)\} \geq t \in \Delta.$$

Thus $x \in \mathcal{S}[h; h(x) \cup h(y)]$ or $y \in \mathcal{S}[h; h(x) \cup h(y)]$. Since h is a $\sup$-hesitant fuzzy ideal of S, we have $\mathcal{S}[h; h(x) \cup h(y)]$ is an ideal of S. Hence $xy \in \mathcal{S}[h; h(x) \cup h(y)]$ which imlpies that

$$\text{Sup}h(xy) \geq \text{Sup}\{h(x) \cup h(y)\} \geq \max\{\text{Sup}h(x), \text{Sup}h(y)\} \geq t.$$

Thus $t \in \mathcal{H}(h;\Delta)(xy)$. Therefore $\mathcal{H}(h;\Delta)(x) \cup \mathcal{H}(h;\Delta)(y) \subseteq \mathcal{H}(h;\Delta)(xy)$. Consequently, we have $\mathcal{H}(h;\Delta)$ is a hesitant fuzzy ideal of S.

($\Leftarrow$) Assume that $\mathcal{H}(h;\Delta)$ is a hesitant fuzzy ideal of S for all $\Delta \in \mathcal{P}[0,1]$. Let $x, y \in S$ and $\Delta \in \mathcal{P}[0,1]$ be such that $y \in \mathcal{S}[h;\Delta]$. Then $\mathcal{H}(h;\Delta)(y) = \Delta$ and by using assumption, we have

$$\Delta = \mathcal{H}(h;\Delta)(y) \subseteq \mathcal{H}(h;\Delta)(x) \cup \mathcal{H}(h;\Delta)(y) \subseteq \mathcal{H}(h;\Delta)(xy)$$

and so $\Delta \subseteq \mathcal{H}(h;\Delta)(xy)$. Similarly, we obtain that $\Delta \subseteq \mathcal{H}(h;\Delta)(yx)$. Thus Sup$h(xy) \geq$ SupΔ and Sup$h(yx) \geq$ SupΔ, which implies that $xy, yx \in \mathcal{S}[h;\Delta]$.

Hence $\mathcal{S}[h;\Delta]$ is an ideal of S, that is h is a Δ-$\sup$-hesitant fuzzy ideal of S. Therefore h is a $\sup$-hesitant fuzzy ideal of S. □

The following theorem, characterizations of $\sup$-hesitant fuzzy ideals of semigroups are shown in terms of sets, fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets.

Theorem 3.1. *For any hesitant fuzzy set h on a semigroup S, the following statements are equivalent.*

(1) *h is a $\sup$-hesitant fuzzy ideal of S.*
(2) *$\mathcal{H}_h$ is a hesitant fuzzy ideal of S.*
(3) *$\mathcal{H}_h$ is a $\sup$-hesitant fuzzy ideal of S.*
(4) *$\mathcal{I}_h$ is an interval-valued fuzzy ideal of S.*
(5) *$\mathcal{I}_h$ is a $\sup$-hesitant fuzzy ideal of S.*
(6) *$\mathcal{I}_h$ is a hesitant fuzzy ideal of S.*
(7) *$\mathcal{F}_h$ is a fuzzy ideal of S.*
(8) *Sup$h(xy) \geq \max\{$Sup$h(x),$ Sup$h(y)\}$ for all $x, y \in S$.*
(9) *The set $U_{\sup}(h;t) = \{x \in S \mid$ Sup$h(x) \geq t\}$ is either empty or an ideal of S for all $t \in [0,1]$.*

Proof.
$(1) \Leftrightarrow (7) \Leftrightarrow (8)$. It follows from Lemma 3.3.
$(1) \Rightarrow (2)$ and $(1) \Rightarrow (6)$. It follows from Lemma 3.4.
$(2) \Rightarrow (3)$ and $(6) \Rightarrow (5)$. It follows from Lemma 3.2.
$(4) \Rightarrow (5)$. It follows from Lemma 3.1.
$(1) \Leftrightarrow (9)$. Straightforward.

$(3) \Rightarrow (1)$. Let $\Delta \in \mathcal{P}[0,1]$ and $x, y \in S$ be such that $y \in \mathcal{S}[h;\Delta]$. By using Remark 3.1, we have Sup$\mathcal{H}_h(y) =$ Sup$h(y) \geq$ SupΔ and so $y \in \mathcal{S}[\mathcal{H}_h;\Delta]$. By assumption, we have $\mathcal{S}[\mathcal{H}_h;\Delta]$ is an ideal of S and then $xy, yx \in \mathcal{S}[\mathcal{H}_h;\Delta]$. By using again Remark 3.1, we see that Sup$h(xy) =$ Sup$\mathcal{H}_h(xy) \geq$ SupΔ and Sup$h(yx) =$ Sup$\mathcal{H}_h(yx) \geq$ SupΔ, which implies that $xy, yx \in \mathcal{S}[h;\Delta]$. Hence $\mathcal{S}[h;\Delta]$ is an ideal of S. Therefore h is a $\sup$-hesitant fuzzy ideal of S.

$(1) \Rightarrow (4)$. Let $x, y \in S$. Then $y \in \mathcal{S}[h;h(y)]$ and by assumption, we have $xy, yx \in \mathcal{S}[h;h(y)]$. Thus Sup$h(y) \leq$ Sup$h(xy)$ and Sup$h(y) \leq$ Sup$h(yx)$. Hence

$$\mathcal{I}_h(y) = [0, \text{Sup}h(y)] \preceq [0, \text{Sup}h(xy)] = \mathcal{I}_h(xy)$$

and so $\mathcal{I}_h(y) \preceq \mathcal{I}_h(xy)$. Similarly, we obtain that $\mathcal{I}_h(y) \preceq \mathcal{I}_h(yx)$. Hence $\mathcal{I}_h$ is an interval-valued fuzzy ideal of S.

$(5) \Rightarrow (1)$. Let $\Delta \in \mathcal{P}[0,1]$ and $x, y \in S$ be such that $y \in \mathcal{S}[h; \Delta]$. By using Remark 3.1, we have $\sup \mathcal{I}_h(y) = \text{Sup}h(y) \geq \text{Sup}\Delta$ and so $y \in \mathcal{S}[\mathcal{I}_h; \Delta]$. By assumption, we have $xy, yx \in \mathcal{S}[\mathcal{I}_h; \Delta]$. By using again Remark 3.1, we get $\text{Sup}h(xy) = \sup \mathcal{I}_h(xy) \geq \text{Sup}\Delta$ and $\text{Sup}h(yx) = \sup \mathcal{I}_h(yx) \geq \text{Sup}\Delta$, which implies that $xy, yx \in \mathcal{S}[h; \Delta]$. Hence $\mathcal{S}[h; \Delta]$ is an ideal of S. Therefore h is a $\sup$-hesitant fuzzy ideal of S. □

For any subset A of a nonempty set X, the characteristic interval-valued fuzzy set $\mathbf{CI}_A$ and the characteristic hesitant fuzzy set $\mathbf{CH}_A$ of A on X are defined by

$$\mathbf{CI}_A \colon X \to \mathcal{D}[0,1], x \mapsto \begin{cases} \tilde{1}, & \text{if } x \in A, \\ \tilde{0}, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{CH}_A \colon X \to \mathcal{P}[0,1], x \mapsto \begin{cases} [0,1], & \text{if } x \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Theorem 3.2. *[5] Let A be a nonempty subset of a semigroup S. Then A is an ideal of S if and only if $\mathbf{CI}_A$ is an interval-valued fuzzy ideal of S.*

Theorem 3.3. *[18] Let A be a nonempty subset of a semigroup S. Then A is an ideal of S if and only if $\mathbf{CH}_A$ is a hesitant fuzzy ideal of S.*

For any subset A of a nonemtpy set X and $\Delta, \Omega \in \mathcal{P}[0,1]$ with $\text{Sup}\Delta < \text{Sup}\Omega$, define a map $\mathbf{H}_A^{(\Delta,\Omega)}$ as follows

$$\mathbf{H}_A^{(\Delta,\Omega)} \colon X \to \mathcal{P}[0,1], x \mapsto \begin{cases} \Omega, & \text{if } x \in A, \\ \Delta, & \text{otherwise.} \end{cases}$$

Then $\mathbf{H}_A^{(\Delta,\Omega)}$ is a hesitant fuzzy set on X, which is called the $\sup(\Delta, \Omega)$**-characteristic hesitant fuzzy set** of A on X. The $\sup(\Delta, \Omega)$-characteristic hesitant fuzzy set with $\Delta = \emptyset$ and $\Omega = [0,1]$ is the characteristic hesitant fuzzy set of A, that is $\mathbf{H}_A^{(\emptyset,[0,1])} = \mathbf{CH}_A$. The $\sup(\Delta, \Omega)$-characteristic hesitant fuzzy set with $\Delta = \tilde{0}$ and $\Omega = \tilde{1}$ is the characteristic interval-valued fuzzy set of A, that is $\mathbf{H}_A^{(\tilde{0},\tilde{1})} = \mathbf{CI}_A$.

Theorem 3.4. *Let A be a nonempty subset of a semigroup S and $\Delta, \Omega \in \mathcal{P}[0,1]$ be such that $\text{Sup}\Delta < \text{Sup}\Omega$. Then A is an ideal of S if and only if $\mathbf{H}_A^{(\Delta,\Omega)}$ is a $\sup$-hesitant fuzzy ideal of S.*

Proof. ($\Rightarrow$) Suppose that there exist $x, y \in S$ such that

$$\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(xy) < \max\{\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(x), \text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(y)\}.$$

Then $\mathbf{H}_A^{(\Delta,\Omega)}(x) = \Omega$ or $\mathbf{H}_A^{(\Delta,\Omega)}(y) = \Omega$, which implies that $x \in A$ or $y \in A$. Since A is an ideal of S, we have $xy \in A$ and so $\mathbf{H}_A^{(\Delta,\Omega)}(xy) = \Omega$. Thus

$$\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(xy) = \max\{\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(x), \text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(y)\},$$

it is a contradiction. Hence

$$\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(xy) \geq \max\{\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(x), \text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(y)\} \text{ for all } x, y \in S$$

and by using Theorem 3.1, we have $\mathbf{H}_A^{(\Delta,\Omega)}$ is a $\sup$-hesitant fuzzy ideal of S.

($\Leftarrow$) Let $a \in A$ and $x \in S$. Then $\mathbf{H}_A^{(\Delta,\Omega)}(a) = \Omega$. Since $\mathbf{H}_A^{(\Delta,\Omega)}$ is a $\sup$-hesitant fuzzy ideal of S and by Theorem 3.1, we have

$$\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(ax) \geq \max\{\text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(a), \text{Sup}\mathbf{H}_A^{(\Delta,\Omega)}(x)\} = \text{Sup}\Omega.$$

Thus $\mathbf{H}_A^{(\Delta,\Omega)}(ax) = \Omega$ which implies that $ax \in A$. Similarly, $xa \in A$. Hence A is an ideal of S. □

Theorem 3.5. *For any nonempty subset A of a semigroup S , the following statements are equivalent.*

(1) *A is an ideal of S.*
(2) ***CI**$_A$ is an interval-valued fuzzy ideal of S.*
(3) ***CI**$_A$ is a sup-hesitant fuzzy ideal of S.*
(4) ***CH**$_A$ is a hesitant fuzzy ideal of S.*
(5) ***CH**$_A$ is a sup-hesitant fuzzy ideal of S.*
(6) *If $\Delta, \Omega \in \mathcal{P}[0, 1]$ and $\sup \Delta < \sup \Omega$, then $\boldsymbol{H}_A^{(\Delta,\Omega)}$ is a sup-hesitant fuzzy ideal of S.*

Proof. It follows from Lemma 3.1-3.2 and Theorem 3.1-3.4. □

Given a hesitant fuzzy set h on a nonempty set X, let

$$\top := 1 - \sup\{\text{Sup}h(x) \mid x \in X\}.$$

Definition 3.2. *[20] Let h be a hesitant fuzzy set on a nonempty set X and $t \in [0, \top]$. A hesitant fuzzy set g of X is called a* $\sup$***-hesitant fuzzy** t**-translation*** *of h if Sup$g(x) =$ Sup$h(x) + t$ for all $x \in X$.*

Given a hesitant fuzzy set h on a nonempty set X and $t \in [0, \top]$, we see that Sup$g(x) = \sup g(x)$ for every $\sup$-hesitant fuzzy t-translation g of h and for all $x \in X$.

Theorem 3.6. *Let h be a $\sup$-hesitant fuzzy ideal of a semigroup S and let $t \in [0, \top]$. Then every $\sup$-hesitant fuzzy t-translation of h is a $\sup$-hesitant fuzzy ideal of S.*

Proof. Asume that g is a $\sup$-hesitant fuzzy t-translation of h. Then

$$\begin{aligned} \sup g(xy) &= \text{Sup}h(xy) + t \\ &\geq \max\{\text{Sup}h(x), \text{Sup}h(y)\} + t \\ &= \max\{\text{Sup}h(x) + t, \text{Sup}h(y) + t\} \\ &= \max\{\sup g(x), \sup g(y)\} \end{aligned}$$

for all $x, y \in S$. It follows from Theorem 3.1 that g is a $\sup$-hesitant fuzzy ideal of S. □

Theorem 3.7. *Let h be a hesitant fuzzy set on a semigroup S such that its $\sup$-hesitant fuzzy t-translation is a $\sup$-hesitant fuzzy ideal of S for some $t \in [0, \top]$. Then h is a $\sup$-hesitant fuzzy ideal of S.*

Proof. Let $t \in [0, \top]$ and g be a $\sup$-hesitant fuzzy ideal of S such that g is a $\sup$-hesitant fuzzy t-translation of h. Then

$$\begin{aligned} \text{Sup}h(xy) &= \sup g(xy) - t \\ &\geq \max\{\sup g(x), \sup g(y)\} - t \\ &= \max\{\sup g(x) - t, \sup g(y) - t\} \\ &= \max\{\text{Sup}h(x), \text{Sup}h(y)\} \end{aligned}$$

for all $x, y \in S$ and by using Theorem 3.1, we have h is a $\sup$-hesitant fuzzy ideal of S. □

Theorem 3.8. *Let h be a hesitant fuzzy set on a semigroup S and $t \in [0, \top]$. Then a $\sup$-hesitant fuzzy t-translation of h is a $\sup$-hesitant fuzzy ideal of S if and only if $U_{\sup}(h; k - t)$ is either empty or an ideal of S for all $k \in [t, 1]$.*

Proof.

($\Rightarrow$) It follows from Theorem 3.1 and Theorem 3.7.

($\Leftarrow$) Let g be a sup-hesitant fuzzy t-translation of h. Suppose that there exist $k \in [t, 1]$ and $a, b \in S$ such that $\sup g(ab) < k \leq \max\{\sup g(a), \sup g(b)\}$. Then Sup$h(a) = \sup g(a) - t \geq k - t$ or Sup$h(b) = \sup g(b) - t \geq k - t$. Hence $a \in U_{\sup}(h; k-t)$ or $b \in U_{\sup}(h; k-t)$. By assumption, we have $ab \in U_{\sup}(h; k-t)$. It follows that $\sup h(ab) \geq k - t$, i.e., $\sup g(ab) \geq k$, a contradiction. Therefore

$$\sup g(ab) \geq \max\{\sup g(a), \sup g(b)\} \text{ for all } a, b \in S,$$

and it follows from Theorem 3.1 that g is a sup-hesitant fuzzy ideal of S. □

Definition 3.3. *[20] Let h and g be hesitant fuzzy sets on a nonempty set X. If* ***Sup***$h(x) \leq$ ***Sup***$g(x)$ *for all $x \in X$, then we say that g is a* sup***-hesitant fuzzy extension*** *of h.*

Given a hesitant fuzzy set h on a nonempty set X and $t \in [0, \top]$, the following statements are true.

(1) Every sup-hesitant fuzzy t-translation of h is a sup-hesitant fuzzy extension of h.
(2) If g_1 and g_2 are sup-hesitant fuzzy t-translations of h and g_3 is a sup-hesitant fuzzy extension of g_1, then g_3 is a sup-hesitant fuzzy extension of g_2.

Definition 3.4. *Let h and g be hesitant fuzzy sets on a semigroup S. Then g is called a* sup***-hesitant fuzzy extension*** *of h based on an ideal of S (briefly,* sup***-hesitant fuzzy I-extension*** *of h) if the following assertions are valid.*

(1) *g is a* sup*-hesitant fuzzy extension of h.*
(2) *If h is a* sup*-hesitant fuzzy ideal of S, then so is g.*

Theorem 3.9. *If h is a* sup*-hesitant fuzzy ideal of a semigroup S, then its* sup*-hesitant fuzzy t-translation is a* sup*-hesitant fuzzy I-extension of h for all $t \in [0, \top]$.*

Proof. Straightforward. □

Theorem 3.10. *Let h be a* sup*-hesitant fuzzy ideal of a semigroup S and $t_1, t_2 \in [0, \top]$. If $t_1 \geq t_2$, then a* sup*-hesitant fuzzy t_1-translation of h is a* sup*-hesitant fuzzy I-extension of a* sup*-hesitant fuzzy t_2-translation of h.*

Proof. Straightforward. □

Theorem 3.11. *Let h be a sup-hesitant fuzzy ideal of a semigroup S and $t \in [0, \top]$. If $\mathcal{G}$ is a* $\sup$*-hesitant fuzzy I-extension of a* $\sup$*-hesitant fuzzy t-translation of h, then $\mathcal{G}$ is a* $\sup$*-hesitant fuzzy I-extension of a* $\sup$*-hesitant fuzzy k-translation of h for some $k \in [t, \top]$.*

Proof. If h is a $\sup$-hesitant fuzzy ideal of a semigroup S and $t \in [0, \top]$, then a $\sup$-hesitant fuzzy t-translation g of h is a $\sup$-hesitant fuzzy ideal of S. Suppose that $\mathcal{G}$ is a $\sup$-hesitant fuzzy I-extension of g. Let $r = \inf\{\text{Sup}\mathcal{G}(x) - \sup g(x) \mid x \in S\}$ and choose $k = r + t$, we have $\top \geq k \geq t$. Define $T_h^k \colon S \to \mathcal{P}[0,1]$ by $T_h^k(x) = \{\text{Sup}h(x) + k\}$ for all $x \in S$. Hence T_h^k is a $\sup$-hesitant fuzzy k-translation of h and a $\sup$-hesitant fuzzy ideal of S. We see that

$$\text{Sup}\mathcal{G}(x) \geq \sup g(x) + r = \text{Sup}h(x) + t + r = \text{Sup}h(x) + k = \sup T_h^k(x)$$

for all $x \in S$. Therefore $\mathcal{G}$ is a $\sup$-hesitant fuzzy I-extension of T_h^k. □

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