Advances in Mathematics: Scientific Journal **10** (2021), no.4, 2213–2221
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.4.35

A KAHLER MANIFOLD ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

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ABSTRACT. In present paper we study the properties of Kahler manifold satisfying the semi - symmetric metric connection. Symmetric and skew-symmetric conditions for Nijenhuis tensor of the connection in Kahler manifold has been discussed. The paper also includes some properties of contravariant almost analytic vector field in a Kahler manifold.

1. INTRODUCTION

Study of differential geometry has been gathering attention from researchers across the world. Many mathematicians delve and studied this field like S. Kobayasi and K. Nomizu [7], Yano [15], R. S. Mishra [8] and many more. In 1970, K. Yano [15] considered a semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically. The semi-symmetric metric connection plays an important role in the study of Riemannian manifolds.

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2020 *Mathematics Subject Classification.* 53B05, 53B50, 53C05, 53C15, 53C55.
Key words and phrases. Kahler manifold, Semi-symmetric metric connection, Nijenhuis tensor, Contravariant almost analytic vector field.
Submitted: 10.04.2021; *Accepted:* 26.04.2021; *Published:* 29.04.2021.

Friedmann and Schouten [6] introduced the notion of semi-symmetric linear connection on a differentiable manifold. O.C. Andonie [2] studied the Riemannian manifold equipped with a semi-symmetric metric connection. M.C. Chaki and A. Konar [3], U.C. De [4] and S.C. Biswas [5] studied a special type of semi-symmetric metric connection on a Riemannian manifold.

M. Parvanovic and N. Pusic [12], studied on manifolds admitting semi-symmetric connection. P.N. Pandey and S.K. Dubey [10] discussed an almost Grayan manifold admitting a semi-symmetric metric connection while P.N. Pandey and B.B. Chaturvedi [9], [11]studied a Kahler manifold equipped with semi-symmetricmetric connection and an almost hermitian manifold with semi-sym- metric recurrent connection. Agashe and Chafle [1] introduced the idea of semi-symmetric non metric connection in Riemannian manifold and studied further.

There are various physical problems involving semi-symmetric metric connection also. For example, if a man is moving on the surface of the earth always facing one defined point, the north pole, then this displacement is semi-symmetric and metric [13], [14].

Let M^n be an even dimensional differentiable manifold of differentiability class C^{r+1}. If there exist a vector valued linear function F of differentiability class C^r such that for any vector field X

$$\bar{\bar{X}} + X = 0 \tag{1.1}$$

$$g(\bar{X}, \bar{Y}) = g(X, Y) \tag{1.2}$$

$$(D_X F)Y = 0, \tag{1.3}$$

where $\bar{X} = FX$, g is non-singular metric tensor and D is Riemannian connection, then M^n is called a Kahler manifold.

Now a linear connection $\bar{\nabla}$ on $\{M^n, g\}$ is said to be a semi-symmetric connection, if the torsion tensor T of the connection $\bar{\nabla}$ and metric tensor g of the manifold satisfy the following conditions:

$$(\bar{\nabla}_z g)(X, Y) = 0, \tag{1.4}$$

and torsion tensor T,

$$T(X, Y) = \eta(Y)X - \eta(X)Y. \tag{1.5}$$

For any vector field X, Y where η is 1-form associated with the torsion tensor of the connection $\bar{\nabla}$. We have

$$\bar{\nabla}_X Y = D_X Y + \eta(Y)X - g(X,Y)\rho \tag{1.6}$$

the 1-form η and vector field ρ are usually called 1-form and vector field associated with tensor field T, where

$$\eta(X) = g(X,\rho). \tag{1.7}$$

2. Properties on a Kahler Manifold with connection $\bar{\nabla}$

Let us consider a Kahler manifold M^n equipped with a semi-symmetric metric connection $\bar{\nabla}$ and define

$$\begin{aligned} 'F(Y,Z) &= g(\bar{Y},Z) \\ 'T*(X,Y,Z) &= g(T*(X,Y),Z) \\ H(X,Y) &= \eta(Y)X \\ 'H(X,Y,Z) &= g(H(X,Y),Z). \end{aligned} \tag{2.1}$$

In view of (2.1) the (1.6) becomes $\bar{\nabla}_X Y = D_X Y + H(X,Y) - g(X,Y)\rho$. For any vector field $\bar{Y}$, (1.6) becomes

$$\bar{\nabla}_X \bar{Y} = D_X \bar{Y} + \eta(\bar{Y})X - g(X,\bar{Y})\rho, \tag{2.2}$$

which implies

$$(\bar{\nabla}_X F)Y + \overline{\bar{\nabla}_X Y} = (D_X F)Y + \overline{D_X Y} + \eta(\bar{Y})X - g(X,\bar{Y})\rho,$$

or

$$(\bar{\nabla}_X F)Y = (D_X F)Y + \overline{D_X Y} - \overline{\bar{\nabla}_X Y} + \eta(\bar{Y})X - g(X,\bar{Y})\rho. \tag{2.3}$$

Operating both sides of (1.6) with F we have

$$\overline{\bar{\nabla}_X Y} = \overline{D_X Y} + \eta(Y)\bar{X} - g(\bar{X},\bar{Y})\rho. \tag{2.4}$$

Using equations (1.3) and (2.4) in (2.3), we have

$$(\bar{\nabla}_X F)Y = \eta(\bar{Y})X - \eta(Y)\bar{X} - g(X,\bar{Y})\rho + g(\bar{X},\bar{Y})\rho. \tag{2.5}$$

Barring X and Y in (2.5) we get,

$$(\bar{\nabla}_{\bar{X}} F)\bar{Y} = \eta(\bar{\bar{Y}})\bar{X} - \eta(\bar{Y})\bar{\bar{X}} - g(\bar{X},\bar{\bar{Y}})\rho + g(\bar{\bar{X}},\bar{\bar{Y}})\rho.$$

Using (1.1) we have,

$$(\bar{\nabla}_{\bar{X}}F)\bar{Y} = -\eta(Y)\bar{X} + \eta(\bar{Y})X + g(\bar{X}, Y)\rho + g(X, Y)\rho. \tag{2.6}$$

Thus we have following theorem

Theorem 2.1. *A Kahler manifold equipped with a semi symmetric connection $\bar{\nabla}$ satisfies the following:*

$$\begin{aligned}(\bar{\nabla}_{\bar{X}}F)\bar{Y} &= (\bar{\nabla}_X F)Y \\ (\bar{\nabla}_X F)Y &= g(X, Y)\rho - g(X, \bar{Y})\rho,\end{aligned} \tag{2.7}$$

if, and only if, $\eta(\bar{Y})X = \eta(Y)\bar{X}$.

Differentiating (2.1) I with respect to connection $\bar{\nabla}$ and using (1.4) and (1.6) and $g(\bar{X}, Y) = -g(X, \bar{Y})$, we have $(\bar{\nabla}_X{}'F)(Y, Z) = (\bar{\nabla}_X g)(\bar{Y}, Z)$,

$$\begin{aligned}&(\nabla_X{}'F)(Y, Z) +' F(\nabla_X Y, Z) +' F(Y, \nabla_X Z) \\ &\quad = (\nabla_X g)(\bar{Y}, Z) + g((\nabla_X F)Y, Z) + g(\overline{\nabla_X Y}, Z) + g(\bar{Y}, \nabla_X Z)\end{aligned}$$

$$(\bar{\nabla}_X{}'F)(Y, Z) = 0 \tag{2.8}$$

$d'F(X, Y, Z)$ is defined as

$$d'F(X, Y, Z) = (\bar{\nabla}_X{}'F)(Y, Z) + (\bar{\nabla}_Y{}'F)(Z, X) + (\bar{\nabla}_Z{}'F)(X, Y). \tag{2.9}$$

So, $d'F(X, Y, Z) = 0$. Thus we have

Theorem 2.2. *On a Kahler manifold equipped with semi-symmetric metric connection $\bar{\nabla}$, we have $d'F(X, Y, Z) = 0$.*

3. Nijenhuis tensor with connection $\bar{\nabla}$

The Nijenhuis tensor with respect to semi-symmetric metric connection $\bar{\nabla}$ is given by

$$N(X, Y) = (\bar{\nabla}_{\bar{X}}F)Y - (\bar{\nabla}_{\bar{Y}}F)X - \overline{(\bar{\nabla}_X F)Y} + \overline{(\bar{\nabla}_Y F)X}. \tag{3.1}$$

From (2.5) and (1.1), we have

$$(\bar{\nabla}_{\bar{X}}F)Y = \eta(\bar{Y})\bar{X} + \eta(Y)X - g(\bar{X}, \bar{Y})\rho - g(X, \bar{Y})\rho. \tag{3.2}$$

Interchanging X and Y in (3.2) we have

$$(\bar{\nabla}_{\bar{Y}}F)X = \eta(\bar{X})\bar{Y} + \eta(X)Y - g(\bar{Y},\bar{X})\rho - g(Y,\bar{X})\rho. \tag{3.3}$$

Operating F on both side of the (2.5) and using (1.1) we get

$$\overline{(\bar{\nabla}_X F)Y} = \eta(\bar{Y})\bar{X} + \eta(Y)X + g(\bar{X},Y)\rho + g(X,Y)\rho. \tag{3.4}$$

Interchanging X and Y in (3.4) we have,

$$\overline{(\bar{\nabla}_Y F)X} = \eta(\bar{X})\bar{Y} + \eta(X)Y + g(\bar{Y},X)\rho + g(Y,X)\rho. \tag{3.5}$$

From equations (3.1), (3.2), (3.3), (3.4) and (3.5) we have

$$N(X,Y) = -2g(X,\bar{Y}) - -2g(\bar{X},Y). \tag{3.6}$$

So we have,

Theorem 3.1. *On a Kahler manifold, Nijenhuis tensor with respect to semi-symmetric metric connection is of the form* $N(X,Y) = -2g(X,\bar{Y}) - 2g(\bar{X},Y)$.

Symmetric and Skew-symmetric condition of Nijenhuis tensor of $\bar{\nabla}$ in a Kahler manifold:

From (3.6) we have

$$N(Y,X) = -2g(Y,\bar{X}) - 2g(\bar{Y},X). \tag{3.7}$$

Hence from equations (3.6) and (3.7) we have

$$N(X,Y) - N(Y,X) = 0. \tag{3.8}$$

Theorem 3.2. *The Nijenhuis tensor* $N(X,Y)$ *of the Kahler manifold with respect to semi-symmetric metric connection is symmetric. Again from equations* (3.6) *and* (3.7) *we have*

$$N(X,Y) + N(Y,X) = -4g(X,\bar{Y}) - 4g(\bar{X},Y). \tag{3.9}$$

If $N(X,Y)$ is skew-symmetric the left part of (3.9) vanishes, which gives:

$$g(X,\bar{Y}) + g(\bar{X},Y) = 0. \tag{3.10}$$

Hence we state following theorem:

Theorem 3.3. *The Nijenhuis tensor* $N(X,Y)$ *of Kahler manifold with respect to semi- symmetric metric connection is skew symmetric, if, and only if,* $g(X,\bar{Y}) + g(\bar{X},Y) = 0$.

4. CONTRAVARIANT ALMOST ANALYTIC VECTOR FIELD ON A KAHLER MANIFOLD

A vector field V is said to be contravariant almost analytic if the Lie derivative of F with respect to V vanishes identically, i.e.,

$$(L_V F)X = 0, \tag{4.1}$$

for all X. (4.1) is equivalent to the equatioin

$$[V, \bar{X}] = \overline{[V, X]}. \tag{4.2}$$

In Kahler manifold, (4.2) becomes

$$D_{\bar{X}} V = \overline{D_X V} \tag{4.3}$$

if, and only if,

$$\overline{D_{\bar{X}} V} + D_X V = 0.$$

Theorem 4.1. *On a Kahler manifold, a contravariant almost analytic vector V with respect to Riemannian connection D is also contravariant almost analytic with respect to semi-symmetric metric connection $\bar{\nabla}$, if, and only if, $g(\bar{X}, V)\rho = g(\bar{X}, \bar{V})\rho$.*

Proof. For any vector field V, (1.6) gives,

$$\bar{\nabla}_X V = D_X V + \eta(V)X - g(X, V)\rho. \tag{4.4}$$

For the vector field $\bar{X}$ (4.4) becomes

$$\bar{\nabla}_{\bar{X}} V = D_{\bar{X}} V + \eta(V)\bar{X} - g(\bar{X}, V)\rho. \tag{4.5}$$

Operating both sides of (4.4) by F we have,

$$\overline{\bar{\nabla}_X V} = \overline{D_X V} + \eta(V)\bar{X} - g(\bar{X}, \bar{V})\rho. \tag{4.6}$$

Subtracting (4.5) from (4.6) we have,

$$\begin{aligned} &\overline{\bar{\nabla}_X V} - \bar{\nabla}_{\bar{X}} V \\ &= \overline{D_X V} - D_{\bar{X}} V - g(\bar{X}, \bar{V})\rho + g(\bar{X}, V)\rho. \end{aligned} \tag{4.7}$$

Since V is contravariant almost analytic with respect to connection D, we have $D_{\bar{X}} V - \overline{D_X V} = 0$, and then $\overline{\bar{\nabla}_X V} - \bar{\nabla}_{\bar{X}} V = 0$, if, and only if, $g(\bar{X}, V)\rho = g(\bar{X}, \bar{V})\rho$, therefore V is contravariant almost analytic with respect to connection $\bar{\nabla}$. This proves the theorem. □

Theorem 4.2. *If V is contravariant almost analytic with respect to Levi Civita connection ∇, then it is also contravariant almost analytic with respect to $\bar{\nabla}$, if, and only if, $'F(X,\rho)=\eta(X)\bar{V}$, where $'F(X,\rho)=g(\bar{X},\rho)$.*

Proof. Let V be contravariant almost analytic vector field and X be any arbitrary vector field. From (1.6) we get,

$$\bar{\nabla}_V X = \nabla_V X + \eta(X)V - g(V,X)\rho \tag{4.8}$$

and

$$\bar{\nabla}_X V = \nabla_X V + \eta(V)X - g(X,V)\rho, \tag{4.9}$$

from equations (4.8) and (4.9) we get,

$$[V,X]_s = [V,X] + \eta(X)V - \eta(V)X, \tag{4.10}$$

where $[,]_s$ is lie bracket with respect to semi-symmetric connection $\bar{\nabla}$.

From (4.10) we get,

$$\begin{aligned}[V,\bar{X}]_s &= [V,\bar{X}] + \eta(\bar{X})V - \eta(V)\bar{X}\\ \overline{[V,X]_s} &= \overline{[V,X]} + \eta(X)\bar{V} - \eta(V)\bar{X}.\end{aligned} \tag{4.11}$$

From equations (2.5), (2.6), (4.1) and (4.2) we get,

$$[V,\bar{X}]_s = \overline{[V,X]}, \tag{4.12}$$

if, and only if,

$$'F(X,\rho)V = \eta(X)\bar{V}. \tag{4.13}$$

From equations (4.12) and (4.13) we get the first part of the theorem.

From (4.11), we get

$$\bar{\nabla}_{\bar{X}} V = (\bar{\nabla}_V F)(X) + \overline{\nabla_X V} + \eta(X)\bar{V} - \eta(\bar{X})V, \tag{4.14}$$

$$g(\bar{\nabla}_{\bar{X}} V, Y) = g((\bar{\nabla}_V F)(X), Y) + g(\overline{\bar{\nabla}_X V}, Y) + \eta(X)g(\bar{V},Y) - \eta(\bar{X})g(V,Y). \tag{4.15}$$

From theorem 2.1, we have

$$(\bar{\nabla}_{\bar{X}} V)(Y) = g(\bar{\nabla}_{\bar{X}} V, Y). \tag{4.16}$$

Further from equations (4.15) and (4.16) we get,

$$\bar{\nabla}_{\bar{X}} V(Y) = (\bar{\nabla}_V F)(X,Y) - \bar{\nabla}_X V(\bar{Y}) + \eta(X)g(\bar{V},Y) - \eta(\bar{X})g(V,Y). \tag{4.17}$$

If 1-form V is almost analytic with respect to connection $\bar{\nabla}$, then we have

$$V(\bar{\nabla}_X F)(Y) - (\bar{\nabla}_Y F)(X) = (\bar{\nabla}_{\bar{X}} V)(Y) - (\bar{\nabla}_X V)(\bar{Y}), \tag{4.18}$$

if, and only if,

$$V(\bar{X})V(Y) = V(X)V(\bar{Y}). \tag{4.19}$$

Using equations (4.17) and (4.18) we get,

$$\begin{aligned} 2\bar{\nabla}_{\bar{X}} V(Y) = (\bar{\nabla}_V {'F})(X,Y) + \bar{\nabla}_X {'F}(Y,V) + \bar{\nabla}_Y {'F}(V,X) + \eta(X)g(\bar{V},Y) \\ - \eta(\bar{X})g(V,Y), \end{aligned} \tag{4.20}$$

which implies

$$2\bar{\nabla}_{\bar{X}} V(Y) = (\bar{d}'F)(X,Y,Z) + \eta(X)g(\bar{V},Y) - \eta(\bar{X})g(V,Y).$$

Finally we have, $2\bar{\nabla}_{\bar{X}} V)Y = (\bar{d}'F)(X,Y,Z)$ if $g(X,\rho)'F(V,Y) = {'F}(X,\rho)$ $g(V,Y)$. □

Theorem 4.3. *If V is contravariant almost analytic vector field with respect t to Levi Civita Connection ∇ then if $g(X,\rho)'F(V,Y) = {'F}(X,\rho)g(V,Y)$. Then we have $2(\bar{\nabla}_{\bar{X}} V)Y = (\bar{d}'F)(X,Y,Z)$ where $V(Y) = g(V,Y)$.*

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