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BUFFON'S COIN AND NEEDLE PROBLEMS FOR THE SNUB HEXAGONAL TILING

Salvatore Vassallo

ABSTRACT. In this paper we consider the snub hexagonal tiling of the plane ($(3^4, 6)$ Archimedean tiling) and compute the probability that a random circle or a random segment intersects a side of the tiling.

1. INTRODUCTION

A *tiling* or *tessellation* in the plane is a collection of disjoint closed sets (the *tiles*) that can intersect only on the boundary, which cover the plane. A tiling is said to be polygonal if the tiles are polygon, a polygonal tiling is said to be *edge-to-edge* if two non disjoint tiles have in common or a vertex or a segment that is an edge for both the polygons. In this case we call any edge of a tile an *edge of the tiling*. An edge-to-edge tiling is called *regular* if it is composed of congruent copies of a single regular polygon. An *Archimedean tessellation* (semi-regular or uniform tessellation) is an edge-to-edge tessellation of the plane made of more than one type of regular polygon so that the same polygons surround each vertex. There are eight different Archimedean tilings and we can classify them giving the types of polygons as they come together at the vertex [10]. The *snub*

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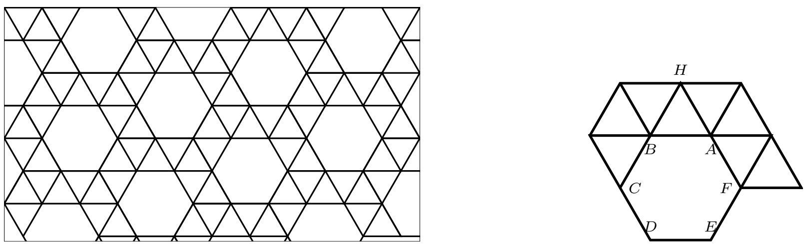
hexagonal tiling is a tiling such that four triangles and an hexagon come together (in clockwise order) in any vertex so it can be called a $(3^4, 6)$ Archimedean tiling (see Figure 1a). Many authors studied Buffon type problems for different lattices



(a) snub hexagonal tiling

(b) The fundamental cell T_0

FIGURE 1. The tiling $\mathcal{R}$

of figures or tilings and different test bodies: See for example [1–9, 11, 12, 15–17, 22].

In particular the cases of the $(3^3, 4^2)$, of the $(3^2, 4, 3, 4)$, of the $(8^2, 4)$ and of the $(3, 6, 3, 6)$ Archimedean tilings (elongated triangular tiling, snub square tiling, truncated square tiling and trihexagonal tiling) are studied in [18–21], respectively.

We will study Buffon type problems for the snub hexagonal tiling and two special test bodies: a circle of constant diameter D and a line segment of length l.

Let E_2 be the Euclidean plane and let $\mathcal{R}$ be the snub hexagonal tiling of E_2 given in figure 1a. We denote by T_0 the *fundamental tile* (or cell) of $\mathcal{R}$ (see figure 1b) and by T_n one of congruent copies of T_0 such that:

(i) $\bigcup_{n\in\mathbb{N}} T_n = E_2$,

(ii) $\mathrm{Int}(T_i) \cap \mathrm{Int}(T_j) = \varnothing, \forall i, j \in \mathbb{N}$ and $i \neq j$,

(iii) $T_n = \gamma_n(T_0), \forall n \in \mathbb{N}$, where γ_n are the elements of a discrete subgroup of the group of motions in E_2 that leaves invariant the tiling $\mathcal{R}$.

The body T_0 can be expressed as the union of a hexagon of side a and eight equilateral triangle of the same side a.

Let us denote by K a convex body (which means here a compact convex set) which we shall call test body. A general problem of Buffon type can be stated as follows: “Which is the probability $p_{K,\mathcal{R}}$ that the random convex body K, or more precisely, a random congruent copy of K, meets some of the boundary points of at least one of the domains T_n? "

If we denote by $\mathcal{M}$ the set of all test bodies K whose centroid is in the interior of T_0 and by $\mathcal{N}$ the set of all test bodies K that are completely contained in one of the 8 triangles or in the hexagon $ABCDEF$, we have

$$p_{K,\mathcal{R}} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}, \tag{1.1}$$

where μ is the Lebesgue measure in the plane E_2.

2. THE TEST BODY IS A CIRCLE

Let us suppose that the test body K is a circle of diameter D. Easy geometrical considerations will lead us to distinguish between the cases $D < \frac{a}{\sqrt{3}}$ (the diameter of the circle inscribed in the triangle) , $\frac{a}{\sqrt{3}} \leq D < a\sqrt{3}$ (the diameter of the circle inscribed in the hexagon) and $D \geq a\sqrt{3}$. It is obvious that if $D \geq a\sqrt{3}$ the circle always meets the boundary of one of the bodies T_n, so we have to study the other two cases.

Proposition 2.1. *The probability that the circle K of diameter D intersects the tiling $\mathcal{R}$ is given by*

$$p_{K,\mathcal{R}} = \begin{cases} \frac{D\left[10\sqrt{3}a-13D\right]}{7a^2}, & \text{if } D < \frac{a}{\sqrt{3}}, \\ \frac{4a^2+2\sqrt{3}aD-D^2}{7a^2}, & \text{if } \frac{a}{\sqrt{3}} \leq D < a\sqrt{3}. \end{cases} \tag{2.1}$$

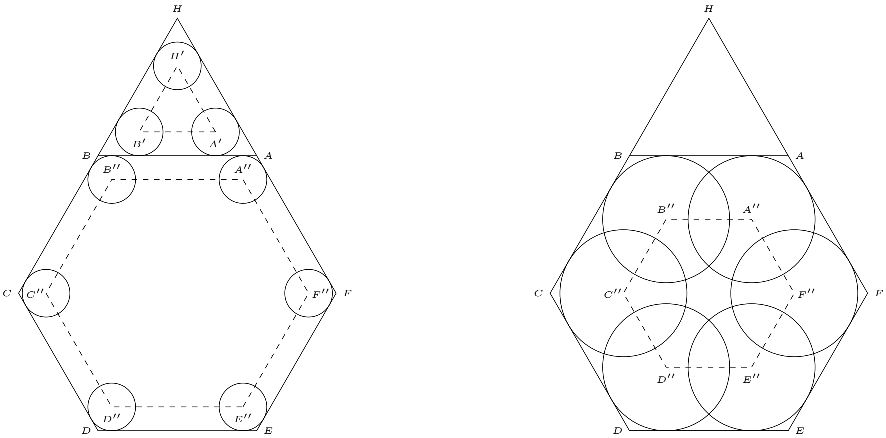
Proof. We compute the measures $\mu(\mathcal{M})$ e $\mu(\mathcal{N})$ with help of the elementary kinematic measure $dK = dx \wedge dy \wedge d\phi$ of E_2 (see [13], [14]) where x and y are the coordinates of the center of K (or the components of a translation), and ϕ is the angle of rotation. We have

$$\mu(\mathcal{M}) = \int_0^\pi d\phi \iint_{(x,y)\in T_0} dxdy = \pi \cdot \text{area}(T_0) = \frac{7}{2}\pi a^2\sqrt{3}.$$

Let $\mathcal{N}_1$ be the set of circles of diameter D that are contained in the triangle ABH and $\mathcal{N}_2$ be the set of circles of diameter D that are contained in the hexagon $ABCDEF$. From ((1.1)) we obtain

$$p_{K,\mathcal{R}} = 1 - \frac{8\mu(\mathcal{N}_1) + \mu(\mathcal{N}_2)}{\frac{7}{2}\pi a^2\sqrt{3}}. \tag{2.2}$$

From figure 2a it is easy to see that $\mu(\mathcal{N}_1)$ is π times the area of the triangle



(a) The case $D < \frac{a}{\sqrt{3}}$ (b) The case $\frac{a}{\sqrt{3}} \leq D < a\sqrt{3}$

FIGURE 2. The case K = circle

A'B'H' whose sides are parallel to the sides of the triangle *ABH* at distance $D/2$ from them (A' is the center of a disk interior to the triangle *ABH* and tangent to the sides *AB* and *AH* and so on). Since the side of the triangle is $a - D\sqrt{3}$ we have:

$$\mu(\mathcal{N}_1) = \frac{\pi\sqrt{3}}{4}\left(a - \sqrt{3}D\right)^2.$$

In the same way we obtain that

$$\mu(\mathcal{N}_2) = \frac{3\pi\sqrt{3}\left(a - \frac{D}{\sqrt{3}}\right)^2}{2}.$$

Then we have for the case $D < \frac{a}{\sqrt{3}}$,

$$p_{K,\mathcal{R}} = \frac{D\left[10\sqrt{3}a - 13D\right]}{7a^2}.$$

Let $\frac{a}{\sqrt{3}} \leq D < a\sqrt{3}$ (see figure 2b). If the center of the circle K is in the triangle *ABH*, the circle always intersects one of the side of the triangle so that

$$\mu(\mathcal{N}_1) = 0.$$

If the center of the circle is in the hexagon *ABCDEF*, the circle does not intersect the side of the hexagon if its center is in the hexagon $A''B''C''D''E''F''$; since the side of this hexagon is $a - \frac{D}{\sqrt{3}}$ we have

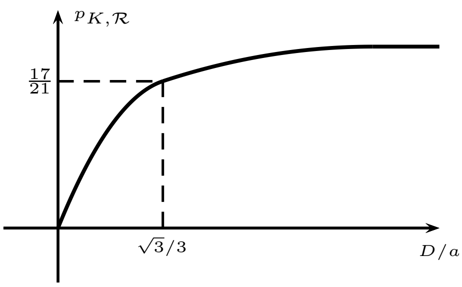
$$\mu(\mathcal{N}_2) = \frac{3\pi\sqrt{3}}{2}\left(a - \frac{D}{\sqrt{3}}\right)^2,$$

and so in this case

$$p_{K,\mathcal{R}} = \frac{4a^2 + 2\sqrt{3}aD - D^2}{7a^2}.$$

□

The graphic of the probability $p_{K,\mathcal{R}}$ is



Let us observe that $p_{K,\mathcal{R}} \geq \frac{1}{2}$ for $D > \frac{10\sqrt{3}-\sqrt{118}}{26}a \approx 0.24837a$ i.e. also for "small" circles.

3. THE TEST BODY IS A LINE SEGMENT

Let us consider now the case K is a line segment of length l. Also in this case easy geometrical considerations give us four cases: $l < \frac{a\sqrt{3}}{2}$ (the minimal width of the triangle), $\frac{a\sqrt{3}}{2} \leq l < a$ (the diameter of the triangle) , $a \leq l < a\sqrt{3}$ (the minimal width of the hexagon), $a\sqrt{3} \leq l \leq 2a$ (the diameter of the hexagon), and $l \geq 2a$. In the last case the segment always intersects the boundary of one of the bodies T_n, so we have to study the other cases. We have

Proposition 3.1. *The probability that the line segment K of length l intersects the tiling $\mathcal{R}$ is given by*

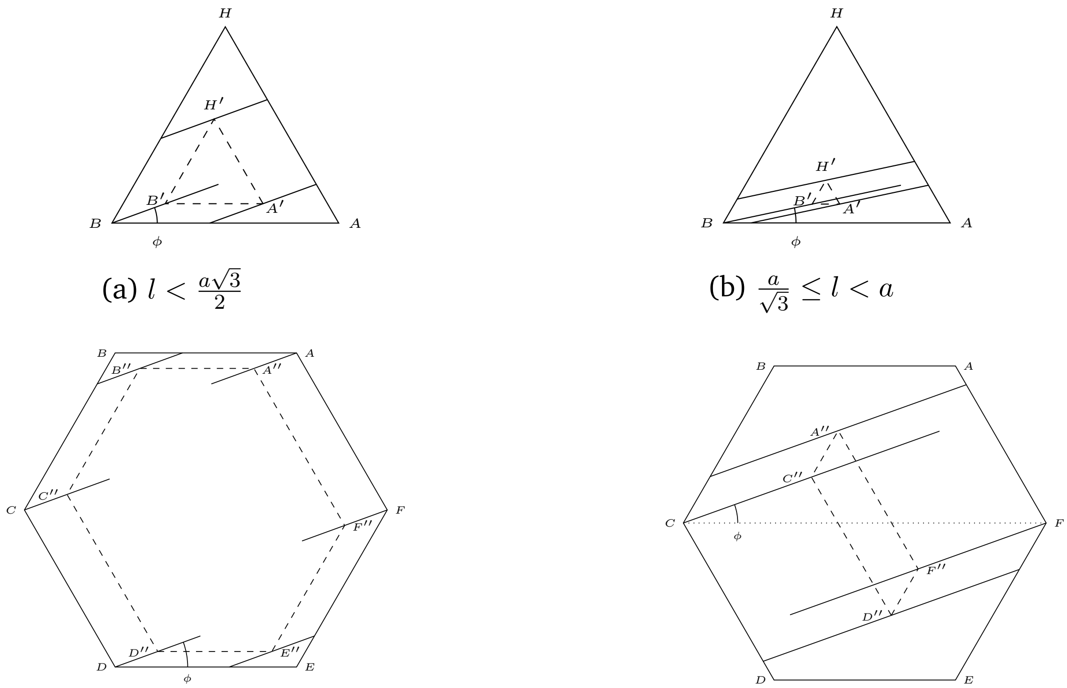
$$
(3.1)\qquad p_{K,\mathcal{R}} = \begin{cases}
\dfrac{l}{21\pi a^2}\left[60a\sqrt{3} - \left(15\sqrt{3} + 7\pi\right) l\right] & \text{if } l < \frac{a\sqrt{3}}{2} \\
\dfrac{1}{21\pi a^2}\left[60\sqrt{3}al - l^2\left(15\sqrt{3} + 7\pi\right)\right. \\
\quad - 36\sqrt{3}a\sqrt{4l^2 - 3a^2} & \text{if } \frac{a\sqrt{3}}{2} \leq l < a \\
\quad \left. +24\left(3a^2 + 2l^2\right)\arccos\left(\dfrac{a\sqrt{3}}{2l}\right)\right] \\
\dfrac{1}{7\pi a^2}\left[3\sqrt{3}a\sqrt{4l^2 - 3a^2} + \right. \\
\quad + 2\left(3a^2 + 2l^2\right)\arcsin\left(\dfrac{a\sqrt{3}}{2l}\right) & \text{if } a \leq l < a\sqrt{3} \\
\quad \left. + \pi\left(2a^2 - l^2\right)\right] \\
\dfrac{1}{21\pi a^2}\left[9a^2\left(2\sqrt{3} + 5\pi\right) + l^2\left(3\sqrt{3} + 2\pi\right)\right. \\
\quad - 30\sqrt{3}a\sqrt{l^2 - 3a^2} & \text{if } a\sqrt{3} \leq l < 2a \\
\quad \left. -6\left(12a^2 + l^2\right)\arcsin\left(\dfrac{a\sqrt{3}}{l}\right)\right]
\end{cases}
$$

Proof. In the following we can always suppose, by symmetry, that the line segment K forms an angle $\phi \leq \frac{\pi}{6}$ with the direction of the side DF.

(i) Let us consider the case $l < \frac{a\sqrt{3}}{2}$. We compute first the measure $\mu(\mathcal{N}_1)$ of the set $\mathcal{N}_1$ of all line segments of length l contained in the triangle *ABH*. For a fixed angle $\phi \in [0, \frac{\pi}{6}[$ we denote by (see figure 3a):
  - B' the midpoint (in ABH) of the line segment of length l with one endpoint in B that makes an angle ϕ with AB;
  - A' the midpoint of the line segment of length l with endpoints on AB and AH that makes an angle ϕ with AB;
  - H' the midpoint of the line segment of length l with endpoints on AH and BH that makes an angle ϕ with the direction of AB.

We compute

$$\text{area}(A'B'H') = \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2,$$



(a) $l < \frac{a\sqrt{3}}{2}$ (b) $\frac{a}{\sqrt{3}} \leq l < a$

(c) $\mathcal{N}_2 =$ hexagon (d) $\mathcal{N}_2 =$ parallelogram

FIGURE 3. The case $K =$ line segment

and, by symmetry, we obtain

$$\mu\left(\mathcal{N}_1\right) = 6\int_0^{\pi/6} \mathrm{area}(A'B'H')d\phi = \int_0^{\pi/6} \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2 d\phi$$

$$= \frac{3\sqrt{3}\pi a^2 - 36al + \left(9 + 2\sqrt{3}\pi\right)l^2}{12}. \tag{3.2}$$

In the same way, if $\phi \in \left[0, \frac{\pi}{6}\right]$, we obtain that K is contained in the hexagon $ABCDEF$ if its centroid is in the hexagon $A''B''C''D''E''F''$ whose sides have length $\overline{A''B''} = a - \frac{2l}{\sqrt{3}}\sin\left(\frac{\pi}{3} - \phi\right)$, $\overline{B''C''} = a - \frac{2l}{\sqrt{3}}\sin\phi$, and $\overline{C''D''} = a$ (see figure 3c). Then

$$\mathrm{area}(A''B''C''D''E''F'') = \frac{3}{2}a^2\sqrt{3} - l\left[2\,a\sin\left(\frac{\pi}{3} + \phi\right) - \frac{2}{3}\,l\sqrt{3}\sin\phi\cos\left(\frac{\pi}{6} + \phi\right)\right],$$

and so have

$$\mu\left(\mathcal{N}_2\right) = 6\int_0^{\pi/6} \mathrm{area}(A''B''C''D''E''F'')d\phi = -6al + \frac{3}{2}l^2 + \frac{3}{2}\sqrt{3}a^2\pi - \frac{1}{6}\sqrt{3}l^2\pi,$$

and then

$$\mu(\mathcal{N}) = \frac{7}{2}\sqrt{3}\pi a^2 - 30al + \left(\frac{15}{2} + \frac{7}{6}\sqrt{3}\pi\right) l^2.$$

Hence we have if $l < \frac{a\sqrt{3}}{2}$,

$$p_{K,\mathcal{R}} = \frac{l\left[60a\sqrt{3} - \left(15\sqrt{3} + 7\pi\right) l\right]}{21\pi a^2}. \tag{3.3}$$

(ii) Let now $\frac{a\sqrt{3}}{2} \leq l < a$. With reference to figure 3b it is easy to see that the line segment can be contained in the triangle ABH only if the angle $\phi \in [0, \pi/6[$ between the line segment and the side AB satisfies $0 \leq \phi < \frac{\pi}{6} - \arccos\frac{\sqrt{3}a}{2l}$.

So the measure of the line segments completely contained in the triangle ABH is, by symmetry,

$$\begin{aligned}\mu\left(\mathcal{N}_1\right) &= 6\int_0^{\frac{\pi}{6}-\arccos\frac{\sqrt{3}a}{2l}} \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2 d\phi = \\ &= \frac{1}{12}\left[9l^2 - 36al + \pi\sqrt{3}\left(3a^2 + 2l^2\right) + 27a\sqrt{4l^2 - 3a^2} - \right. \\ &\left. -6\sqrt{3}\left(3a^2 + 2l^2\right)\arccos\left(\frac{a\sqrt{3}}{2l}\right)\right].\end{aligned} \tag{3.4}$$

The measure of the line segment completely contained in the hexagon $ABCDEF$ is the same as in the case above:

$$\mu\left(\mathcal{N}_2\right) = -6al + \frac{3}{2}l^2 + \frac{3}{2}\sqrt{3}a^2\pi - \frac{1}{6}\sqrt{3}l^2\pi.$$

Hence we have if $\frac{a\sqrt{3}}{2} \leq l < a$,

$$\begin{aligned}p_{K,\mathcal{R}} &= \frac{1}{21\pi a^2}\left[60\sqrt{3}al - l^2\left(15\sqrt{3} + 7\pi\right) - 36\sqrt{3}a\sqrt{4l^2 - 3a^2} + \right. \\ &\left. +24\left(3a^2 + 2l^2\right)\arccos\left(\frac{a\sqrt{3}}{2l}\right)\right].\end{aligned}$$

(iii) Let now $a \leq l < a\sqrt{3}$. It is easy to see that in this case, if the centroid of the line segment is in the triangle ABH, the line segment always meets one of the side of the triangle and so $\mu(\mathcal{N}_1) = 0$.

The segment K does not intersect the sides of the hexagon if its centroid is in the hexagon $A''B''C''D''E''F''$ when the angle ϕ satisfies

$\frac{\pi}{3} - \arcsin\left(\frac{a\sqrt{3}}{2l}\right) \le \phi < \frac{\pi}{6}$ and if its centroid is in the parallelogram $A''C''D''F''$ when the angle ϕ is in $\left[0, \frac{\pi}{3} - \arcsin\left(\frac{a\sqrt{3}}{2l}\right)\right]$ (see Figure 3c and Figure 3d).

The area of the hexagon $A''B''C''D''E''F''$ is the same as above; since the sides of the parallelogram $A''C''D''F''$ have lengths $\overline{C''D''} = 2a - \frac{2l\sin\left(\frac{\pi}{3}-\phi\right)}{\sqrt{3}}$ and $\overline{A''C''} = 2a - \frac{2l\left[\sin\left(\frac{\pi}{3}-\phi\right)+\sin\phi\right]}{\sqrt{3}}$ and the angle of the parallelogram is $\frac{\pi}{3}$ we obtain

$$\text{area}(A''C''D''F'') = 2\,a^2\sqrt{3} - 2\,\sqrt{3}al\cos\phi + \frac{1}{2}\,\sqrt{3}l^2\cos^2\phi - \frac{1}{6}\,\sqrt{3}l^2\sin^2\phi.$$

The measure of the line segments completely contained in the hexagon $ABCDEF$ is given by:

$$\begin{aligned}
&\mu(\mathcal{N}) = \mu(\mathcal{N}_2) = \\
&= 6\left[\int_0^{\frac{\pi}{3}-\arcsin\left(\frac{a\sqrt{3}}{2l}\right)} \text{area}(A''C''D''F'')d\phi\right.\\
&\quad \left.+ \int_{\frac{\pi}{3}-\arcsin\left(\frac{a\sqrt{3}}{2l}\right)}^{\pi/6} \text{area}(A''B''C''D''E''F'')d\phi\right] = \\
&= \frac{\sqrt{3}\pi}{2}\left(l^2+5a^2\right) - \frac{9}{2}a\sqrt{4l^2-3a^2} - \sqrt{3}\left(3a^2+2l^2\right)\arcsin\left(\frac{a\sqrt{3}}{2l}\right).
\end{aligned}$$

Hence we have if $a \le l < a\sqrt{3}$

(3.5)
$$p_{K,\mathcal{R}} = \frac{1}{7\pi a^2}\left[\pi\left(2a^2-l^2\right) + 3\sqrt{3}a\sqrt{4l^2-3a^2} + 2\left(3a^2+2l^2\right)\arcsin\left(\frac{a\sqrt{3}}{2l}\right)\right]$$

(iv) Finally if $a\sqrt{3} \le l < 2a$ the segment K does not intersect $\mathcal{R}$ if and only if its centroid is in the parallelogram $A''C''D''F''$ and the angle ϕ satisfies $0 \le \phi \le \arcsin\left(\frac{a\sqrt{3}}{l}\right) - \frac{\pi}{3}$. Since the area of the parallelogram is the same as above we have:

$$\mu(\mathcal{N}) = \mu(\mathcal{N}_2) = 6\left[\int_0^{\arcsin\left(\frac{a\sqrt{3}}{l}\right)-\frac{\pi}{3}} \text{area}(A''C''D''F'')d\phi\right] =$$

$$= \sqrt{3}\left(l^2+12a^2\right)\arcsin\left(\frac{a\sqrt{3}}{l}\right) + 15a\sqrt{l^2-3a^2} - a^2\left(4\sqrt{3}\pi+9\right)$$

$$-\frac{l^2}{6}\left(9+2\sqrt{3}\pi\right).$$

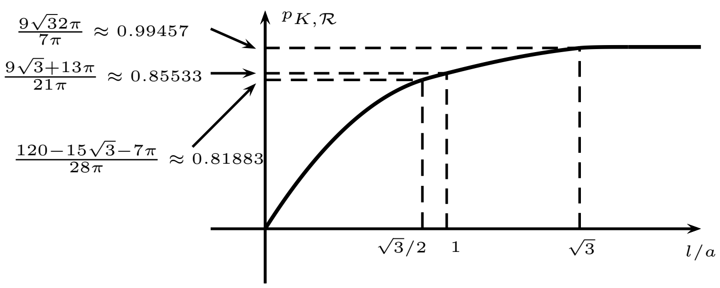
and so:

$$p_{K,\mathcal{R}} = \frac{1}{21\pi a^2}\left[9a^2\left(2\sqrt{3}+5\pi\right) + l^2\left(3\sqrt{3}+2\pi\right) - 30\sqrt{3}a\sqrt{l^2-3a^2}\right.$$

$$\left.-6\left(12a^2+l^2\right)\arcsin\left(\frac{a\sqrt{3}}{l}\right)\right].$$

□

This is the probability distribution of $p_{K,\mathcal{R}}$



Let us observe that $p_{K,\mathcal{R}} \geq \frac{1}{2}$ for $l \geq \frac{60\sqrt{3}-\sqrt{10800-630\sqrt{3}\pi-294\pi^2}}{30\sqrt{3}+14\pi}a \approx 0.3863a$ i.e. also for "small" needles.

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DIPARTIMENTO DI MATEMATICA PER LE SCIENZE ECONOMICHE, FINANZIARIE ED ATTUARIALI, UNIVERSITÀ CATTOLICA DEL S. CUORE, LARGO GEMELLI 1, MILAN, ITALY.

Email address: salvatore.vassallo@unicatt.it