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SOLUTION OF INTEGRAL EQUATIONS BY BESSEL WAVELET TRANSFORM

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ABSTRACT. The integral equations of the first kind arise in many areas of science and engineering fields such as image processing and electromagnetic theory. The wavelet transform technique to solve integral equation allows the creation of very fast algorithms when compared with known algorithms. Various wavelet methods are used to solve certain type of integral equations. To find the most accurate and stable solution of the integral equation Bessel wavelet is the appropriate method. To study the properties of solution of integral equations on distribution spaces Bessel wavelet transform is also used. In this paper, we accomplished the concept of Hankel convolution and continuous Bessel wavelet transform to solve certain types of integral equations (Volterra integral equation of first kind, Volterra integral equation of second kind and Abel integral equation). Also distributional wavelet transform and generalized convolution will be applied to find the solution of certain Integral equations.

1. INTODUCTION

Harmonic analysis has been used [1–4] to obtain solution of certain type of integral equations. During the preceding years, wavelet analysis involving special functions and generalized integral transform has been used in different areas

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of mathematics [5]. In pure mathematics, generalized convolution are used to study distribution and function spaces [6], which are useful in obtaining solutions of integral and differential equations. Several authors has solved differential equations by using classical wavelet transform.

Classical wavelets are used to solve certain integral equations, e.g. Pathak has studied solutions of Abel integral equation by using Gegenbauer wavelets [7], Mandal [8] has find solution of certain integral equations by using the technique of fractional calculus and fractonal convolution also Pathak et. al. [9] studied solutions of generalized Abel's integral equation by using Legendre wavelet and associated convolution.

Bessel functions are useful to olve certain type of differential equations. it is widely used to solve differential equations aries in quantum physics. Pathak and Dixit [10] has studied the construction of Bessel wavelet transform by using the theory of Hankel transform and associated convolution. Motivated from the above works study the solution of certain type of integral equations by using the theory of Hankel transform and Bessel wavelet transform.

In this paper, Bessel wavelet transform and distributional Bessel wavelet transform are used to obtain solutions of certain integral equations.

2. Preliminaries

Define

$$\sigma(u) = \frac{u^{2\gamma}}{2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)}, \gamma > 0$$

and

$$j(u) = C_\gamma u^{\frac{1}{2}-\gamma} J_{\gamma-\frac{1}{2}}(u), C_\gamma = 2^{\gamma-\frac{1}{2}}\Gamma\left(\gamma+\frac{1}{2}\right),$$

where $J_{\gamma-\frac{1}{2}}(u)$ denotes the Bessel function of order $\gamma - \frac{1}{2}$.

We define $L_{p,\sigma}(R_+), 1 \leq P \leq \infty$, as the space of those real functions f on the set of positive real numbers R_+ for which

$$\|f\|_{p,\sigma} = \left[\int_0^\infty \mid f(u) \mid^p d\sigma(u)\right]^{\frac{1}{p}} < \infty, 1 \leq p < \infty.$$

For each $f \in L_{1,\sigma}(R_+)$, the Hankel transform of f is defined by [11]

$$\hat{f}(u) = \int_0^\infty j(ut)f(t)d\sigma(t). \tag{2.1}$$

For $f \in L_{1,\sigma}(R_+)$ the inverse of (2.1) is given by [11]

$$f(u) = \int_0^\infty j(ut)\hat{f}(u)d\sigma(t). \tag{2.2}$$

For $f, g \in L_{1,\sigma}(R_+)$ the Hankel convolution is defined by [11]

$$(f * g)(u) = \int_0^\infty (\tau_u f)(\nu) g(\nu) d\sigma(\nu),$$

where the Hankel translation τ_u is given by

$$(\tau_u f)(\nu) = \int_0^\infty H(u, \nu, \omega) f(u) d\sigma(\omega). \tag{2.3}$$

$H(u, \nu, \omega)$ is symmetric in u, ν, ω. Applying (2.2) and (2.3) we get the formula

$$\int_0^\infty j(\omega t) H(u, \nu, \omega) d\sigma(\omega) = j(ut) j(\nu t).$$

Setting $t = 0$, we get

$$\int_0^\infty H(u, \nu, \omega) d\sigma(\omega) = 1.$$

Trimeche [12] has shown that the integral is convergent

$$\|\hat{f}(u, \nu)\|_{1,\sigma} \leq \|f\|_{1,\sigma}.$$

From (2.3) we have

$$(f * g)(u) = \int_0^\infty \int_0^\infty H(u, \nu, \omega) f(u) g(\nu) d\sigma(\omega) d\sigma(\nu).$$

Properties of Hankel convolution:

(i) If $f, g \in L_{1,\sigma}(R_+)$ then from (2.2)

$$\|f * g\|_{1,\sigma} \leq \|f\|_{1,\sigma} \|g\|_{1,\sigma}.$$

(ii) $(f * g)^\wedge(u) = \hat{f}(u)\hat{g}(u)$.

(iii) Let $f \in L_{1,\sigma}(R_+)$ and $g \in L_{p,\sigma}(R_+), P \geq 1$. Then $(f * g)$ exists and continuous. Also

$$\|f * g\|_{p,\sigma} \leq \|f\|_{1,\sigma} \|g\|_{p,\sigma}. \tag{2.4}$$

(iv) Let $f \in L_{p,\sigma}(R_+)$ and $g \in L_{q,\sigma}(R_+), \frac{1}{P} + \frac{1}{q} = 1$. Then $(f * g)$ exists and continuous. Also

$$\|f * g\|_{\infty,\sigma} \leq \|f\|_{p,\sigma} \|g\|_{q,\sigma}.$$

(v) Let $f \in L_{p,\sigma}(R_+)$ and $g \in L_{q,\sigma}(R_+), \frac{1}{r} = \frac{1}{P} + \frac{1}{q} - 1$. Then $(f * g)$ exists and continuous. Also

$$\|f * g\|_{r,\sigma} \leq \|f\|_p \|g\|_q.$$

(vi) Let $f \in L_{p,\sigma}(R_+), g \in L_{q,\sigma}(R_+$ and $h \in L_{r,\sigma}(R_+$. Then

$$\left| \int_0^\infty f(u)(g * h)(u) d\sigma(u) \right| \leq \|f\|_{p,\sigma} \|g\|_{q,\sigma} \|h\|_{r,\sigma}.$$

The following Parseval's identity [11] also holds

$$\int_0^\infty f(u)g(u)d\sigma(u) = \int_0^\infty \hat{f}(\nu)\hat{g}(\nu)d\sigma(\nu).$$

For $\phi \in L_{1,\sigma}(R_+)$, using translation τ given in (2.3) and dilation

$$D_{\rho_1} f(u,\nu) = f\left(\rho_1 u, \rho_1 \nu\right).$$

The Bessel wavelet (2.4)

$$\hat{\phi}\left(\frac{t}{\rho_1}, \frac{\rho_2}{\rho_1}\right) = D_{\frac{1}{\rho_1}} \tau_{\rho_2}\phi(t) = \int_0^\infty \phi(\omega) H\left(\frac{t}{\rho_1}, \frac{\rho_2}{\rho_1}, \omega\right) d\sigma(\omega)$$

The continuous Bessel wavelet transform of a function f for the wavelet ϕ is given by [10]

$$(2.5) \qquad \left(B_\phi f\right)(\rho_2, \rho_1) = \rho_1^{-2\gamma-1} \int_0^\infty f(t)\overline{\phi\left(\frac{t}{\rho_1}, \frac{\rho_2}{\rho_1}\right)} d\sigma(t), \rho_1 > 0, \rho_2 \geq 0,$$

where

$$\phi_{\rho_2,\rho_1}(t) = \rho_1^{-2\gamma-1}\phi\left(\frac{\rho_2}{\rho_1}, \frac{u}{\rho_1}\right).$$

When $f \in L_{p,\sigma}(R_+)$ and $\phi \in L_{q,\sigma}(R_+)$ and $\frac{1}{P} + \frac{1}{q} = 1$, then

$$\mid \left(B_\phi f\right)(\rho_2, \rho_1) \mid \leq \rho_1^{(2\gamma+1)\left(\frac{1}{q}-1\right)} \|f\|_{p,\sigma} \|\phi\|_{q,\sigma}.$$

The convolution product of continuous Bessel wavelet transform is given by [11]

$$(2.6) \qquad B_\phi \left(f \otimes g\right)(\rho_2, \rho_1) = \left(B_\phi f\right)(\rho_2, \rho_1)\left(B_\phi g\right)(\rho_2, \rho_1).$$

Whereas the associated convolution is defined by

$$\left(f \otimes g\right)(\omega) = \int_0^\infty \left(\tau_{z,\rho_1} f\right)(\nu) g(\nu) d\sigma(\nu)$$

$$\left(f \otimes g\right)(\omega) = \int_0^\infty \int_0^\infty f(u)g(\nu)H_{\rho_1}(u,\nu,\omega)d\sigma(u).$$

Pathak and Dixit [10] defined the inversion formula for Bessel wavelet transform. From [10], let $\phi \in L_{2\sigma}(R_+)$ be a basic wavelet defines Bessel wavelet transform (2.5). Then, for

$$C_\phi = \int_0^\infty \omega^{-2\gamma-1} \mid \hat{\phi}(\omega) \mid^2 d\omega > 0,$$

we have

$$\int_0^\infty \int_0^\infty \left(B_\phi f\right)(\rho_2, \rho_1)\left(B_\phi g\right)(\rho_2, \rho_1)\rho_1^{-2\gamma-1} d\sigma(\rho_1) d\sigma(\rho_2) = C_\phi \langle f, g \rangle$$

for all $f, g \in L_{2\sigma}(R_+)$.

3. Solution of Integral Equation by Bessel Wavelet Transform:

In this section we use Bessel wavelet transform and associated convolution to find the solution if certain integral equation. This idea is also extended on distribution spaces.

(a) Volterra Integral Equation of First Kind:

Let

$$f(u) = \int_0^u \tau_s k(t) g(t) dt = (k * g), \tag{3.1}$$

where $\tau_s k(t)$ is the translation operator. Taking the Bessel wavelet transform of (3.1) and by using convolution product (2.6), the expression (3.1) yields

$$(B_\phi f)(\rho_2, \rho_1) = (B_\phi k)(\rho_2, \rho_1)(B_\phi g)(\rho_2, \rho_1).$$

Therefore

$$(B_\phi g)(\rho_2, \rho_1) = \frac{(B_\phi f)(\rho_2, \rho_1)}{(B_\phi k)(\rho_2, \rho_1)}. \tag{3.2}$$

Taking inverse Bessel wavelet transform of equation (3.2), we have

$$g(t) = B_\phi^{-1}\left[\frac{(B_\phi f)(\rho_2,\rho_1)}{(B_\phi k)(\rho_2,\rho_1)}\right] = B_\phi^{-1} h(\rho_2, \rho_1),$$

where

$$h(\rho_2, \rho_1) = \left[\frac{(B_\phi f)(\rho_2,\rho_1)}{(B_\phi k)(\rho_2,\rho_1)}\right].$$

By using the inverse Bessel wavelet transform

$$g(t) = \frac{1}{C_\phi}\int\int_R h(\rho_2, \rho_1)\, \phi_{\rho_2,\rho_1}(t) \frac{d\rho_1 d\rho_2}{\rho_1^{-2\gamma-1}},$$

where

$$C_\phi = \int_0^\infty \frac{|\hat{\phi}(\omega)|^2}{\omega^{-2\gamma-1}} d\omega > 0.$$

The RHS of equation (3.2) can be evaluated by using the above formula.

(b) Volterra Integral Equation of Second Kind:

Let

$$g(u) = f(u) + \int_0^u \tau_s k(t) g(t) dt = f(u) + (k * g). \tag{3.3}$$

By taking the Bessel wavelet transform of (3.3) and using equation (2.6) we have

$$(B_\phi g)(\rho_2,\rho_1) = (B_\phi f)(\rho_2,\rho_1) + (B_\phi k)(\rho_2,\rho_1)(B_\phi g)(\rho_2,\rho_1).$$

Therefore

$$(B_\phi g)(\rho_2,\rho_1)\left[1 - (B_\phi k)(\rho_2,\rho_1)\right] = (B_\phi f)(\rho_2,\rho_1).$$

Therefore

$$(B_\phi g)(\rho_2,\rho_1) = \frac{(B_\phi f)(\rho_2,\rho_1)}{\left[1 - (B_\phi k)(\rho_2,\rho_1)\right]}. \tag{3.4}$$

Taking inverse Bessel wavelet transform equation (3.4), we have

$$g(t) = B_\phi^{-1}\left[\frac{\left(B_\phi f\right)(\rho_2,\rho_1)}{\left[1-\left(B_\phi k\right)(\rho_2,\rho_1)\right]}\right] = B_\phi^{-1}\eta(\rho_2,\rho_1).$$

where

$$\eta(\rho_2,\rho_1) = \frac{\left(B_\phi f\right)(\rho_2,\rho_1)}{\left[1-\left(B_\phi k\right)(\rho_2,\rho_1)\right]}.$$

By using the inverse Bessel wavelet transform

$$g(t) = \frac{1}{C_\phi}\int\int_R \eta(\rho_2,\rho_1)\,\phi_{\rho_2,\rho_1}(t)\frac{d\rho_1 d\rho_2}{\rho_1^{-2\gamma-1}}.$$

This gives the solution of (3.3).

(c) Abel Integral Equation:

Let

$$f(u) = \int_0^u \frac{g(u)}{(\tau - u)^\beta}du = \left(g * t^{-\beta}\right), 0 < \beta < 1. \tag{3.5}$$

When $B_\phi\left[t^{-\beta}\right]$ is known and the convolution of the Bessel wavelet transform is applied on equation (3.5), then

$$(B_\phi f)(\rho_2,\rho_1) = (B_\phi g)(\rho_2,\rho_1)\left(B_\phi t^{-\beta}\right)(\rho_2,\rho_1), \tag{3.6}$$

where $B_\phi\left[t^{-\beta}\right]$ is the Bessel wavelet transform of the function $t^{-\beta}$. By taking suitable mother wavelet ϕ, the Bessel wavelet transform of $t^{-\beta}$ is given by

$$\left(B_\phi t^{-\beta}\right)(\rho_2,\rho_1) \tag{3.7}$$

$$= \frac{c_\gamma \Gamma\left(\frac{-\beta+2\gamma+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)2^{2\left(\frac{\gamma}{2}-\beta\right)}}{\Gamma\left(\gamma+\frac{1}{2}\right)2^{\beta-\gamma-\frac{1}{2}}\Gamma\left(\gamma-\frac{\beta}{2}+\frac{1}{2}\right)} \tag{3.8}$$

$$\int_0^\infty (\rho_2+\omega\rho_1)^{-\beta}2F_1\left(\frac{\beta}{2};2\gamma;\frac{4\rho_2\omega\rho_1}{(\rho_2+\omega\rho_1)^2}\right)\phi(\omega)d\sigma(\omega)$$

$$= \varphi(\rho_2,\rho_1).$$

By using (3.7) in (3.6), we have

$$(B_\phi f)(\rho_2,\rho_1) = (B_\phi g)(\rho_2,\rho_1)\,\varphi(\rho_2,\rho_1).$$

Therefore

$$(B_\phi g) = \frac{\left(B_\phi f\right)(\rho_2,\rho_1)}{\varphi(\rho_2,\rho_1)}.$$

Now equation (3.6) can be evaluated by inverse Bessel wavelet transform which is given by

$$g(t) = B_\phi^{-1}\left[\frac{\left(B_\phi f\right)(\rho_2,\rho_1)}{\varphi(\rho_2,\rho_1)}\right] = B_\phi^{-1}\mu\left(\rho_2,\rho_1\right),$$

where

$$\mu = \left[\frac{\left(B_\phi f\right)(\rho_2,\rho_1)}{\varphi(\rho_2,\rho_1)}\right].$$

If $f(t)$ is given function, then one can evaluate $g(t)$ by using inverse wavelet transform.

$$g(t) = \frac{1}{C_\phi}\int\int_R \left[\frac{\left(B_\phi f\right)(\rho_2,\rho_1)}{\varphi(\rho_2,\rho_1)}\right]\phi_{\rho_2,\rho_1}(t)\frac{d\rho_1 d\rho_2}{\rho_1^{-2\gamma-1}}.$$

To find the solution of integral Equations on different distribution and function spaces. We need to study generalized convolution and translation operator on distribution and function spaces. Several authors has worked in this direction. A. Prasad [14] has studied the Bessel wavelet transform as distribution spaces. Pathak [15] has obtained the construction of wavelets on generalized Sobolev spaces. He studied the generalized translation and generalized convolution structure for the generalized Integral transform. The solutions of Integral equations can be obtained by using the concept of generalized convolution.

For example, the space D_k, which is the space of compactly supported functions [1] that can be classified with the distribution space $D'(R)$ whose support is $[\rho_1,\rho_2)$. Also $S(R^n)$ is the space of tempered distribution whose support is contained in $[\rho_1,\infty)$.

If both f and g are functions of compact supports, then $(f * g)$ is always defined. If f and g are generalized functions, then the generalized convolution [13] of f and g is given by the equation

$$(f * g)(\omega) = \int_a^b \left(\tau_\omega g\right) g(\nu) d\nu.$$

Also the generalized Fourier transforms of convolution of two distributions f and g is given by

$$I_\phi(f * g)(t) = \left(I_\phi f\right)(t)\left(I_\phi g\right)(t).$$

Moreover the solutions of integral equations obtained above by generalized Bessel wavelet transform can be applied for generalized functions, either by considering integral equations on distribution spaces or the solutions of integral equations, which can be calculated by using distributional wavelet transform where $f(t), g(t)$ and $t^{-\beta}$ are generalized functions.

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