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THE FOURIER-JACOBI WAVELET CONVOLUTION PRODUCT

C. P. Pandey[1], Pranami Phukan, and Mopi Ado

ABSTRACT. The convolution product associated with the Fourier-Jacobi wavelet transformation is investigated. Certain norm inequalities for the convolution product are established.

1. INTRODUCTION

May authors studied the Fourier-Jacobi convolution for the following form of the Fourier-Jacobi transformation of a function $f \in L^1(\mu)$ and $L^1(\mu) = \{f : \int_0^\infty \mid f(x) \mid d\mu(x) < \infty\}$. Namely,

$$(j_\mu f)(x) = \hat{f}(x) = \int_0^\infty f(t)\varphi_\lambda(t) d\mu(t), \tag{1.1}$$

where

$$d\mu(t) = \frac{\Delta(t)}{(2\pi)^{\frac{1}{2}}} dt,$$
$$\varphi_\lambda(t) = (e^t - e^{-t})^{i\lambda-\rho} F\left(\frac{\beta-\alpha+1-i\lambda}{2}, \frac{\rho-i\lambda}{2}; 1-i\lambda; -\frac{1}{(sht)^2}\right)$$

we say that $f \in L^1(\mu), 1 \le p < \infty$, if

$$\|f\|_p = \left(\int_0^\infty \mid f \mid^p d\mu\right)^{\frac{1}{p}} < \infty.$$

[1]*corresponding author*

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If $f \in L^1(\mu)$ and $(j_\mu f) \in L^1(\mu)$ then the inverse Fourier-Jacobi transform is given by

$$f(x) = \left(j_\mu^{-1}\left[\hat{f}\right]\right)(x) = \int_0^\infty \hat{f}(x)\varphi_\lambda(t)d\nu(\lambda), \tag{1.2}$$

where $f \in L^1(\mu), g \in L^1(\mu)$ then the Fourier-Jacobi convolution is defined by

$$\begin{aligned}(f*g)(x) &= \int_0^\infty \int_0^\infty f(z)g(y)K(x,y,z)d\mu(z)d\mu(y) \\ (f*g)(x) &= \int_0^\infty (\tau_x f)(y)g(y)d\mu(y),\end{aligned} \tag{1.3}$$

where the Fourier-Jacobi translation τ_x is given by

$$\begin{aligned}(\tau_x f)(y) &= f(x,y) = \int_0^\infty f(z)K(x,y,z)d\mu(z), 0 < x, y < \infty \\ K(x,y,z) &= \int_0^\infty \varphi_\lambda(x)\varphi_\lambda(y)\varphi_\lambda(z)d\nu(\lambda) \\ &= \frac{2^{\frac{1}{2}-2\rho}\Gamma(\alpha+1)(chxchychz)^{\alpha-\beta-1}}{\Gamma\left(\alpha+\frac{1}{2}\right)(shxshyshz)^{2\alpha}} \times F\left(\alpha+\beta, \alpha-\beta; \alpha+\tfrac{1}{2}; \tfrac{1-B}{2}\right),\end{aligned} \tag{1.4}$$

with

$$B = \frac{(chx)^2+(chy)^2+(chz)^2-1}{2chxchychz}, \mid x-y \mid < z < x_1 + x_2.$$

Here we note that $K(x,y,z)$ is symmetric in x, y, z. Applying (1.2) and (1.4), we get

$$\int_0^\infty \varphi_\lambda(z)K(x,y,z)d\mu(z) = \varphi_\lambda(x)\varphi_\lambda(y).$$

Setting $t = 0$, we get

$$\int_0^\infty K(x,y,z)d\mu(z) = 1.$$

Therefore in view of (1.4)

$$\|\hat{f}(x,y)\|_1 \leq \|f\|_1. \tag{1.5}$$

Now, using (1.4) awe can write (1.3) in the following form

$$(f*g)(x) = \int_0^\infty \int_0^\infty K(x,y,z)f(z)g(y)d\mu(z)d\mu(y).$$

Some important properties of Fourier-Jacobi convolution that are relevant are:

1. If $f, g \in L^1(\mu)$ then from [1]

$$\|f*g\|_1 \leq \|f\|_1\|g\|_1. \tag{1.6}$$

2. With the same assumptions

$$j_\mu(f*g)(x) = (j_\mu f)(x)(j_\mu g)(x). \tag{1.7}$$

3. Let $f \in L^1(\mu)$ and $g \in L^p(\mu), p \geq 1$. Then $(f*g)$ exists, is continuous and from [1], we get the inequality

$$\|f*g\|_p \leq \|f\|_1 \|g\|_p. \tag{1.8}$$

4. Let $f \in L^p(\mu)$ and $g \in L^q(\mu), \frac{1}{p}+\frac{1}{q}=1$. Then $(f*g)$ exists, is continuous and from [1], we have

$$\|f*g\|_\infty \leq \|f\|_p \|g\|_q. \tag{1.9}$$

5. Let $f \in L^p(\mu)$ and $g \in L^q(\mu), \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$. Then $(f*g)$ exists, is continuous and from [1], we get the inequality

$$\|f*g\|_r \leq \|f\|_p \|g\|_q. \tag{1.10}$$

6. Let $f \in L^p(\mu), g \in L^q(\mu)$ and $h \in L^r(\mu)$. Then the weighted norm inequality

$$\mid \int_0^\infty f(x)(g*h)(x)d\mu(x) \mid \leq \|f\|_p \|g\|_q \|h\|_r$$

holds for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$.

As indicated above, the proof of property 1 to 1 are well known. Hence we next give the proof of 1.

Using Holder's inequality, we get

$$\mid \int_0^\infty f(x)(g*h)(x)d\mu(x) \mid \leq \|f\|_p \|g*h\|_s, \frac{1}{p}+\frac{1}{s}=1.$$

Therefore using (1.9) we have

$$\mid \int_0^\infty f(x)(g*h)(x)d\mu(x) \mid \leq \|f\|_p \|g\|_q \|h\|_s, \frac{1}{s}=\frac{1}{q}+\frac{1}{r}-1.$$

From paper, j_μ is isometric on $L^2(\mu)$, $\left(j_\mu^{-1} j_\mu f\right)=f$, then Parsevals formula of the Fourier-Jacobi transformation for $f,g \in L^2(\mu)$ given by

$$(f*g)^\wedge(x) = \hat{f}(x)\hat{g}(x)$$
$$\Rightarrow \int_0^\infty f(x)(g)(x)d\mu(x) = \int_0^\infty (j_\mu f)(y)(j_\mu g)(y)d\mu(y).$$

Furthermore, the relation also holds for $f,g \in L^1(\mu)$, from [7].

For $\psi \in L^1(\mu)$, using translation τ given in equation (1.4) and dilation $D_a f(x,y) = f(ax, ay)$,

$$\psi\left(\frac{t}{a}, \frac{b}{a}\right) = D_{\frac{1}{a}} \tau_b \psi(t) = \int_0^\infty \psi(z) K\left(\frac{t}{a}, \frac{b}{a}, z\right) d\mu(z). \tag{1.11}$$

Then the continuous Fourier-Jacobi wavelet transform [5] of a function $f \in L^1(\mu)$ with respect to wavelet $\psi \in L^1(\mu)$ is defined by

$$\begin{aligned} (J_\psi f)(b,a) &= \int_0^\infty f(t) \bar{\psi}_{b,a}(t) d\mu(t) \\ &= \int_0^\infty f(t) \bar{\psi}\left(\frac{t}{a}, \frac{b}{a}\right) d\mu(t), a > 0. \end{aligned} \tag{1.12}$$

By simple modification of (1.12), we get

$$(J_\psi f)(b,a) = (f * \psi)\left(\tfrac{b}{a}\right), a > 0.$$

From (1.3) and (1.4) the continuous Fourier-Jacobi wavelet transform of a function $f \in L^1(\mu)$ can be written in the form

$$\begin{aligned} (J_\psi f)(b,a) &= (f * \psi)\left(\frac{b}{a}\right) \\ \Rightarrow j_\mu \left[(J_\psi f)(b,a)\right] &= j_\mu \left[(f * \psi)\left(\frac{b}{a}\right)\right] \\ \Rightarrow j_\mu \left[(J_\psi f)(b,a)\right] &= (j_\mu f)\left(\frac{b}{a}\right)(j_\mu \psi)\left(\frac{b}{a}\right) \\ \Rightarrow \left[(J_\psi f)(b,a)\right] &= j_\mu^{-1}\left[(j_\mu f)\left(\frac{b}{a}\right)(j_\mu \psi)\left(\frac{b}{a}\right)\right] \\ \Rightarrow \left[(J_\psi f)(b,a)\right] &= \int_0^\infty \varphi_\lambda(b)(j_\mu f)(a,\lambda)(j_\mu \psi)(\lambda) d\nu(\lambda). \end{aligned} \tag{1.13}$$

Now we state that the Parseval formula of the Fourier-Jacobi wavelet transform from [5],

$$\int_0^\infty (J_\psi f)(b,a) \overline{(J_\psi g)(b,a)} \frac{d\mu(a) d\mu(b)}{a} = C_\psi \langle f, g \rangle, \tag{1.14}$$

for $f \in L^2(\mu)$ and $g \in L^2(\mu)$.

Now, we also state from [6] which is useful for our approximation results.

Theorem 1.1. *Suppose that*

1. $M_n(x) \geq 0, 0 < x < \infty$.
2. $\int_0^\infty M_n(x) d\mu(x) = 1, n = 0, 1, 2, 3, \ldots$.

3. $\lim_{n\to\infty}\int_\delta^\infty M_n(x)d\mu(x) = 0, for\ each\ \delta > 0.$
4. $\phi(x) \in L^\infty(\mu)$.
5. $\phi\ is\ continuous\ at\ x_0, x_0 \in [x-\delta, x+\delta]\ and\ \delta > 0.$

Then $\lim_{n\to\infty}(\phi * M_n)(x_0) = \phi(x_0)$.

Corollary 1.1. *With the same assumptions on* $M_n(x)$*, if* $f(x) \in L^\infty(\mu)$ *then*

$$\lim_{n\to\infty}\|f * M_n - f\|_1 = 0.$$

In this paper, Motivated from [4] we defined convolution product for Bessel wavelet transform and study some of its properties.

2. The Bessel Wavelet Convolution Product

In this section, using properties (1.5), (**??**) and (1.11), we formally define the convolution product for Fourier-Jacobi wavelet transform by the relation

$$J_\psi(f \otimes g)(b,a) = (J_\psi f)(b,a)\,(J_\psi g)(b,a). \tag{2.1}$$

And investigate its boundedness and approximation properties. This in turn implies that the product of two Fourier-Jacobi wavelet transform could be wavelet transform under certain conditions.

Theorem 2.1. *Let* $f, g, \psi \in L^1(\mu)$ *and* $j_\mu(\psi)(\omega) \neq 0$*. Then the Fourier-Jacobi avelet convolution can be written in the form*

$$(f \otimes g)(z) = \int_0^\infty (\tau_{z,a}f)(y)g(y)d\mu(y),$$

where

$$(\tau_{z,a}f)(y) = \int_0^\infty f(x)K_a(x,y,z)d\mu(x),$$

(2.2)

$$K_a(x,y,z) = \int_0^\infty\int_0^\infty \varphi_x(t)\varphi_y(\xi)\,(j_\mu\psi)(at)\,(j_\mu\psi)(a\xi)L_a(t,\xi,z)d\mu(\xi)d\mu(t),$$

$$L_a(t,\xi,z) = \int_0^\infty \varphi_\lambda(t)\varphi_\lambda(\xi)Q_a(\lambda,z)d\nu(\lambda),$$

and

$$Q_a(\lambda,z) = \int_0^\infty \frac{\varphi_\lambda(\omega)\varphi_\lambda(z)}{(j_\mu\psi)(a\omega)}d\nu(\omega). \tag{2.3}$$

Proof. From (1.13) we have

$$j_\mu\left[(J_\psi f)(b,a)\right](\omega) = (j_\mu f)(\lambda)\,(j_\mu\psi)(a\lambda). \tag{2.4}$$

Using (2.1) and (2.4), we get

$$\begin{aligned} &j_\mu \left[J_\psi \left(f \otimes g\right)(b,a)\right](\omega) = j_\mu \left[\left(J_\psi f\right)(b,a)\left(J_\psi g\right)(b,a)\right](\omega) \\ &= j_\mu \left[j_\mu^{-1}\left(\left(j_\mu f\right)(\cdot)\left(j_\mu \psi\right)(a,\cdot)\right) j_\mu^{-1}\left(\left(j_\mu g\right)(\cdot)\left(j_\mu \psi\right)(a,\cdot)\right)\right](\omega). \end{aligned}$$

By property (1.7) of the Fourier-Jacobi convolution, we have

$$j_\mu \left[J_\psi \left(f \otimes g\right)(b,a)\right](\omega) = \left[\left(j_\mu \psi\right)(a,\cdot)\left(j_\mu f\right)(a) * \left(j_\mu \psi\right)(a,\cdot)\left(j_\mu g\right)(\cdot)\right](\omega).$$

Therefore by (2.4), we get

$$\left(j_\mu \psi\right)(a\omega) j_\mu \left(f \otimes g\right)(\omega) = \left[\left(j_\mu \psi\right)(a,\cdot)\left(j_\mu f\right)(a) * \left(j_\mu \psi\right)(a,\cdot)\left(j_\mu g\right)(\cdot)\right](\omega). \tag{2.5}$$

This gives the relation between the Fourier-Jacobi wavelet transform convolution and Fourier-Jacobi transformation convolution.

Let us set

$$\begin{aligned} F_a &= \left(j_\mu \psi\right)(a,\cdot)\left(j_\mu f\right)(\cdot). \\ G_a &= \left(j_\mu \psi\right)(a,\cdot)\left(j_\mu g\right)(\cdot). \end{aligned}$$

Then by (1.3) and (1.4), we get

$$\begin{aligned} &\left(j_\mu \psi\right)(a\omega) j_\mu \left(f \otimes g\right)(\omega) \\ = &\int_0^\infty \left(\tau_\omega G_a\right)(\eta) d\mu(\eta) \\ = &\int_0^\infty F_a(\eta) \left\{ \int_0^\infty K(\omega,\eta,\xi) G_a(\xi) d\mu(\xi) \right\} d\mu(\eta) \\ = &\int_0^\infty \int_0^\infty F_a(\eta) G_a(\xi) K(\omega,\eta,\xi) d\mu(\xi) d\mu(\eta) \\ = &\int_0^\infty \int_0^\infty F_a(\eta) G_a(\xi) \left(\int_0^\infty \varphi_\lambda(\omega)\varphi_\lambda(\eta)\varphi_\lambda(\xi) d\nu(\lambda) \right) d\mu(\xi) d\mu(\eta) \\ = &\int_0^\infty \left(\int_0^\infty F_a(\eta)\varphi_\lambda(\eta) d\mu(\eta) \right) \left(\int_0^\infty G_a(\xi)\varphi_\lambda(\xi) d\mu(\xi) \right) \varphi_\lambda(\omega) d\nu(\lambda) \\ = &\int_0^\infty \left(j_\mu F_a\right)(\lambda)\left(j_\mu G_a\right)(\lambda) \varphi_\lambda(\omega) d\nu(\lambda). \end{aligned} \tag{2.6}$$

Therefore by the inversion formula of the Fourier-Jacobi wavelet transform (1.2), we have

$$
\begin{aligned}
j_\mu (f \otimes g)(z) &= \frac{1}{(j_\mu \psi)(a\omega)} \int_0^\infty (j_\mu F_a)(\lambda)(j_\mu G_a)(\lambda)\varphi_\lambda(\omega) d\nu(\lambda) \\
\Rightarrow (f \otimes g)(z) &= j_\mu^{-1} \left[\frac{\varphi_\lambda(z)}{(j_\mu \psi)(a\omega)} \int_0^\infty (j_\mu F_a)(\lambda)(j_\mu G_a)(\lambda)\varphi_\lambda(\omega) d\nu(\lambda) \right] \\
&= \int_0^\infty (j_\mu F_a)(\lambda)(j_\mu G_a)(\lambda) \left(\int_0^\infty \frac{\varphi_\lambda(z)\varphi_\lambda(\omega)}{(j_\mu \psi)(a\omega)} d\nu(\omega) \right) d\nu(\lambda) \\
&= \int_0^\infty (j_\mu F_a)(\lambda)(j_\mu G_a)(\lambda) Q_a(\lambda, z) d\nu(\lambda),
\end{aligned}
$$

where $Q_a(\lambda, z)$ is given by (2.3).

Then by the definition of the Fourier-Jacobi transformation (1.1)

$$
\begin{aligned}
&(f \otimes g)(z) \\
&= \int_0^\infty \int_0^\infty \varphi_\lambda(t)(j_\mu \psi)(at)(j_\mu f)(t) d\mu(t) \\
&\quad \left(\int_0^\infty \varphi_\lambda(\xi)(j_\mu \psi)(a\xi)(j_\mu g)(\xi) d\mu(\xi) \right) Q_a(\lambda, z) d\nu(\lambda) \\
&= \int_0^\infty \int_0^\infty (j_\mu \psi)(at)(j_\mu \psi)(a\xi)(j_\mu f)(t)(j_\mu g)(\xi) \\
&\quad \left(\int_0^\infty \varphi_\lambda(t)\varphi_\lambda(\xi) Q_a(\lambda, z) d\nu(\lambda) \right) d\mu(\xi) d\mu(t) \\
&= \int_0^\infty \int_0^\infty (j_\mu \psi)(at)(j_\mu \psi)(a\xi)(j_\mu f)(t)(j_\mu g)(\xi) L_a(t, \xi, z) d\mu(\xi) d\mu(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(f \otimes g)(z) \\
&= \int_0^\infty \int_0^\infty (j_\mu \psi)(at)(j_\mu \psi)(a\xi) \left(\int_0^\infty \varphi_x(t) f(x) d\mu(x) \int_0^\infty \varphi_y(\xi) g(y) d\mu(y) \right) \\
&\quad L_a(t, \xi, z) d\mu(\xi) d\mu(t) \\
&= \int_0^\infty \int_0^\infty f(x) g(y) \left(\int_0^\infty \int_0^\infty \varphi_x(t)\varphi_y(\xi)(j_\mu \psi)(at) \right. \\
&\quad \left. (j_\mu \psi)(a\xi) L_a(t, \xi, z) d\mu(\xi) d\mu(t) \right) d\mu(x) d\mu(y) \\
&= \int_0^\infty \int_0^\infty f(x) g(y) K_a(x, y, z) d\mu(x) d\mu(y),
\end{aligned}
$$

where

$$K_a(x,y,z)=\int_0^\infty\int_0^\infty \varphi_x(t)\varphi_y(\xi)\left(j_\mu\psi\right)(at)\left(j_\mu\psi\right)(a\xi)L_a(t,\xi,z)d\mu(\xi)d\mu(t).$$

If we define the generalized translation by

$$F_a(z,y)=\left(\tau_{z,a}f\right)(y)=\int_0^\infty f(x)K_a(x,y,z)d\mu(x),$$

then

$$\left(f\otimes g\right)(z)=\int_0^\infty \left(\tau_{z,a}f\right)(y)g(y)d\mu(y).$$

□

Theorem 2.2. *Assume that* $\inf_\omega \mid \left(j_\mu\psi\right)(a\omega)\mid = B_\psi(a)>0$. *Then*

$$\mid K_a(x,y,z)\mid \le \tfrac{a^{-2}}{B_\psi(a)}\left[\|\psi\|_{1,\mu}\right]^2.$$

Proof. From (2.2) we have

$$\begin{aligned}
&K_a(x,y,z)\\
=\;&\int_0^\infty\int_0^\infty \varphi_x(t)\varphi_y(\xi)\left(j_\mu\psi\right)(at)\left(j_\mu\psi\right)(a\xi)L_a(t,\xi,z)d\mu(\xi)d\mu(t)\\
=\;&\int_0^\infty\int_0^\infty \varphi_x(t)\varphi_y(\xi)\left(j_\mu\psi\right)(at)\left(j_\mu\psi\right)(a\xi)\\
&\quad\left(\int_0^\infty \varphi_\lambda(t)\varphi_\lambda(\xi)Q_a(\lambda,z)d\nu(\lambda)\right)d\mu(\xi)d\mu(t)\\
=\;&\int_0^\infty\left(\int_0^\infty \varphi_x(t)\varphi_\lambda(t)\left(j_\mu\psi\right)(at)d\mu(t)\right)\\
&\quad\left(\int_0^\infty \varphi_y(\xi)\varphi_\lambda(\xi)\left(j_\mu\psi\right)(a\xi)d\mu(\xi)\right)Q_a(\lambda,z)d\nu(\lambda)\\
=\;&\int_0^\infty j_\mu\left[\varphi_x(t)\left(j_\mu\psi\right)(at)\right](\lambda)j_\mu\left[\varphi_y(\xi)\left(j_\mu\psi\right)(a\xi)\right](\lambda)Q_a(\lambda,z)d\nu(\lambda)\\
=\;&\int_0^\infty j_\mu\left[\varphi_x(t)\left(j_\mu\psi\right)(at)*\varphi_y(\xi)\left(j_\mu\psi\right)(a\xi)\right](\lambda)\int_0^\infty \frac{\varphi_\lambda(\omega)\varphi_\lambda(z)}{\left(j_\mu\psi\right)(a\omega)}d\nu(\omega)d\nu(\lambda)\\
=\;&\int_0^\infty \left[\varphi_x(\cdot)\left(j_\mu\psi\right)(a\cdot)*\varphi_y(\cdot)\left(j_\mu\psi\right)(a\cdot)\right](\lambda)\varphi_\lambda(\omega)\left[\left(j_\mu\psi\right)(a\omega)\right]^{-1}d\nu(\omega).
\end{aligned}$$

Now, set $F_a(t)=\varphi_\lambda(t)(j_\mu\psi)(at)$ and assume that $\inf_\omega \mid \left(j_\mu\psi\right)(a\omega)\mid = B_\psi(a)>0$. Since $\mid j_\mu(z)\mid\le 1$, we have

$$\mid K_a(x,y,z)\mid \le \tfrac{1}{B_\psi(a)}\int_0^\infty \mid \left(F_a*F_a\right)\mid d\mu(\omega).$$

Using (1.6), we have

$$\begin{aligned} \mid K_a(x,y,z) \mid &\leq \frac{a^{-2}}{B_\psi(a)} \left[\|\psi\|_{1,\mu}\right]^2 \\ &\leq \frac{1}{B_\psi(a)} \left[\int_0^\infty \mid \varphi_\lambda(x)(j_\mu\psi)(a\lambda)d\mu(\lambda) \mid\right]^2 \\ &\leq \frac{1}{B_\psi(a)} \left[\int_0^\infty \mid \psi(a\lambda))d\mu(\lambda) \mid\right]^2 \\ &\leq \frac{1}{B_\psi(a)} \left[\|\psi_a\|_{1,\mu}\right]^2 \\ &\leq \frac{a^{-2}}{B_\psi(a)} \left[\|\psi\|_{1,\mu}\right]^2. \end{aligned}$$

In order to obtain Plancharal formula for the Fourier-Jacobi wavelet transform, we define the space

$$W^2(I\times I) = \left\{ f(b,a) : \|f\|_{W^2} = \left(\int_0^\infty \int_0^\infty \mid f(b,a) \mid^2 \frac{d\mu(a)d\mu(b)}{a^2}\right)^2 < \infty \right\}.$$

□

Theorem 2.3. *Let* $f \in L^2(\mu)$ *and let* $g \in L^2(\mu)$*. Then*

$$\| \left(j_\psi f\right)(b,a) \|_{\omega^2} = \sqrt{C_\psi}\|f\|_{2,\mu}.$$

Proof. Putting $f = g$ in (1.14), we prove the above theorem. □

Theorem 2.4. *Let* $f, g \in L^2(\mu)$ *and let* $\psi \in L^2(\mu)$ *be a Bessel wavelet which satisfies* $C_\psi = \int_0^\infty \mid \left(j_\psi f\right)(b,a) \mid^2 \frac{d\mu(a)}{a} > 0$*. Then*

$$\|f \otimes g\|_{2,\mu} \leq \|f\|_{2,\mu}\|g\|_{2,\mu}\|\psi\|_{2,\mu}.$$

Proof. Using theorem 2.3 and (2.1),

$$\begin{aligned} \sqrt{C_\psi}\|f\otimes g\|_{2,\mu} &= \|J_\psi\left(f\otimes g\right)\|_{W^2} \\ &= \|J_\psi f(b,a) J_\psi g(b,a)\|_{W^2} \\ &= \left(\int_0^\infty \int_0^\infty \mid J_\psi f(b,a) J_\psi g(b,a) \mid^2 \frac{d\mu(a)d\mu(b)}{a^2}\right)^{\frac{1}{2}}. \qquad (2.7) \end{aligned}$$

From (1.13) and (2.3), we have

$$\mid J_\psi g(b,a) \mid \leq \mid \left(g(a\cdot) * \psi(\cdot)\right)\left(\frac{b}{a}\right) \mid \leq \|g\|_{2,\mu}\|\psi\|_{2,\mu}. \qquad (2.8)$$

Applying (2.7) in (2.8), we have

$$\sqrt{C_\psi}\|f\otimes g\|_{2,\mu}\leq\|g\|_{2,\mu}\|\psi\|_{2,\mu}\left(\int_0^\infty\int_0^\infty\mid J_\psi f(b,a)J_\psi g(b,a)\mid^2\frac{d\mu(a)d\mu(b)}{a^2}\right)^{\frac{1}{2}}.$$

From 2.3, we obtain

$$\sqrt{C_\psi}\|f\otimes g\|_{2,\mu}\leq\|g\|_{2,\mu}\|\psi\|_{2,\mu}\sqrt{C_\psi}\|f\|_{2,\mu}.$$

Hence

$$\|f\otimes g\|_{2,\mu}\leq\|g\|_{2,\mu}\|\psi\|_{2,\mu}\|f\|_{2,\mu}.$$

□

3. Weighted Sobolev Space

In this section we study certain properties of the Fourier-Jacobi wavelet convolution on a Weighted Sobolev space defined below:

Definition 3.1. *The Zemanian space $H(\sigma)$, is the set of all infinitely differentiable functions ϕ on $(-\infty,\infty)$ such that*

$$\gamma^\sigma_{m,k}(\phi)=\sup_{x\in(0,\infty)}\mid x^m\left(x^{-1}\frac{d}{dx}\right)^k x^{-\sigma^2}\phi(x)\mid<\infty,$$

for all $m,k\in N_0$. Then $f\in H'(\sigma)$ is defined by the following way:

$$\langle f,\phi\rangle=\int_0^\infty f(x)\phi(x)dx,\phi\in H(\sigma),$$

Definition 3.2. *Let $k(\omega)$ be an arbitrary weight function. Then a function $\Phi\in[H(\sigma)]'$ is said to belong to the weighted sobolev space $G^p_k(\sigma)$ for $1\leq p<\infty$, if it satisfies*

$$\|\Phi\|_{p,\sigma,\mu,k}=\left(\int_{-\infty}^\infty\mid k(\omega)\left(H(\phi)\right)(\omega)\mid^p d\mu(w)\right)^{\frac{1}{p}},\tag{3.1}$$

where $a>0$ and $\Phi\in L^p(\sigma)$.

In what follows we shall assume that $k(\omega)=\mid(j_\mu\psi)(a\omega)\mid$ for fixed $a>0$.

Theorem 3.1. *Let $f\in G^1_k(\sigma)$ and $g\in G^1_k(\sigma),p\geq 1$. Then*

$$\|f\otimes g\|_{p,\sigma,\mu,k}=\left(\|f\|_{1,\sigma,\mu,k}\|g\|_{p,\sigma,\mu,k}\right).$$

Proof. In view of (3.1) we have

$$\|f\otimes g\|_{p,\sigma,\mu,k}=\left(\int_{-\infty}^\infty\mid k(\omega)j_\mu\left(f\otimes g\right)(\omega)\mid^p d\mu(\omega)\right)^{\frac{1}{p}}.$$

By (1.8) and (1.11) we have

$$\begin{aligned}\|f\otimes g\|_{p,\sigma,\mu,k} &\leq \|F_a(\omega)\|_{1,\sigma,\mu,k}\|G_a(\omega)\|_{p,\sigma,\mu,k}\\ &\leq \|\left(j_\mu\psi\right)(a\omega)\left(j_\mu f\right)(\omega)\|_{1,\sigma,\mu,k}\|\left(j_\mu\psi\right)(a\omega)\left(j_\mu g\right)(\omega)\|_{p,\sigma,\mu,k}.\end{aligned}$$

From Definition 3.2, we have

$$\|f\otimes g\|_{p,\sigma,\mu,k} = \|f\|_{1,\sigma,\mu,k}\|g\|_{p,\sigma,\mu,k}.$$

□

Theorem 3.2. *Let $f \in G_k^p(\sigma)$ and $g \in G_k^q(\sigma)$, with $1 \leq p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then*

$$\|f\otimes g\|_{r,\sigma,\mu,k} = \|f\|_{p,\sigma,\mu,k}\|g\|_{q,\sigma,\mu,k}. \tag{3.2}$$

Proof. Using (1.10) and (3.1) we get (3.2). □

Approximation properties of the Fourier-Jacobi wavelet convolution are given next.

Theorem 3.3. *Let $\psi_{n,a}(\omega) = \psi_n(a\omega), n = 0,1,2,....$ be the sequence of basic wavelet functions such that*

1. $\psi_{n,a}(\omega) \geq 0, 0 < \omega < \infty$.
2. $\int_0^\infty \psi_{n,a}(\omega)d\mu(\omega) = 1$.
3. $\lim_{n\to\infty}\int_\varepsilon^\infty \psi_{n,a}(\omega)d\mu(\omega) = 0,$ *for each* $\varepsilon > 0$.
4. $\left(j_\mu\psi_{n,a}\right)(\omega) \in L_\mu^1(I)$.
5. $j_\mu^{-1}\left[\left(j_\mu\psi_{n,a}\right)(\omega)\right] = \psi_{n,a}(\omega)$.

Then

$$\lim_{n\to\infty}\|f(b) - \left(J_{\psi_n}f\right)(b,a)\|_{1,\mu} = 0.$$

Proof. Refer in [6]. □

Theorem 3.4. *Let $k_n(\omega) = \left(j_\mu\psi\right)(a\omega)\left(j_\mu g_n\right)(\omega)$ for fixed $a > 0, n \in N$, and $\phi(\omega) = \left(j_\mu\psi\right)(a\omega)\left(j_\mu f\right)(\omega)$ satisfy:*

1. $k_n(\omega) \geq 0, 0 < \omega < \infty$.
2. $\int_{-\infty}^\infty k_n(\omega)d\mu(\omega) = 1, \omega = 0,1,2,3,\ldots.$
3. $\lim_{n\to\infty}\int_\delta^\infty k_n(\omega)d\mu(\omega) = 0$*, for each* $\delta > 0$.
4. $\phi(\omega) \in L^\infty(\mu)$.
5. ϕ *is continuous at* ω_0 *and* $\left(j_\mu\psi\right)(a\omega_0) \neq 0$ *for* $\omega_0 \in [\omega-\delta, \omega+\delta], \delta > 0$.

Then

$$\lim_{n\to\infty} j_\mu\left(f\otimes g_n\right)(\omega_0)=\left(j_\mu f\right)(\omega_0).$$

Proof. In view of relation (2.5) we have

$$\left(j_\mu\psi\right)(a\omega)j_\mu\left(f\otimes g_n\right)(\omega)=\left(\phi*k_n\right)(\omega).$$

Now, using Theorem 1.1, we have

$$\begin{aligned}\lim_{n\to\infty}\left(j_\mu\psi\right)(a\omega_0)j_\mu\left(f\otimes g_n\right)(\omega_0) &= \lim_{n\to\infty}\left(\phi*k_n\right)(\omega_0)\\ &= \phi(\omega_0)\\ &= \left(j_\mu\psi\right)(a\omega_0)\left(j_\mu f\right)(\omega_0).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} j_\mu\left(f\otimes g_n\right)(\omega_0)=\left(j_\mu f\right)(\omega_0).$$

□

Theorem 3.5. *Let $f,\psi\in L^2(\mu)$ and $k_n(\omega)$ be the same as Theorem 3.4, which satisfies all the properties of Theorem 3.3. Then*

$$\lim_{n\to\infty}\|\left(j_\mu\psi\right)(a\omega_0)\left(j_\mu f\right)(\omega_0)-\left(j_\mu\psi\right)(a\omega_0)j_\mu\left(f\otimes g_n\right)(\omega_0)\|_{1,\mu}=0.$$

Proof. Using (2.5), we have

$$\begin{aligned}&\lim_{n\to\infty}\|\left(j_\mu\psi\right)(a\omega_0)\left(j_\mu f\right)(\omega_0)-\left(j_\mu\psi\right)(a\omega_0)j_\mu\left(f\otimes g_n\right)(\omega_0)\|_{1,\mu}\\ =&\lim_{n\to\infty}\|\left(j_\mu\psi\right)(a\omega_0)\left(j_\mu f\right)(\omega_0)-[\left(j_\mu\psi\right)(a\cdot)\left(j_\mu f\right)(\cdot)\\ &\quad *\left(j_\mu\psi\right)(a\cdot)\left(j_\mu g_n\right)(\cdot)](\omega_0)\|_{1,\mu}\\ =&\lim_{n\to\infty}\|\psi(\omega_0)-\left(\psi*k_n\right)(\omega_0)\|_{1,\mu}.\end{aligned}$$

Since $f,\psi_a\in L^2(\mu),\psi(\omega)=\left(j_\mu f\right)\left(j_\mu\psi_a\right)=j_\mu\left(f*\psi_a\right)\in L^1(\mu)$. Therefore using the tools of [6], we have

$$\lim_{n\to\infty}\|\left(j_\mu\psi\right)(a\omega_0)\left(j_\mu f\right)(\omega_0)-\left(j_\mu\psi\right)(a\omega_0)j_\mu\left(f\otimes g_n\right)(\omega_0)\|_{1,\mu}=0.$$

□

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DEPARTMENT OF MATHEMATICS
NORTH EASTERN REGIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY
NIRJULI, PAPUMPARE, ARUNACHAL PRADESH, 791109
INDIA.
Email address: drcppandey@gmail.com

DEPARTMENT OF MATHEMATICS
NORTH EASTERN REGIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY
NIRJULI, PAPUMPARE, ARUNACHAL PRADESH, 791109
INDIA.
Email address: pranamiphukan94@gmail.com

DEPARTMENT OF MATHEMATICS
NORTH EASTERN REGIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY
NIRJULI, PAPUMPARE, ARUNACHAL PRADESH, 791109
INDIA.
Email address: mopiado007@gmail.com